

EE 1004 Foundation of Information Systems and Data Analysis

Part II

Chapter 9: Vectors

Content

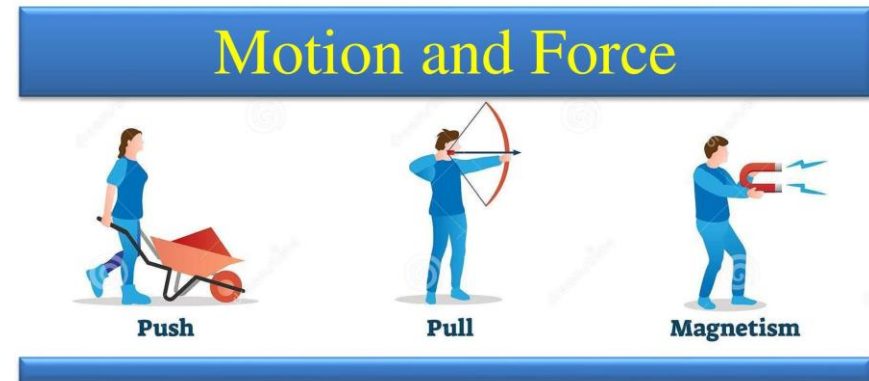
- Vector
 - Definition
 - Terminologies
 - Linear Combination
 - Dot product & Cross product
 - Linear Transformation
- Eigenvalue & Eigenvector
 - Definition
 - Calculation

What is Scalar? What is Vector?

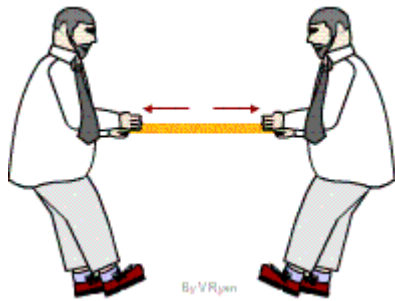
- **Scalar A** is a quantity represented by a **single number**.



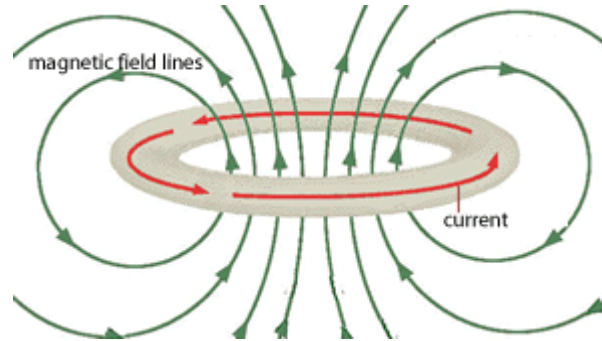
- **Vector $\vec{A}$** is a quantity represented by a **magnitude & direction**.
 - **Magnitude** of vector $\vec{A} = |\vec{A}|$
 - **Direction** of vector $\hat{A} = \frac{\vec{A}}{|\vec{A}|}$



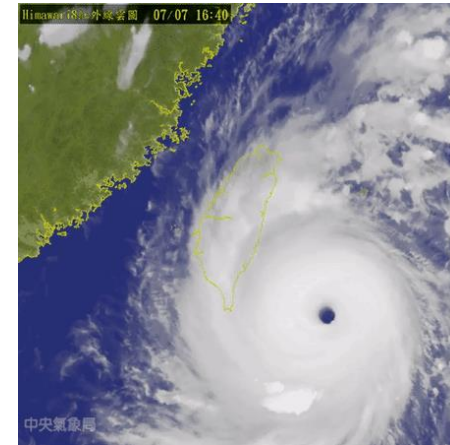
When we use vector?



Force



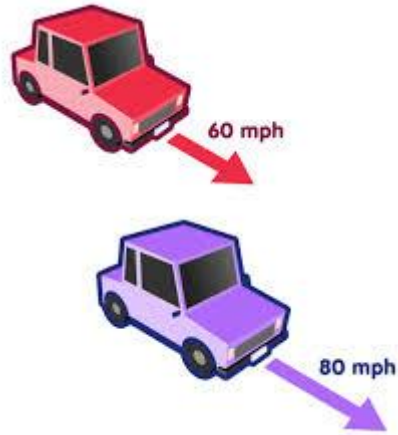
Magnetic field



Typhoon

Vector vs Scalar

Speed

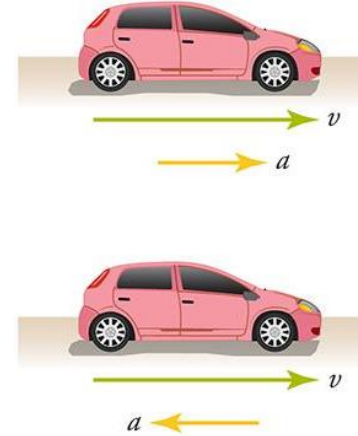


Scalar

Mass



Velocity



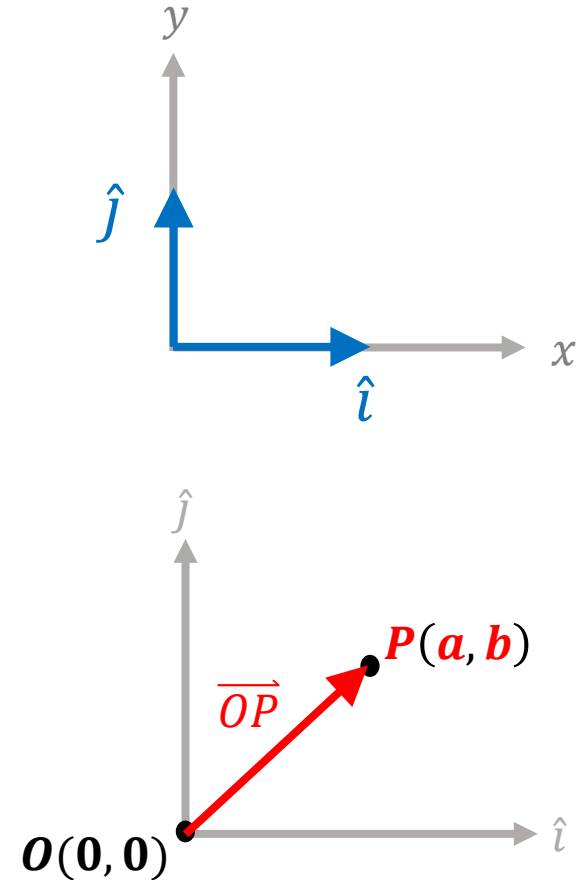
Vector

Weight



Representations in 2D-plane $\mathbf{R}^2$

- $\mathbf{R}^2$ is the **real coordinate space** of dimension 2.
- Unit Vectors
 - $\hat{i}, \hat{j}$ are orthogonal (perpendicular) to each other
 - commonly used in Cartesian coordinate system
 - indicates the directions in axes
- In $\mathbf{R}^2$ with the origin $O(0,0)$ and point $P(a, b)$, we can express as
 - $\overrightarrow{OP} = \begin{pmatrix} a \\ b \end{pmatrix} = a\hat{i} + b\hat{j}$



Representations in 3D-space $\mathbf{R}^3$

- Unit vectors in $\mathbf{R}^3$ are $\hat{i}$, $\hat{j}$ & $\hat{k}$.
- In $\mathbf{R}^3$ with the origin $O(0, 0, 0)$ and point $P(A_1, A_2, A_3)$, we can express as

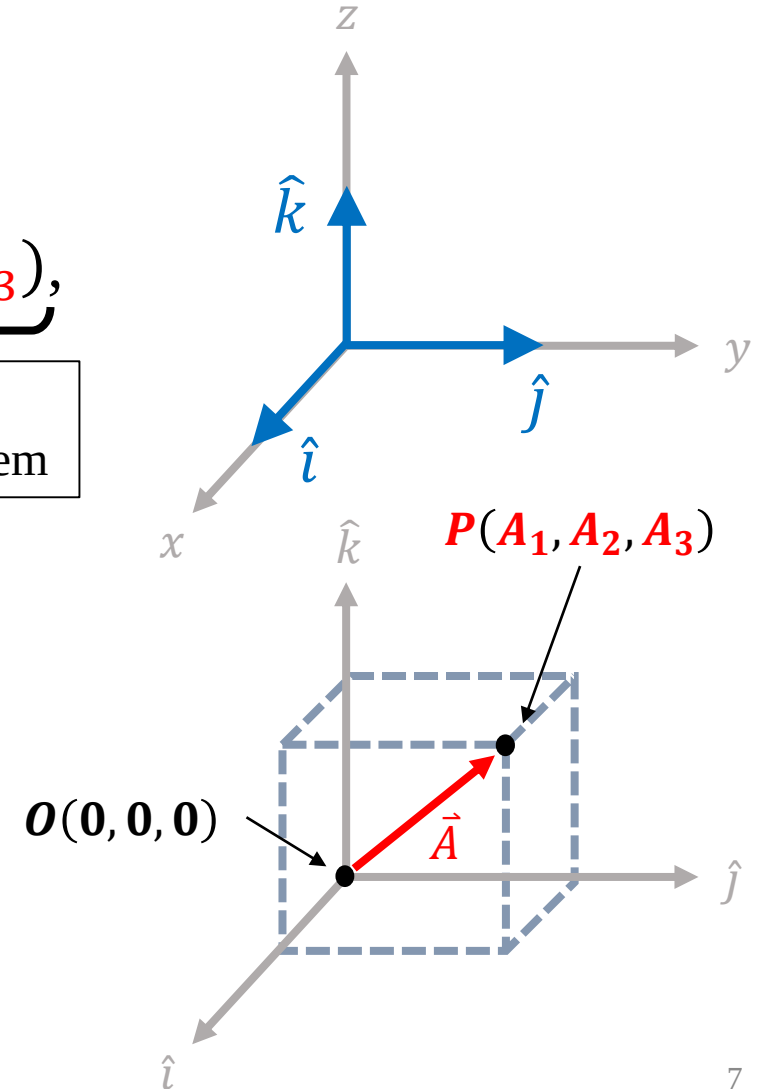
$$\bullet \quad \overrightarrow{OP} = \vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

Vector form

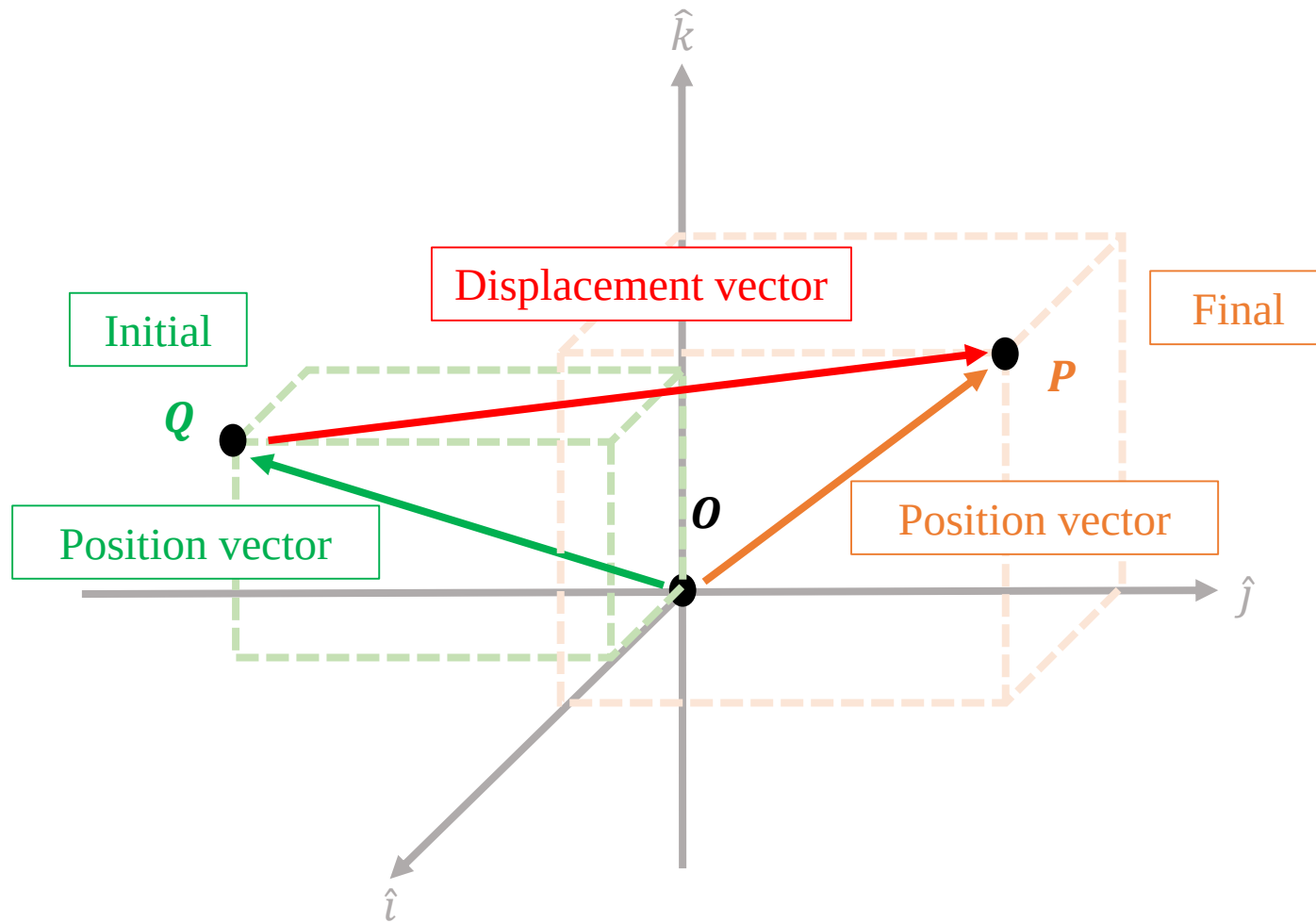
Component form

Location in
coordinate system

- A_1 is the 1st element of the vector
- Vector $\vec{A}$ is a 3-dimensional vector (a vector of size 3)

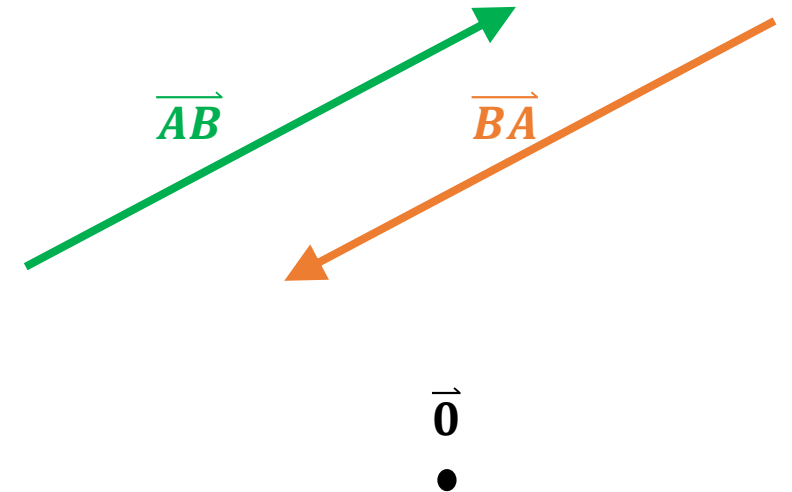
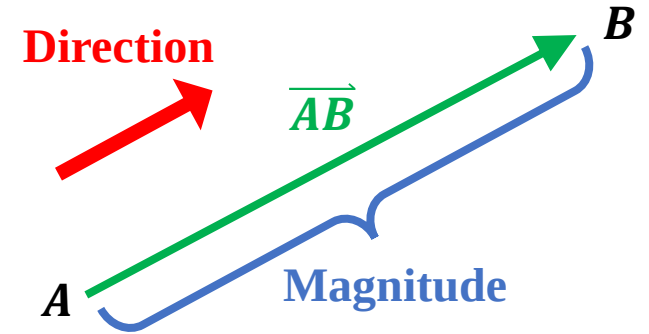


Types of vector



Graphically Operations of Vector

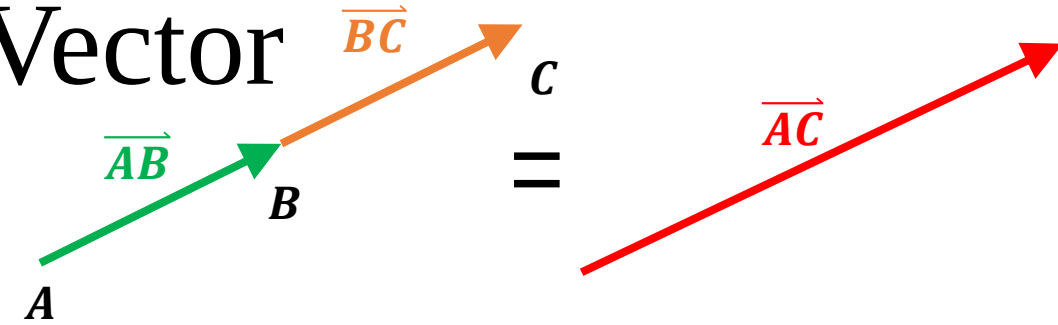
- A line segment joining two points A to B , we can denote a vector as $\overrightarrow{AB}$
 - Length of the line = magnitude
 - Arrow indicates the direction
 - $\overrightarrow{AB} = |\overrightarrow{AB}| \cdot \hat{AB}$
- Negative of a vector means opposite direction
 - $\overrightarrow{AB} \neq \overrightarrow{BA}$
 - $\overrightarrow{BA} = -\overrightarrow{AB}$
- Zero vector $\vec{0}$ means magnitude = 0 and no direction



Graphically Operations of Vector

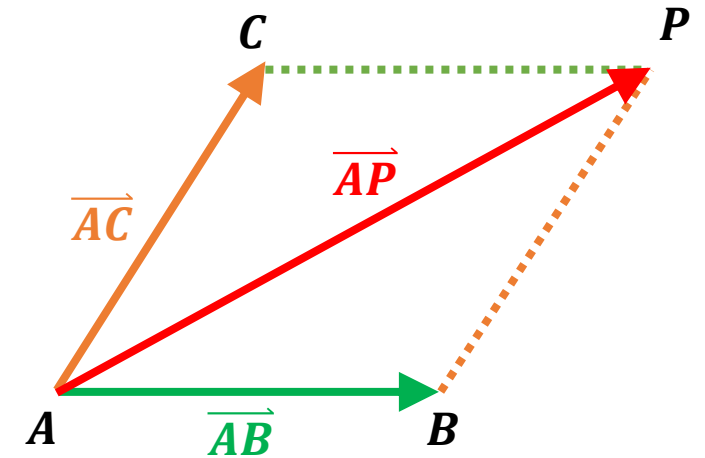
- Addition of collinear vectors

- $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
- $\overrightarrow{AB}$ & $\overrightarrow{BC}$ are collinear which means they are parallel



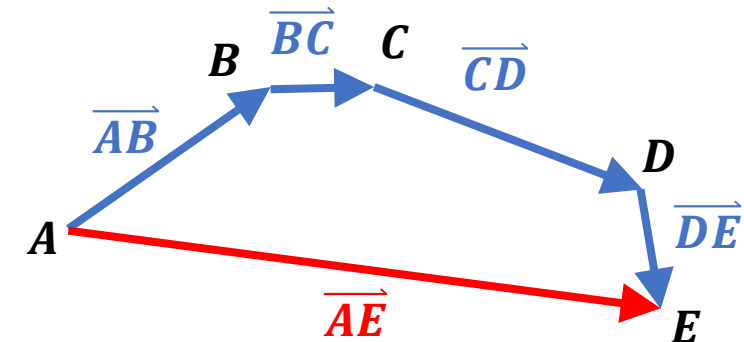
- Addition of vectors pointing to different directions

- $\overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{AC}$
- Using Parallelogram law



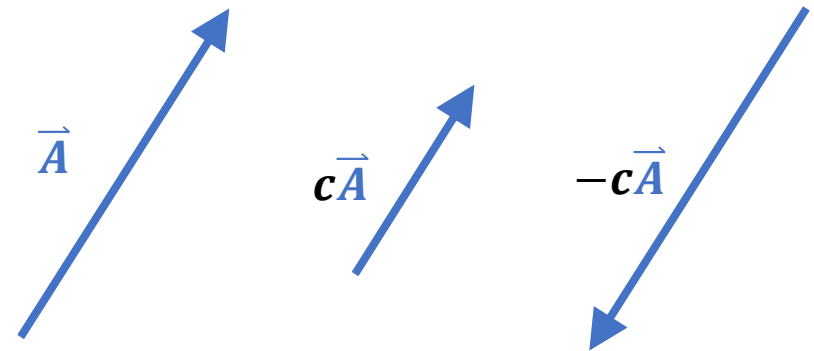
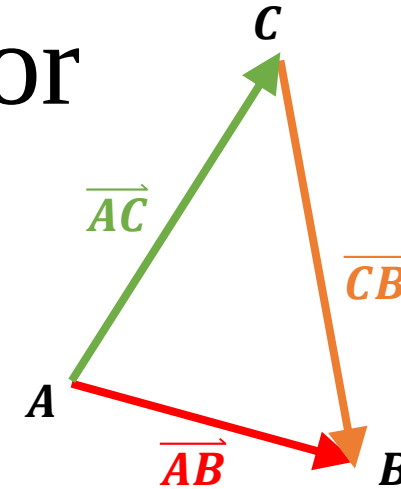
- Addition of multiple vectors (displacement)

- $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$
- Using Head-to-tail method



Graphically Operations of Vector

- Subtraction of vectors
 - $\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{CB} = \overrightarrow{AC} + (-\overrightarrow{BC})$
 - Same as addition of vectors
- Scalar multiplication of a vector
 - c is a scalar
 - $c > 0$: same direction
 - $c < 0$: different direction



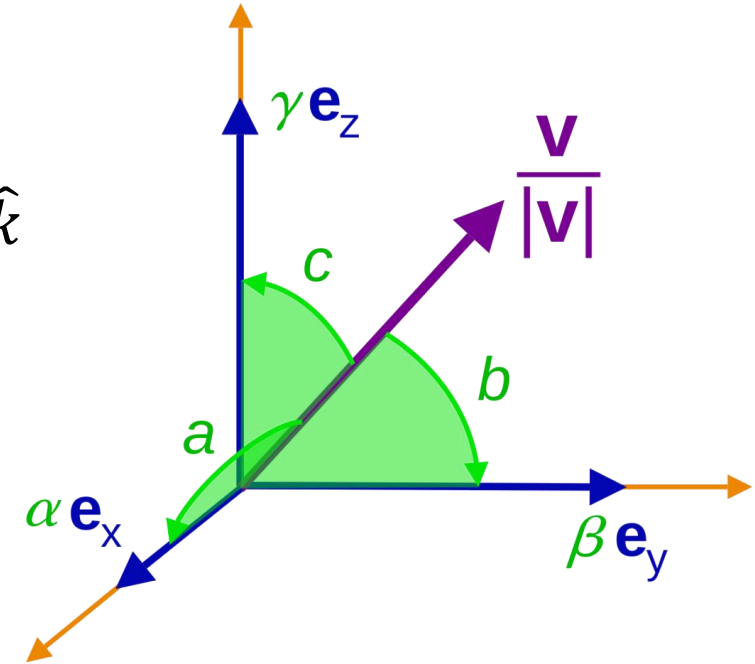
Properties of Vector

- Magnitude (displacement) of a vector $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

- Unit vector (direction) of a vector $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{A_1}{\sqrt{A_1^2 + A_2^2 + A_3^2}}\hat{i} + \frac{A_2}{\sqrt{A_1^2 + A_2^2 + A_3^2}}\hat{j} + \frac{A_3}{\sqrt{A_1^2 + A_2^2 + A_3^2}}\hat{k}$$



Properties of Vector

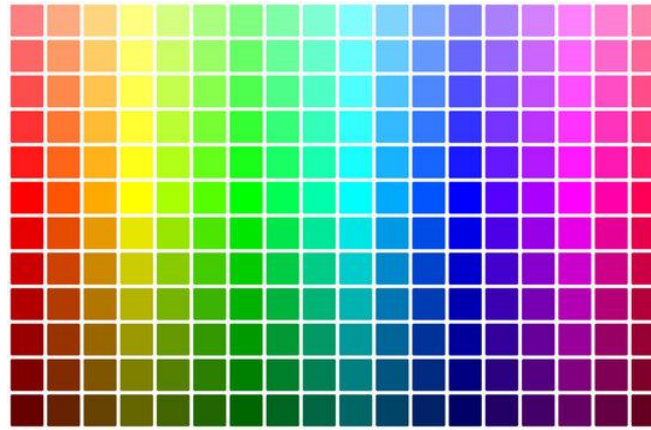
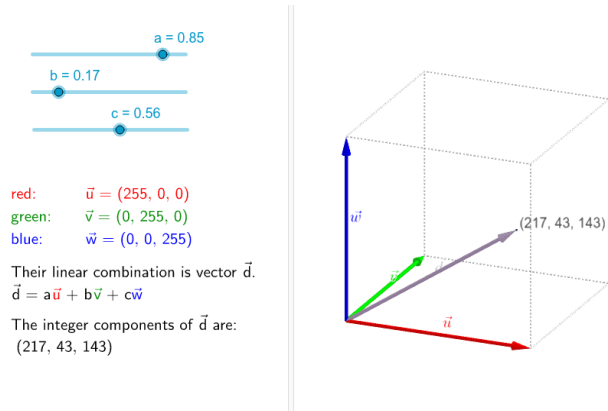
Let $\vec{A}$, $\vec{B}$, $\vec{C}$ be three vectors and k be constant.

- $\vec{A} + \vec{B} = \vec{B} + \vec{A}$ (Commutativity)
- $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$ (Associativity)
- $k(\vec{A} + \vec{B}) = k\vec{A} + k\vec{B}$ (Distributive Law)

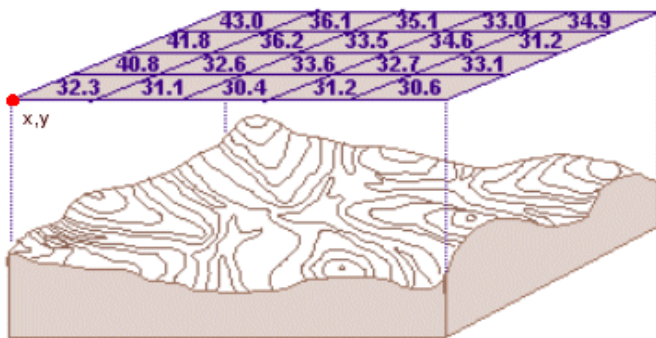
Let P , S and Q be three points in $\mathbf{R}^3$.

- $\overrightarrow{PQ} = \overrightarrow{PS} + \overrightarrow{SQ}$
- $\overrightarrow{SQ} = \overrightarrow{PQ} + \overrightarrow{PS}$

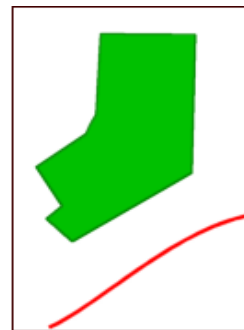
Applications of Vector



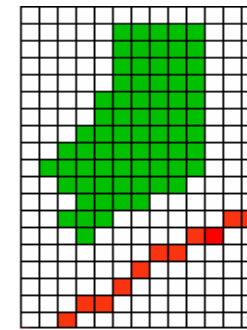
RGB



Real World

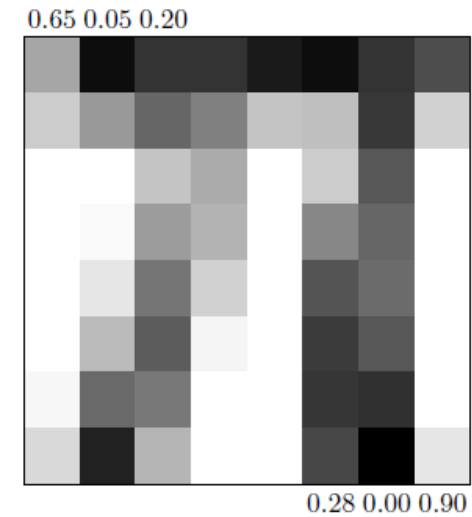


Vector



Raster

Geographic data model



6x6 image and the grayscale levels at six pixels.

Imaging

Example I

Let $P(6, 9, -13)$ and $Q(0, 16, 8)$ be two points in $\mathbf{R}^3$.

(a) Find $\overrightarrow{QP}$.

(b) Find a vector $\vec{A}$ which has a magnitude of 25 and has the same direction of $\overrightarrow{PQ}$.

Solution:

$$(a) \overrightarrow{QP} = \underbrace{(6\hat{i} + 9\hat{j} - 13\hat{k})}_{\overrightarrow{OP}} - \underbrace{(0\hat{i} + 16\hat{j} - 8\hat{k})}_{\overrightarrow{OQ}} = 6\hat{i} - 7\hat{j} - 5\hat{k}$$

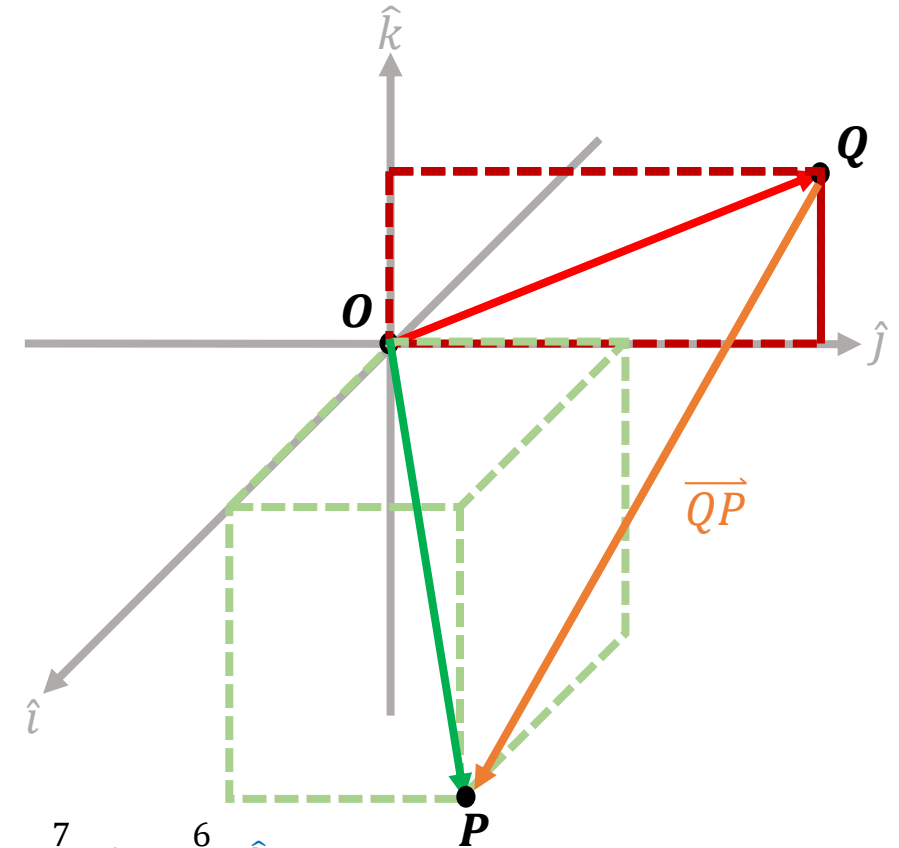
$$(b) \vec{A} = |\vec{A}| \cdot \hat{A}$$

$$|\vec{A}| = 25$$

$$\hat{A} = \overrightarrow{PQ} = -\overrightarrow{QP} = \frac{\overrightarrow{QP}}{|\overrightarrow{QP}|} = \frac{6\hat{i} - 7\hat{j} - 5\hat{k}}{\sqrt{6^2 + (-7)^2 + (-5)^2}} = \frac{6}{\sqrt{110}}\hat{i} - \frac{7}{\sqrt{110}}\hat{j} - \frac{5}{\sqrt{110}}\hat{k}$$

$$\vec{A} = 25 \cdot \left(\frac{6}{\sqrt{110}}\hat{i} - \frac{7}{\sqrt{110}}\hat{j} - \frac{5}{\sqrt{110}}\hat{k} \right)$$

$$= \frac{150}{\sqrt{110}}\hat{i} - \frac{175}{\sqrt{110}}\hat{j} - \frac{125}{\sqrt{110}}\hat{k}$$



Example II

Let $P(1, 3, 10)$, $Q(-7, 11, -4)$ and S be three points in $\mathbf{R}^3$.

(a) Find $\overrightarrow{QP}$.

(b) $\overrightarrow{QS}$ has the same direction as $\overrightarrow{QP}$ with a magnitude of $|\overrightarrow{QP}| + 5$. Find the coordinates of S .

Solution:

$$(a) \overrightarrow{QP} = \underbrace{(1\hat{i} + 3\hat{j} + 10\hat{k})}_{\overrightarrow{OP}} - \underbrace{(-7\hat{i} + 11\hat{j} - 4\hat{k})}_{\overrightarrow{OQ}} = 8\hat{i} - 8\hat{j} + 14\hat{k}$$

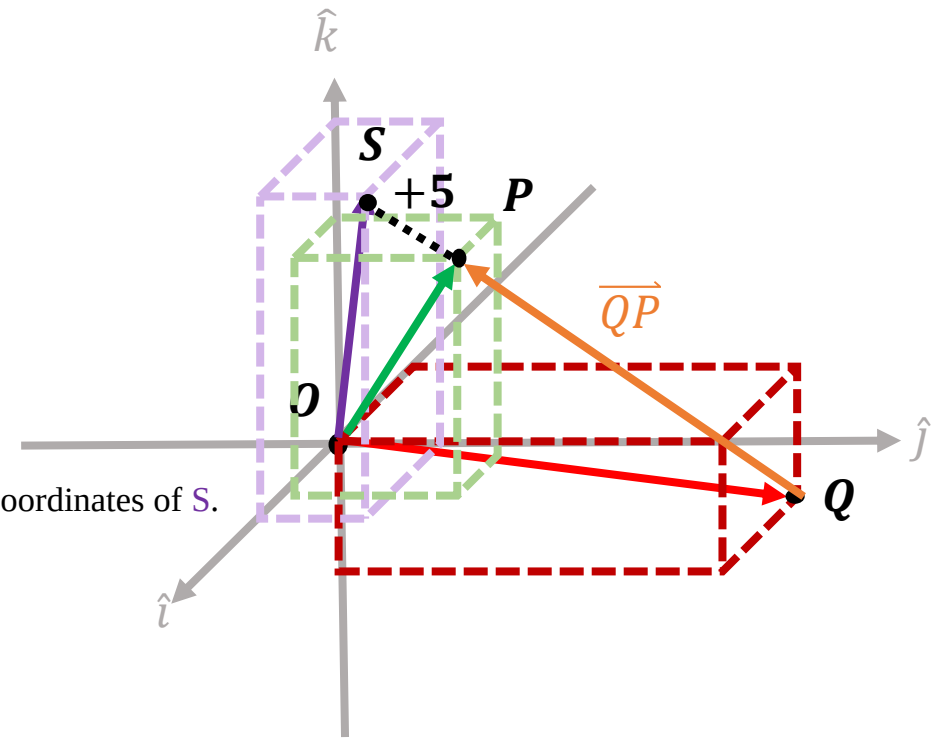
(b)

$$\overrightarrow{QS} = \overrightarrow{QP} = \frac{\overrightarrow{QP}}{|\overrightarrow{QP}|} = \frac{8\hat{i} - 8\hat{j} + 14\hat{k}}{\sqrt{(8)^2 + (-8)^2 + (14)^2}} = \frac{8}{18}\hat{i} - \frac{8}{18}\hat{j} + \frac{14}{18}\hat{k}$$

$$|\overrightarrow{QS}| = |\overrightarrow{QP}| + 5 = \sqrt{(8)^2 + (-8)^2 + (14)^2} + 5 = 18 + 5 = 23$$

$$\overrightarrow{OS} = \overrightarrow{OQ} + \overrightarrow{QS} = (-7\hat{i} + 11\hat{j} - 4\hat{k}) + \left(23 \cdot \left(\frac{8}{18}\hat{i} - \frac{8}{18}\hat{j} + \frac{14}{18}\hat{k} \right) \right) = \frac{29}{9}\hat{i} + \frac{14}{18}\hat{j} + \frac{125}{9}\hat{k}$$

Then coordinates of S is $\left(-\frac{29}{9}, \frac{14}{18}, -\frac{125}{9}\right)$.



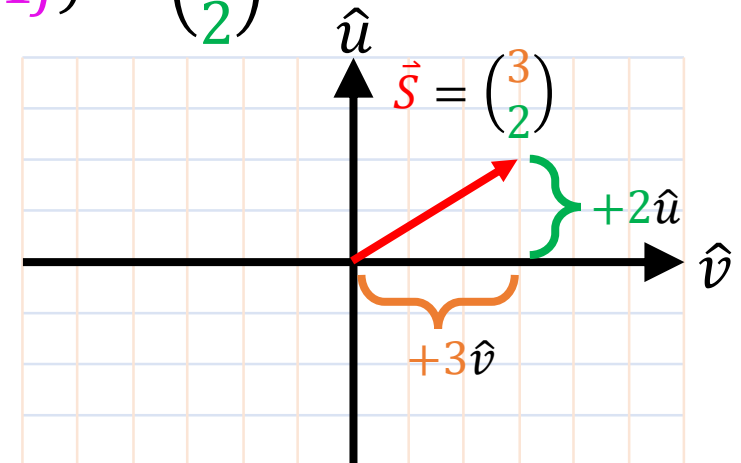
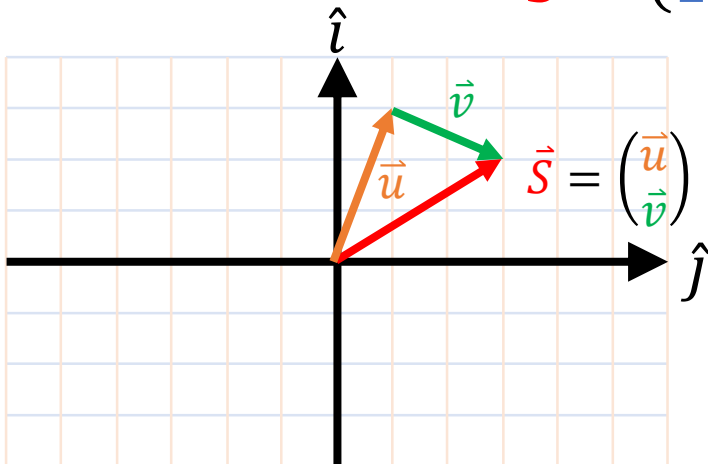
Linear Combination

- Vector can be expressed as the combination of two vectors.

$$\vec{S} = k_1 \vec{u} + k_2 \vec{v} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}$$

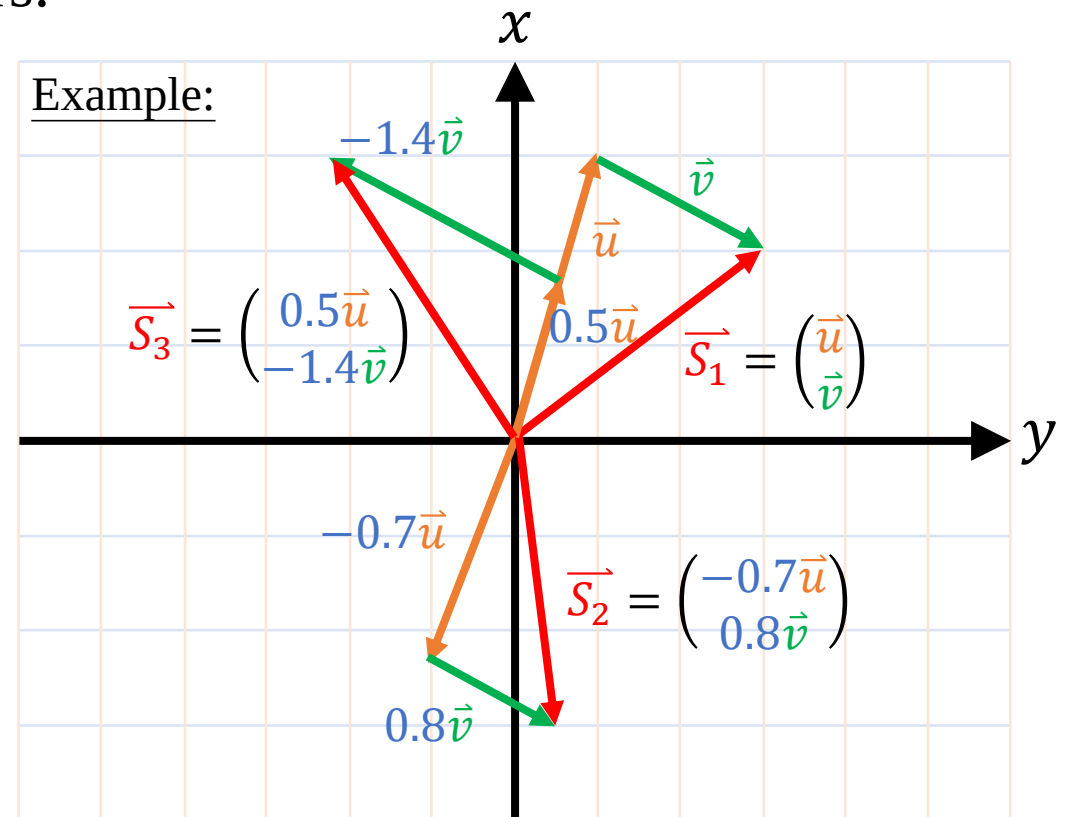
- k_1 & k_2 are the **scalar coefficients** of the corresponding vectors.
- $\vec{S}$ is the linear combination of $\vec{u}$ & $\vec{v}$.
- If we choose certain values for $k_1 \vec{u}$ & $k_2 \vec{v}$,

$$\vec{S} = (1)(1\hat{i} + 3\hat{j}) + (1)(2\hat{i} - 1\hat{j}) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$



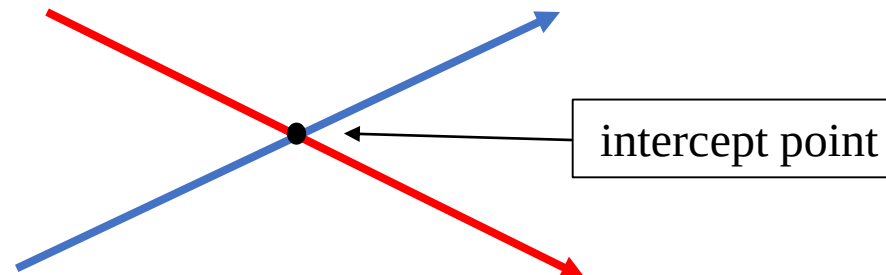
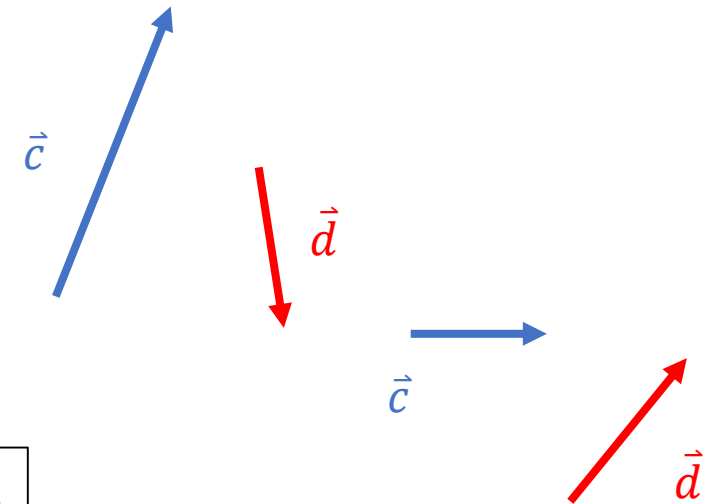
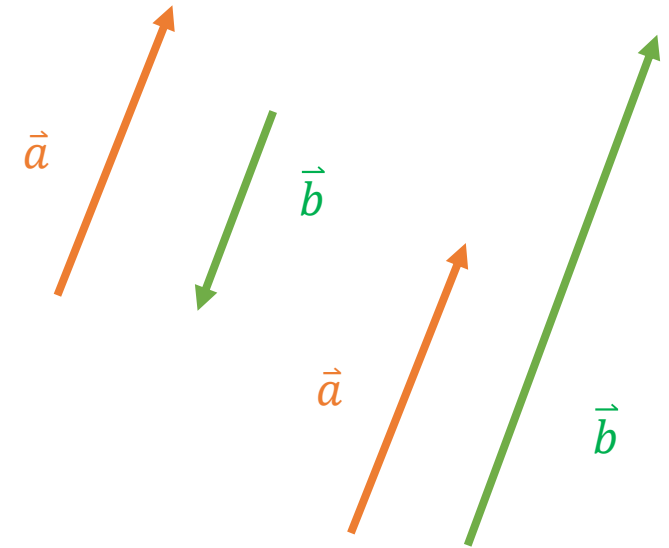
Linear Combination

- Since vector can be controlled by k_1 & k_2 , different combinations creates infinite vectors.
- $\vec{s}$ is the linearly dependent with $\vec{u}$ & $\vec{v}$
- $\vec{u}$ & $\vec{v}$ are linearly independent.



Linear Combination

- If $\vec{a}$ & $\vec{b}$ are linearly dependent, it implies that
 - they are collinear
 - they can be lied on the same straight line or parallel
- If $\vec{c}$ & $\vec{d}$ are linearly independent, it implies that
 - they are non-collinear
 - if they extend infinitely long, there must be intercept point



Dot Product (Scalar Product)

Given vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

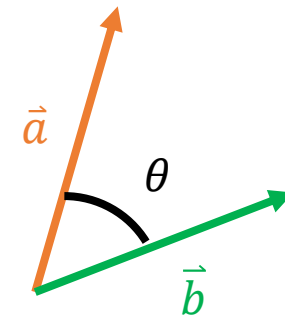
- The dot product of these two vectors is

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

- Geometric definition of dot product is

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

where $0 \leq \theta \leq \pi$ is the angle between two vectors.



Projection vector

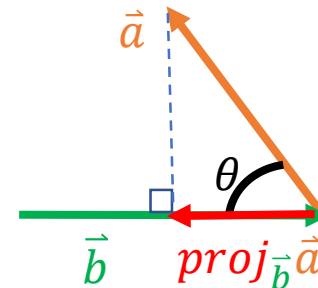
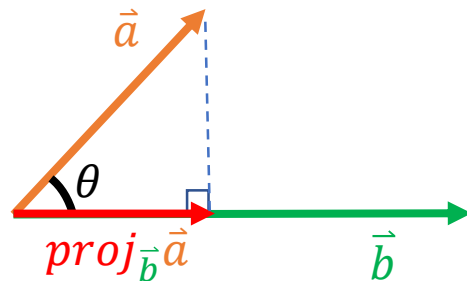
- Projection vector $\text{proj}_{\vec{b}} \vec{a}$ is defined as the perpendicular projection of $\vec{a}$ on $\vec{b}$.

$$\text{proj}_{\vec{b}} \vec{a} = \underbrace{|\vec{a}| \cos \theta}_{\text{Magnitude}} \cdot \underbrace{\hat{b}}_{\text{Direction}} = |\vec{a}| \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \cdot \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$

- Depending the angle calculated from dot product, we can determine the direction of projection vector.

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \Rightarrow \begin{cases} \theta < 90^\circ \text{ if } \cos \theta > 0 \\ \theta > 90^\circ \text{ if } \cos \theta < 0 \end{cases}$$

$\cos \theta > 0$:
Magnitude = $|\vec{a}| \cos \theta$
Direction = $\hat{b}$



$\cos \theta < 0$:
Magnitude = $-|\vec{a}| \cos \theta$
Direction = $-\hat{b}$

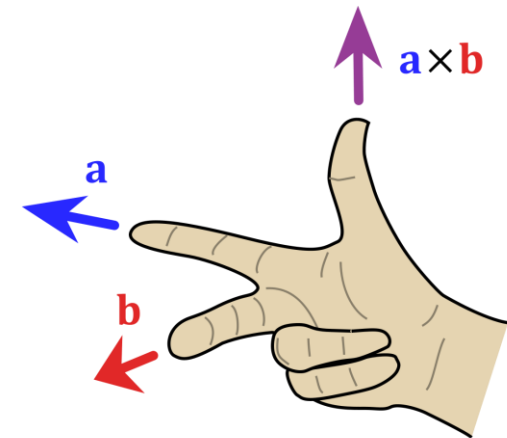
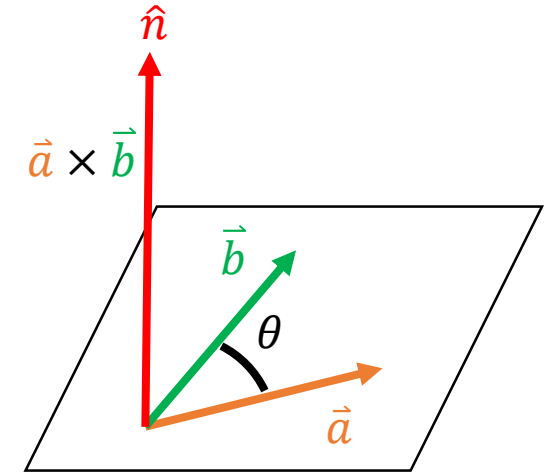
Cross Product (Vector Product)

Given vector $\vec{a}$ & $\vec{b}$.

- The cross product of these two vectors is

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \cdot \hat{n}$$

- The direction of $\hat{n}$ is determined by **right hand rule**.



Cross Product (Vector Product)

Given vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

Cross product using matrix notation:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

Example I

Given a vector $\vec{a} = 2\hat{j} + 4\hat{k}$. Find the projection vector of $\vec{b} = 3\hat{i} + \hat{k}$ on $\vec{a}$.

Solution:

$$\begin{aligned} \text{proj}_{\vec{a}} \vec{b} &= \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} \\ &= \frac{(3\hat{i} + \hat{k}) \cdot (2\hat{j} + 4\hat{k})}{(\sqrt{(2)^2 + (4)^2})^2} (2\hat{j} + 4\hat{k}) \\ &= \frac{(3)(0) + (0)(2) + (1)(4)}{20} (2\hat{j} + 4\hat{k}) \\ &= \frac{4}{20} (2\hat{j} + 4\hat{k}) \\ &= \frac{2}{5} \hat{j} + \frac{4}{5} \hat{k} \end{aligned}$$

Example II

Find a vector perpendicular to the plane containing the points $a(1, 2, 3)$, $b(-1, 4, 8)$ & $c(5, 1, -2)$.

$$\overrightarrow{ab} = (-1 - 1)\hat{i} + (4 - 2)\hat{j} + (8 - 3)\hat{k} = -2\hat{i} + 2\hat{j} + 5\hat{k}$$

$$\overrightarrow{ac} = (5 - 1)\hat{i} + (1 - 2)\hat{j} + (-2 - 3)\hat{k} = 4\hat{i} - 1\hat{j} - 5\hat{k}$$

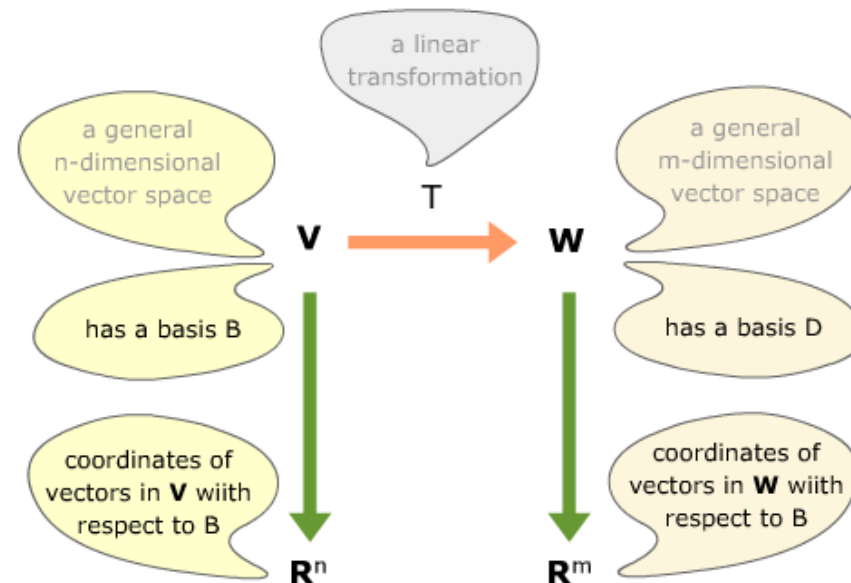
$$\overrightarrow{ab} \times \overrightarrow{ac} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 5 \\ 4 & -1 & -5 \end{vmatrix}$$

$$= [2(-5) - (-1)5]\hat{i} - [(-2)(-5) - 4(5)]\hat{j} + [(-2)(-1) - 4(2)]\hat{k}$$

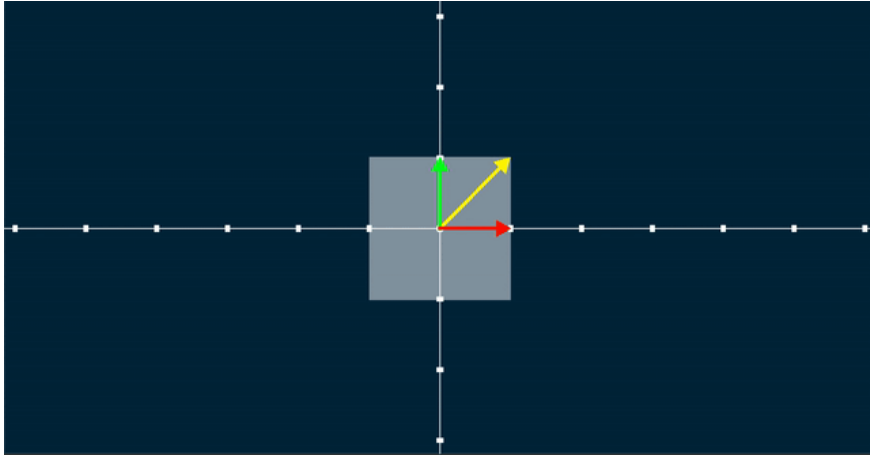
$$= -5\hat{i} + 10\hat{j} - 6\hat{k}$$

Linear Transformation

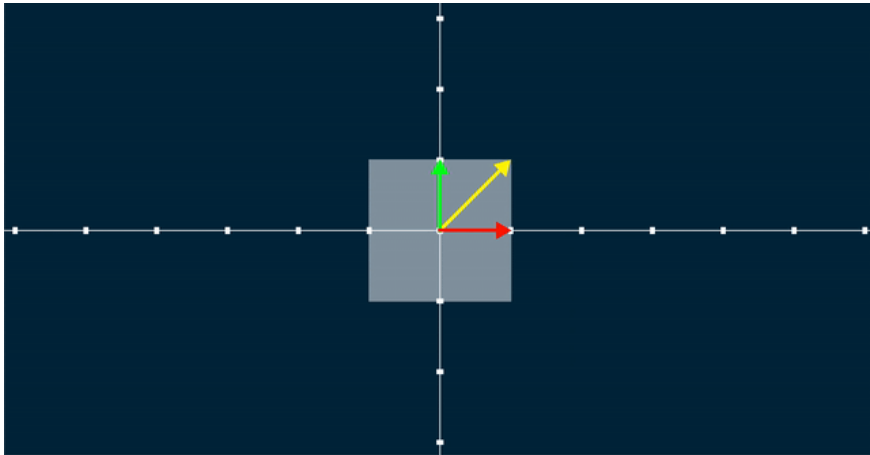
- Transformation
 - Changing the basis of a coordinate system.
 - Mapping a given vector to a transformed coordinate system = Transformed Vector
- Linear
 - During transformation, the position of origin is fixed.



Linear Transformation

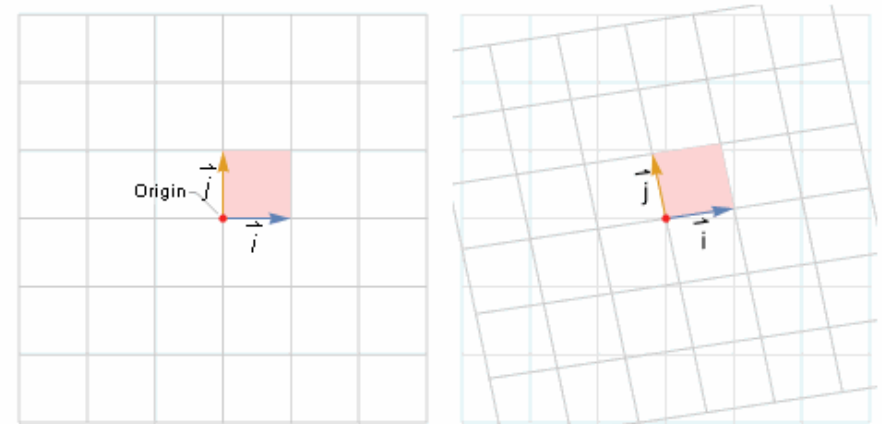


Scaling up by y-axis transformation



Scaling up by x- & y-axis transformation

矩阵 $A = \begin{pmatrix} 1 & -2 \\ 1.5 & 0.5 \end{pmatrix}$ 所对应的变换



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1. & -0.2 \\ 0.15 & 0.95 \end{pmatrix}$$

WeChat ID: meetmath

Linear transformation rotation

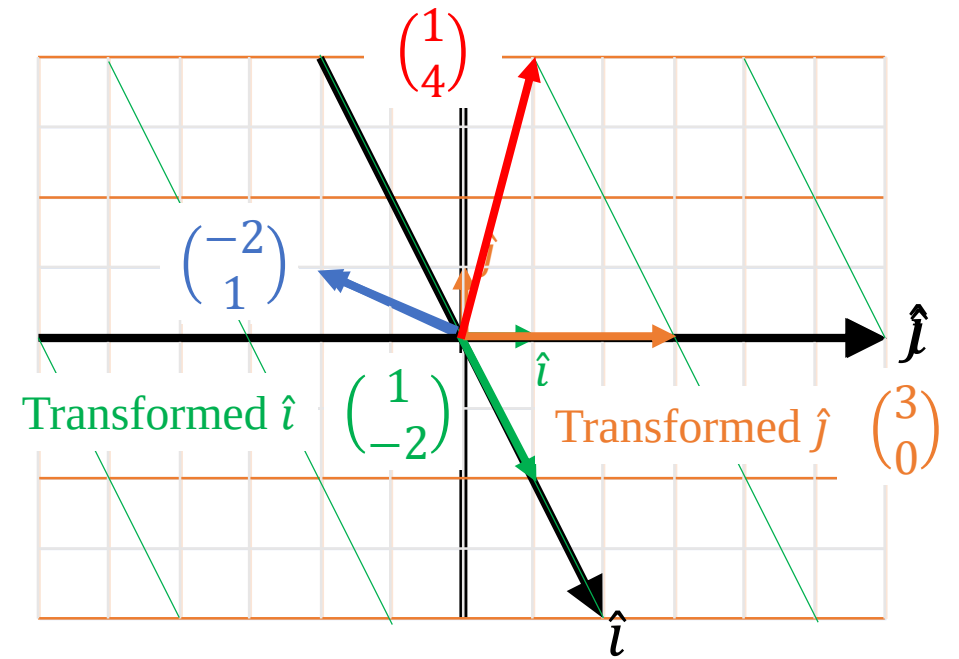
Linear Transformation

Given a vector $\vec{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Applying linear transformation by the influence of $\hat{i}$ & $\hat{j}$,

Provided that $\hat{i} \rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ & $\hat{j} \rightarrow \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Find the transformation of vector $\vec{v}$.



$$\text{Transformed } \vec{v} = -2(\text{Transformed } \hat{i}) + 1(\text{Transformed } \hat{j})$$

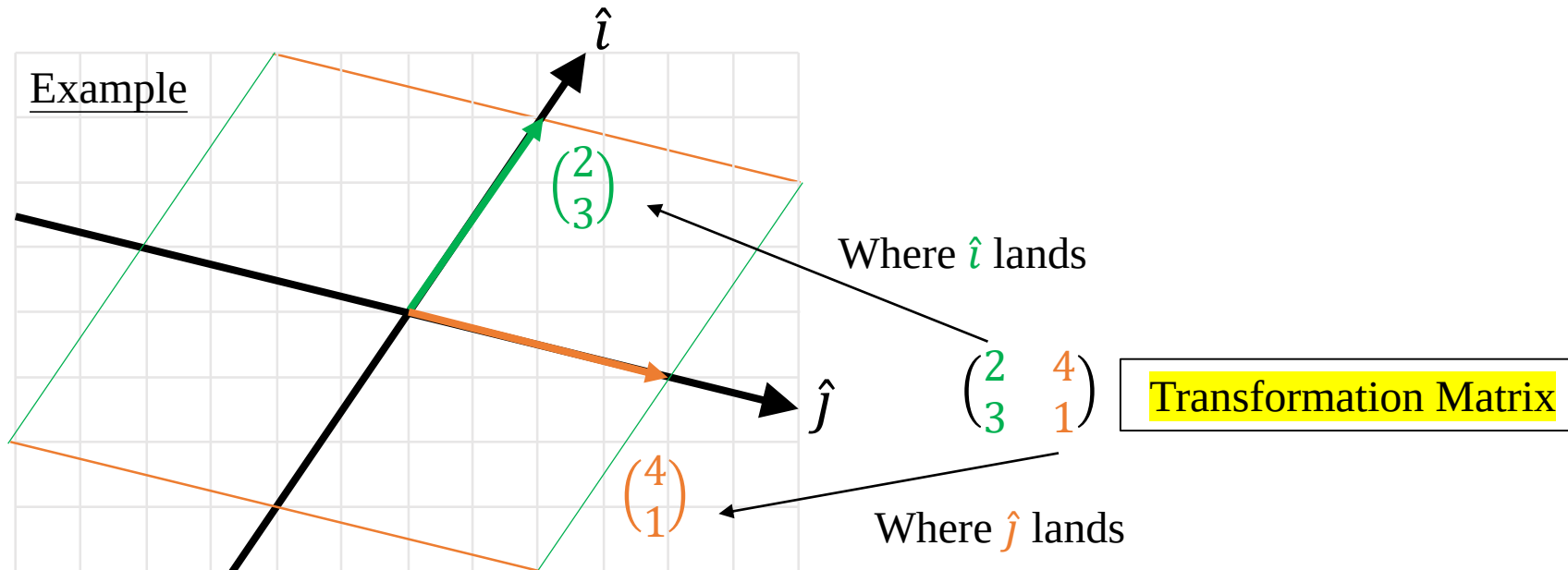
$$= -2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2(1) + 1(3) \\ -2(-2) + 1(0) \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Linear Transformation

- Once you know the transformation, you can find out the position of a new vector with respect to the transformed coordinate system.
- But we need to put the transformed vectors into a 2×2 Matrix.



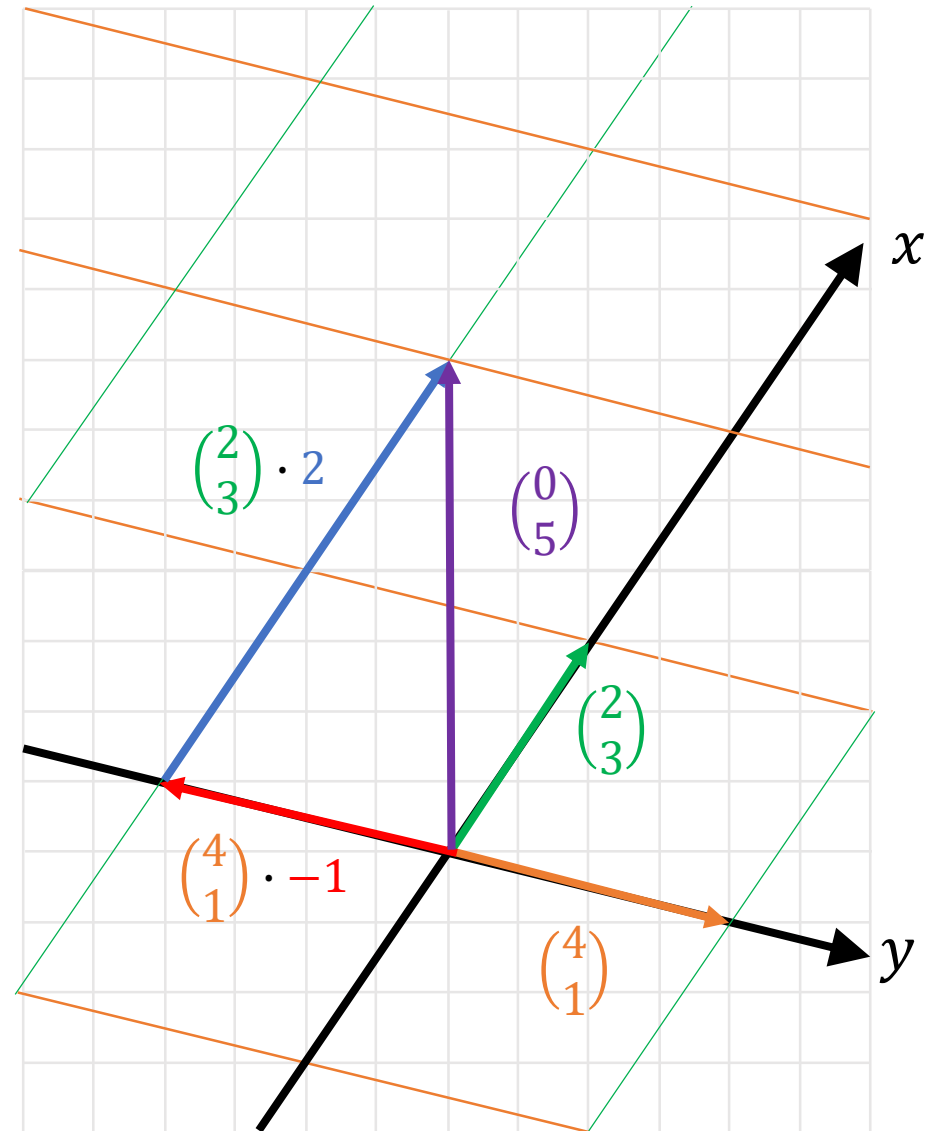
Linear Transformation

Given a Vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ with
a transformation matrix $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$.

Find a transformed vector:

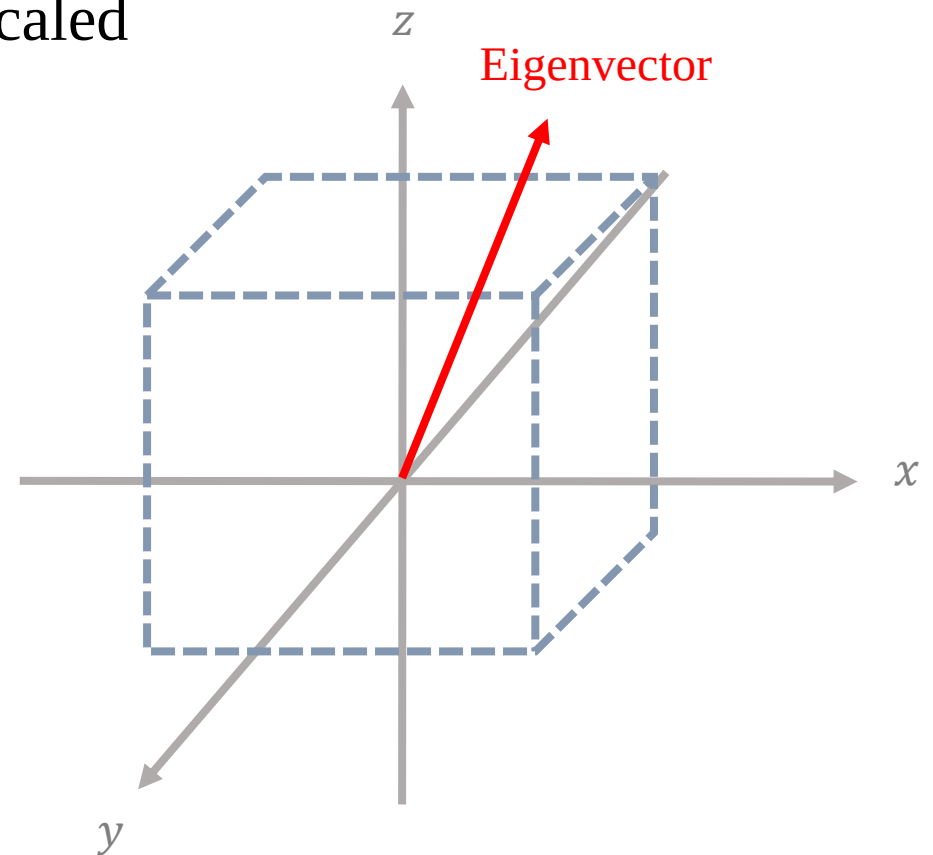
$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2(2) + (-1)(4) \\ 2(3) + (-1)(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

Transformation Matrix Original Vector Transformed Vector

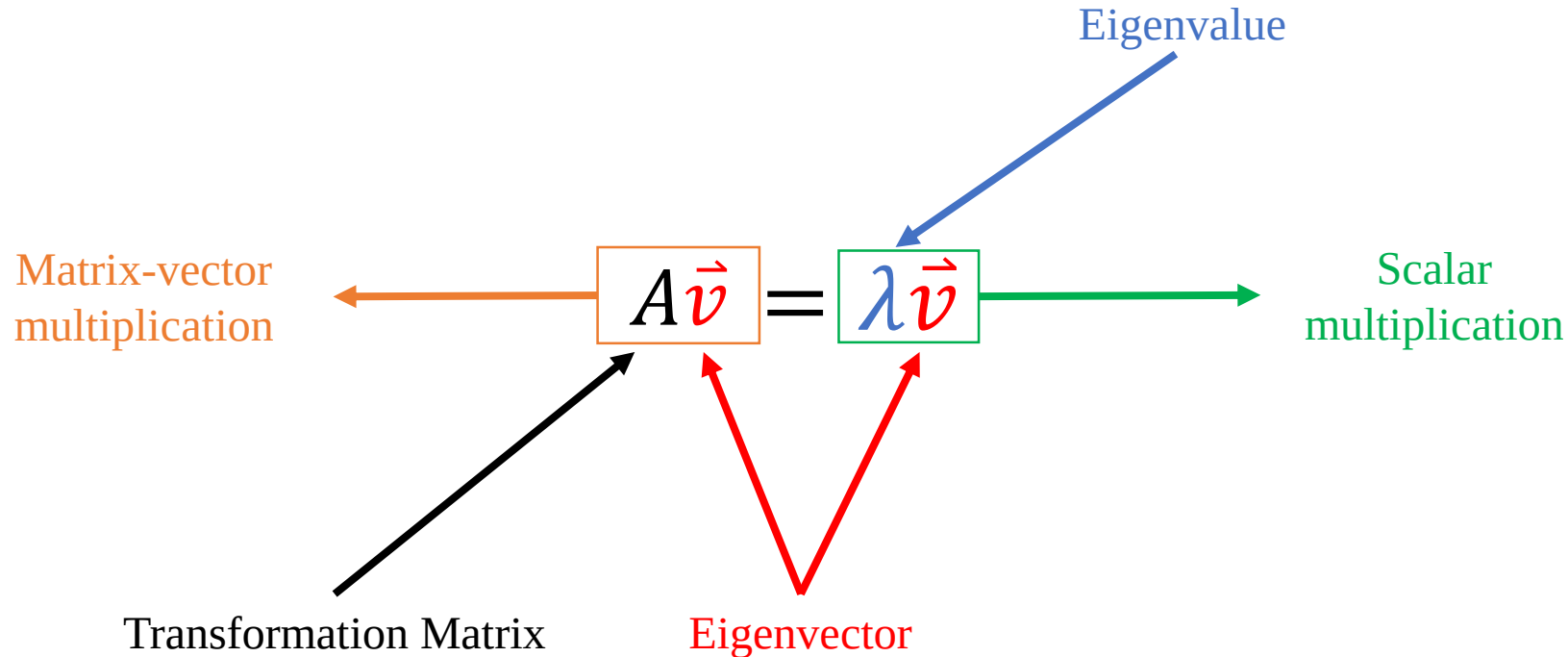


Definition of Eigenvalue & Eigenvector

- Eigenvector is a nonzero vector of a linear transformation.
- Eigenvalue is the factor which the eigenvector is scaled when linear transformation is applied to it.
- What does eigenvector & eigenvalue mean?
- In $\mathbf{R}^3$, eigenvector (vector) is the axis of rotation while eigenvalue (scalar) is the scale.



Calculation of Eigenvalue & Eigenvector



Are they equal?
Of course, they are not!
Only if $\vec{v}$ is eigenvector

Calculation of Eigenvalue & Eigenvector

- To make both side as **Matrix-vector multiplication**, we multiply a diagonal matrix I in R.H.S.

Diagonal matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Given matrix

$$A = \begin{pmatrix} 3 & 4 & 5 \\ 8 & 2 & 9 \\ 7 & 8 & 1 \end{pmatrix}$$

$$A\vec{v} = \lambda I\vec{v}$$
$$\underbrace{(A - \lambda I)}_{\text{3-dimensional transformation matrix}} \vec{v} = 0$$

$$\begin{pmatrix} 3 - \lambda & 4 & 5 \\ 8 & 2 - \lambda & 9 \\ 7 & 8 & 1 - \lambda \end{pmatrix}$$

3-dimensional transformation matrix means transformation in $\vec{i}$, $\vec{j}$ and $\vec{k}$.

- We are searching for a nonzero solution for $\vec{v}$.

Calculation of Eigenvalue & Eigenvector

Given a transformation matrix $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

Step 1.

To fulfill $(A - \lambda I)\vec{v} = 0$, we need to find out

$$\det(A - \lambda I) = 0 \quad \leftarrow \text{determinant}$$

$$\det\begin{pmatrix} 3 - \lambda & 1 \\ 2 & 2 - \lambda \end{pmatrix} = (3 - \lambda)(2 - \lambda) - 1(2) = \lambda^2 - 5\lambda + 6 - 2$$

$$= \lambda^2 - 5\lambda + 4 \quad \leftarrow \text{Characteristic polynomial}$$

Step 2.

Factorizing the characteristic polynomial, we find out roots (eigenvalue)

$$\lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$$

$$\lambda = 4 \text{ and } \lambda = 1 \quad \leftarrow \text{eigenvalue}$$

Calculation of Eigenvalue & Eigenvector

Step 3.

Substitute the eigenvalue back into $(A - \lambda I)\vec{v} = 0$, using **linear elimination** and try to simplify the matrix into **row echelon form**.

$$\begin{pmatrix} 3 - 4 & 1 \\ 2 & 2 - 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array} \right) \xrightarrow{R_2 + 2R_1} \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow -x_1 + x_2 = 0$$

$$x_1 = x_2$$

row echelon form

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ 0 & 0 & a_9 \end{pmatrix}$$

The corresponding **eigenvectors** of A for $\lambda = 4$ are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $k \neq 0$. (k is scalar constant)

Eigenvalue

Eigenvector

Linear Elimination

(1) Interchange two rows.

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right)$$

(2) Multiply a row by $c \neq 0$.

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{2R_2} \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & 2 & 0 \end{array}\right)$$

(3) Addition of a scalar multiple of any row to another row.

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 3 & 0 & 0 \end{array}\right)$$

Applications of Eigenvector

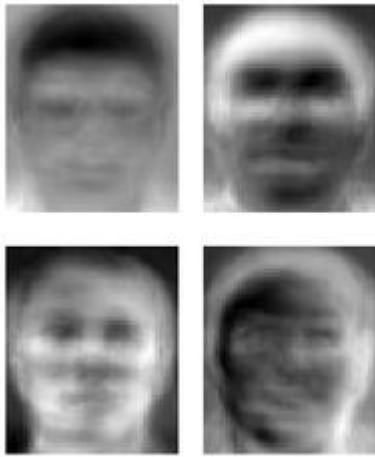
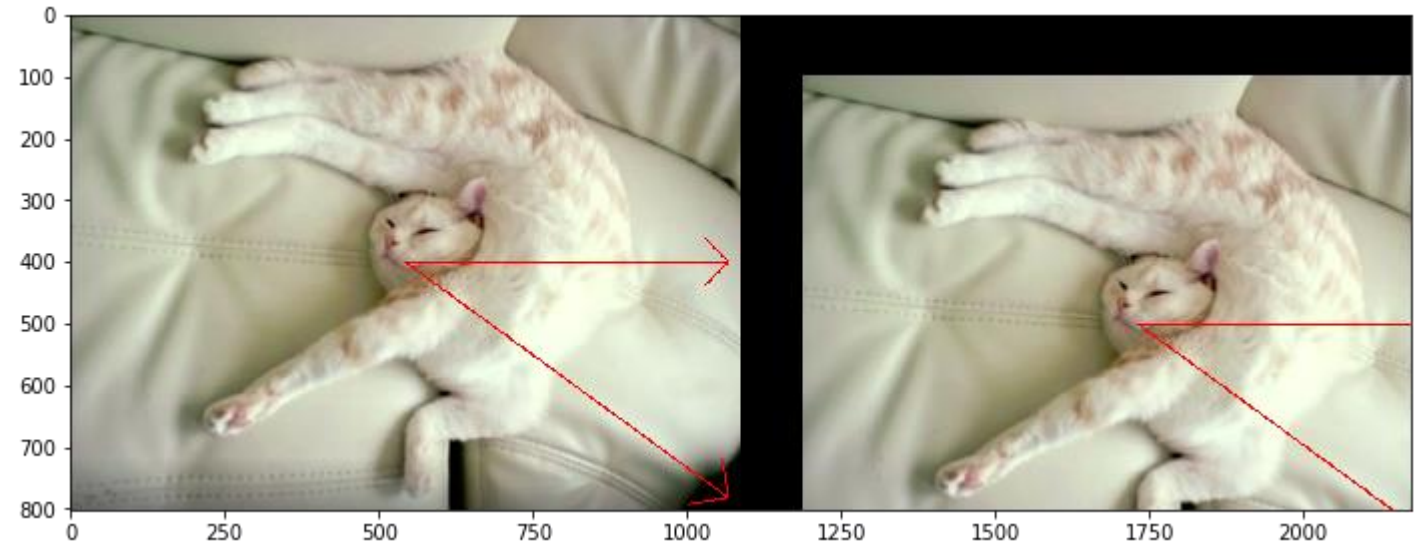
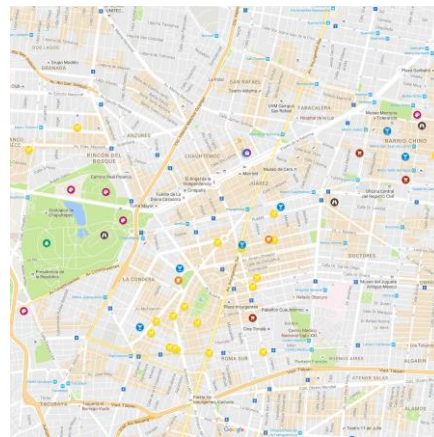


Image processing



Machine learning



Google Map

Examples I

Find the eigenvalues of $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ and the corresponding eigenvectors.

Solution:

$$A - \lambda I = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{vmatrix} = (1 - \lambda)(2 - \lambda) - 3(2) = \lambda^2 - 3\lambda + 2 - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda - 4)(\lambda + 1) \end{aligned}$$

The eigenvalues are $\lambda = 4$ and $\lambda = -1$.

Examples I

When $\lambda = 4$,

$$\begin{pmatrix} 1 - 4 & 2 \\ 3 & 2 - 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array} \right) \xrightarrow{R_2 + R_1} \left(\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow -3x_1 = 2x_2$$

$$x_1 = -\frac{2}{3}x_2$$

The corresponding eigenvectors of A for $\lambda = 4$ are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $k \neq 0$. (k is scalar constant)

Examples I

When $\lambda = -1$,

$$\begin{pmatrix} 1 - (-1) & 2 \\ 3 & 2 - (-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array} \right) \xrightarrow{R_2 - \frac{3}{2}R_1} \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$\Rightarrow x_2$ can be any values, $x_1 = -x_2$

The corresponding eigenvectors of A for $\lambda = -1$ are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \end{pmatrix}, k \neq 0$.

Examples II

Find the eigenvalues of $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ and the corresponding eigenvectors.

Solution:

$$A - \lambda I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(1-\lambda) - (1-\lambda) - (1-\lambda) \\ &= (1-\lambda)(\lambda^2 - 3\lambda + 2 - 2) \\ &= \lambda(1-\lambda)(\lambda - 3) \end{aligned}$$

The eigenvalues are $\lambda = 0$, $\lambda = 1$ and $\lambda = 3$.

Examples II

When $\lambda = 0$,

$$\begin{pmatrix} 1 - 0 & -1 & 0 \\ -1 & 2 - 0 & -1 \\ 0 & -1 & 1 - 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array} \right) \xrightarrow{R_2 + R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array} \right) \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow x_1 = x_3, x_1 = x_2$$

$$x_1 = x_2 = x_3$$

The corresponding eigenvectors of A for $\lambda = 0$ are $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, k \neq 0$.

Examples II

When $\lambda = 1$,

$$\begin{pmatrix} 1 - \textcolor{blue}{1} & -1 & 0 \\ -1 & 2 - \textcolor{blue}{1} & -1 \\ 0 & -1 & 1 - \textcolor{blue}{1} \end{pmatrix} \begin{pmatrix} \textcolor{red}{x}_1 \\ \textcolor{red}{x}_2 \\ \textcolor{red}{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow \textcolor{red}{x}_2 = 0, \textcolor{red}{x}_1 = \textcolor{red}{x}_2 - \textcolor{red}{x}_3$$

$$\textcolor{red}{x}_1 = -\textcolor{red}{x}_3$$

The corresponding eigenvectors of A for $\lambda = 1$ are $\begin{pmatrix} \textcolor{red}{x}_1 \\ \textcolor{red}{x}_2 \\ \textcolor{red}{x}_3 \end{pmatrix} = \textcolor{green}{k} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \textcolor{green}{k} \neq 0.$

Examples II

When $\lambda = 3$,

$$\begin{pmatrix} 1 - \textcolor{blue}{3} & -1 & 0 \\ -1 & 2 - \textcolor{blue}{3} & -1 \\ 0 & -1 & 1 - \textcolor{blue}{3} \end{pmatrix} \begin{pmatrix} \textcolor{red}{x}_1 \\ \textcolor{red}{x}_2 \\ \textcolor{red}{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & -1 & -2 & 0 \end{array} \right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array} \right) \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{R_1 + R_2} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$\Rightarrow \textcolor{red}{x}_2 = -2\textcolor{red}{x}_3, \textcolor{red}{x}_1 = +\textcolor{red}{x}_3, \textcolor{red}{x}_3$ can be any values

The corresponding eigenvectors of A for $\lambda = 3$ are $\begin{pmatrix} \textcolor{red}{x}_1 \\ \textcolor{red}{x}_2 \\ \textcolor{red}{x}_3 \end{pmatrix} = \textcolor{green}{k} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \textcolor{green}{k} \neq 0$.