

# EE3114 - Homework 7 - Solutions

7-1 Consider the uncontrolled Double Integrator described by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

Find its

- a) State transition matrix  $\Phi(t)$
- b) Characteristic polynomial
- c) Eigenvalues (i.e., poles)
- d) Solution with initial value  $x(0) = [0 \ 1]^T$
- e) Is this solution stable? (Namely, is it convergent as  $t \rightarrow \infty$ ?)

**Answer: (4 marks)**

- a) (1 mark)  $\Phi(t) = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ ,
- b) (1 mark)  $\det(\lambda I - A) = 0 \leftrightarrow \lambda^2 = 0$ , hence the characteristic polynomial is  $\lambda^2$ ,
- c) (0.5 mark) eigenvalues  $\lambda_1 = \lambda_2 = 0$ ,
- d) (0.5 mark)  $x(t) = \Phi(t)x(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}$ ,
- e) (1 mark) no, the system is not stable since  $\lim_{t \rightarrow \infty} \|x(t)\|_2 = \infty$ .

7-2 Consider the uncontrolled Harmonic Oscillator described by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

Find its

- a) State transition matrix  $\Phi(t)$
- b) Characteristic polynomial
- c) Eigenvalues (i.e., poles)
- d) Solution with initial value  $x(0) = [1 \ 0]^T$
- e) Is this solution stable? (Namely, is it convergent as  $t \rightarrow \infty$ ?)

**Answer: (4 marks)**

a) (1 mark)  $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ,  $A^2 = -I$ ,  $A^3 = -A$ ,  $A^4 = I$ ,  $A^5 = A$ , ...

$$\begin{aligned} \text{Hence } \Phi(t) &= I \left( 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots \right) + A \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots \right) \\ &= I \left( \sum_{i=0}^{\infty} \frac{(-1)^i t^{2i}}{(2i)!} \right) + A \left( \sum_{i=1}^{\infty} \frac{(-1)^{i+1} t^{2i-1}}{(2i-1)!} \right) \\ &= I \cos(t) + A \sin(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

where the last equality holds due to the fact that the two sums are the Taylor series of  $\cos(t)$  and  $\sin(t)$ ,

b) (1 mark)  $\det(\lambda I - A) = 0 \Leftrightarrow \lambda^2 + 1 = 0$ , hence the characteristic polynomial is  $\lambda^2 + 1$ ,

c) (0.5 mark) eigenvalues  $\lambda_1 = -j$ ,  $\lambda_2 = j$ ,

d) (0.5 mark)  $x(t) = \Phi(t)x(0) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$ ,

e) (1 mark) the system is marginally stable, it is neither stable nor unstable (“not stable” was also accepted, but not “unstable”).

**7-3** Given a transfer function:

$$G(s) = \frac{s+1}{s(s-1)(s+2)^2}$$

write out its space-space representation.

**Answer: (2 marks)**

In time domain, we have

$$y^{(4)}(t) + 3\ddot{y}(t) - 4\dot{y}(t) = \dot{u}(t) + u(t)$$

Choosing the variables

$$x_1 = y, x_2 = \dot{y}, x_3 = \ddot{y}, x_4 = \ddot{y} - u$$

We have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} u, \quad y = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0] u$$

(Since this representation is not unique, other equivalent answers were also accepted).