
EE 3121
Differential Equations for Electrical Engineering

Tutorial 7: Laplace Transform



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Summary

- The Laplace transform $F(s)$ of a function $f(t)$ is

$$F(s) = L \{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

- The transform $F(s)$ exists provided the function $f(t)$ is **piecewise continuous** and of **exponential order** (i.e., for $t > M$, $|f(t)| < Ke^{at}$ for some K , a , and M).
- A linear operation,

$$L \{af(t) + bg(t)\} = aL \{f(t)\} + bL \{g(t)\}.$$

Summary

- This lecture computed the transform of a number of common functions, including functions multiplied by $e^{-\lambda t}$, functions delayed by time T , derivatives of a function, the convolution of two functions, and periodic functions.
- If $f(t)$ is continuous and $F(s)$ is its transform, then there is no other continuous function with that transform.
- Our process for solving ODEs is:
 - Transform each element of the given ODE from a function of time into an algebraic function of s .
 - Solve the resulting algebraic equation.

Summary

- Our process for solving ODEs is:
 - Manipulate the algebraic solution into a linear combination of forms that appear in the right-hand column of the **table** of Laplace transforms.
 - Move from right- to left-hand column to determine the function of time corresponding to the algebraic solution.
- Assuming zero initial conditions and that $L\{g(t)\} = G(s)$, the Laplace transform $Y(s)$ of the solution $y(t)$ to

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \dots + a_n y = g(t)$$

is

$$Y(s) = H(s)G(s),$$

Summary

- where

$$H(s) = \frac{1}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

is called the transfer function of the system. The transfer function is the Laplace transform of the kernel.

- The Dirac delta function is defined by

$$\int_0^{\infty} \delta(t - t_0) f(t) dt = f(t_0).$$

The limit of a sequence of impulses having unit integral but higher magnitude and shorter duration as the sequence progresses.

Summary

- The solution of the ODE

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \dots + a_n y = \delta(t)$$

is

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\left\{\frac{1}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}\right\}$$

- $h(t)$ is called the kernel of the ODE, and also called its impulse response.

Summary

- The initial and final value theorems:

- Let $F(s)$ be the Laplace transform of a function $f(t)$ continuous everywhere except possibly at the origin and let $|df/dt| \leq M$ for all t . Then

$$\lim_{t \rightarrow 0} f(t) = f(0^+) = \lim_{s \rightarrow \infty} sF(s).$$

- Let $F(s)$ be the Laplace transform of a continuous function $f(t)$ that has a limit as $t \rightarrow \infty$. Then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s).$$

Summary

- The response of a stable and damped LTI system to a periodic input eventually becomes periodic.
- A periodic input can cause resonance if the periodicity of the input is commensurate with the system's natural frequency, no matter the waveform.

Summary: Solved Problems

■ SP7.1 Use Laplace transforms to solve

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0, \quad y(0) = -3, \quad \dot{y}(0) = 1.$$

■ *Solution:*

■ Using Laplace transform:

$$L \left\{ \frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y \right\} = L \left\{ \frac{d^2y}{dt^2} \right\} + 10L \left\{ \frac{dy}{dt} \right\} + 25L \{y\} = 0.$$

■ Define $Y(s) = L \{y(t)\}$.

$$\blacksquare L \left\{ \frac{dy}{dt} \right\} = sL \{y(t)\} - y(0) = sY + 3.$$

$$\blacksquare L \left\{ \frac{d^2y}{dt^2} \right\} = sL \left\{ \frac{dy}{dt} \right\} - \dot{y}(0) = s(sY + 3) - 1 = s^2Y + 3s - 1.$$

Summary: Solved Problems

■ *Solution:*

■ We have

$$(s^2Y + 3s - 1) + 10(sY + 3) + 25Y = 0.$$

■ Rearranging: $(s^2 + 10s + 25)Y = -3s - 29$.

■ Solving:

$$Y(s) = \frac{-3s - 29}{s^2 + 10s + 25}.$$

■ Partial fraction expansion:

$$Y(s) = -3\frac{1}{s+5} - 14\frac{1}{(s+5)^2}.$$

■ The inverse Laplace transform:

$$y(t) = L^{-1}\{Y(s)\} = -3L^{-1}\left\{\frac{1}{s+5}\right\} - 14L^{-1}\left\{\frac{1}{(s+5)^2}\right\} = -3e^{-5t} - 14te^{-5t}.$$

Summary: Solved Problems

- **SP7.2** Use Laplace transforms to find the kernel $K(t-\tau)$ for the system governed by

$$\frac{d^3y}{dt^3} + 16\frac{d^2y}{dt^2} + 81\frac{dy}{dt} + 126y = g(t).$$

Compare the difficulty of this method with that of the method used in Tutorial 6, SP6.2.

- *Solution:*

- The kernel of an LTI system is its impulse response.
- The impulse response is the solution to ODE when $g(t) = \delta(t)$.
- Using Laplace transform and assuming zero initial conditions:

$$(s^3 + 16s^2 + 81s + 126)Y(s) = 1.$$

- Solving:

$$Y(s) = \frac{1}{s^3 + 16s^2 + 81s + 126}.$$

Summary: Solved Problems

■ *Solution:*

■ Partial fraction expansion:

$$\begin{aligned} Y(s) &= \frac{1}{s^3 + 16s^2 + 81s + 126} = \frac{1}{(s+3)(s+6)(s+7)} \\ &= \frac{c_1}{s+3} + \frac{c_2}{s+6} + \frac{c_3}{s+7} \\ &= \frac{(c_1 + c_2 + c_3)s^2 + (13c_1 + 10c_2 + 9c_3)s + (42c_1 + 21c_2 + 18c_3)}{(s+3)(s+6)(s+7)} \end{aligned}$$

$$c_1 + c_2 + c_3 = 0, \quad 13c_1 + 10c_2 + 9c_3 = 0, \quad 42c_1 + 21c_2 + 18c_3 = 1.$$

■ $c_1 = 1/12, \quad c_2 = -4/12, \quad c_3 = 3/12.$

■ Hence,

$$Y(s) = \frac{1}{12} \left(\frac{1}{s+3} - \frac{4}{s+6} + \frac{3}{s+7} \right).$$

Summary: Solved Problems

■ *Solution:*

- The kernel is given by:

$$K(t) = L^{-1} \{Y(s)\} = \frac{1}{12}(e^{-3t} - 4e^{-6t} + 3e^{-7t}).$$

- Even taking into account that we did not write out the steps in solving the three linear equations in three unknowns, this procedure is **simpler** than that used in Tutorial 6, Problem SP6.2. The difference is more marked when the fundamental solutions are not just simple exponentials.

Summary: Solved Problems

- **SP7.3** In Lecture 6, to analyze the stability of an Earth's satellite launched into a low-altitude circular orbit, it derived coupled differential equations for the radial deviation $r(t)$ and angular velocity deviation $v(t)$ and then used time domain techniques that were admittedly not easy to apply to find a differential equation in $r(t)$ alone. Our objective here is to show that Laplace transforms make this process very easy. We will solve equations in the same form but for clarity we will use simple coefficients. The equations for $r(t)$ and $v(t)$ were in the form

$$\begin{aligned}\ddot{r} &= -a\dot{r} + br + cv, \\ \dot{v} &= -d\dot{r} + er - fv - g,\end{aligned}$$

where a, b, c, d, e, f , and g are constants.

- **Solution:**

- Using Laplace transform and assuming initial conditions:

$$L\{r(t)\} = \hat{r}(s), \quad L\{v(t)\} = \hat{v}(s).$$

Summary: Solved Problems

■ *Solution:*



$$s^2\hat{r} = -as\hat{r} + b\hat{r} + c\hat{v},$$

$$s\hat{v} = -ds\hat{r} + e\hat{r} - f\hat{v} - g/s$$

or

$$(s^2 + as - b)\hat{r} - c\hat{v} = 0,$$

$$(ds - e)\hat{r} + (s + f)\hat{v} = -g/s$$

■ Express $\hat{v}$ in terms of $\hat{r}$,

$$\hat{v} = \hat{r} \frac{s^2 + as - b}{c}$$

■ Substituting this into the second equation:

$$(ds - e)\hat{r} + (s + f) \frac{s^2 + as - b}{c} \hat{r} = -g/s.$$

■ Multiplying through by c and multiplying out the product:

$$c(ds - e)\hat{r} + (s + f)(s^2 + as - b)\hat{r} = -cg/s.$$

Summary: Solved Problems

■ *Solution:*

- Combining terms in powers of s :

$$(s^3 + (a + f)s^2 + (cd + af - b)s - (ce + bf))\hat{r} = -cg/s.$$

- Using the inverse Laplace transform:

$$\frac{d^3y}{dt^3} + (a + f)\frac{d^2y}{dt^2} + (cd + af - b)\frac{dy}{dt} - (ce + bf)$$

Summary: Solved Problems

- **SP7.4** Using the initial value theorem, find the value of $y(t)$ just after $t = 0$ given that

$$\frac{dy}{dt} + ay = b\delta(t), \quad y(0) = y_0$$

where a and b are constants and $\delta(t)$ is the Dirac delta function.

- *Solution:*

- Using Laplace transform:

$$sY - y_0 + aY = b, \quad Y(s) = \frac{y_0 + b}{s + a},$$



$$y(0^+) = \lim_{s \rightarrow \infty} s \left(\frac{y_0 + b}{s + a} \right) = y_0 + b.$$