

EE6617: Tutorial 8

8.1 We want to detect if a predetermined signal $s[0], \dots, s[N-1]$ is present in a noisy received signal $x[0], \dots, x[N-1]$. The corresponding hypotheses are

$$\begin{aligned}\mathcal{H}_0 : \quad & \mathbf{x} = \mathbf{w} \\ \mathcal{H}_1 : \quad & \mathbf{x} = \mathbf{s} + \mathbf{w}.\end{aligned}$$

and $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$. The $\mathbf{C}$ is positive definite. From the class, we already know the optimal NP detector is

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s} \underset{0}{\geq} \gamma'$$

and for the target false-alarm rate α , the threshold γ' is determined by $\gamma' = \sqrt{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}} Q^{-1}(\alpha)$ and the detection probability is

$$P_D = Q\left(Q^{-1}(\alpha) - \sqrt{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}}\right).$$

What is the optimal $\mathbf{s}$ subject to $\|\mathbf{s}\| = \xi$ that maximize P_D ? You may assume $\mathbf{C}^{-1} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^T$ where $\mathbf{U} = [\mathbf{u}_1 \ \mathbf{u}_2 \ \dots \ \mathbf{u}_N] \in \mathbb{R}^{N \times N}$ is an unitary matrix such that $\mathbf{U}^T \mathbf{U} = \mathbf{U} \mathbf{U}^T = \mathbf{I}$ and $\mathbf{\Lambda}$ is a diagonal matrix with diagonal entries $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$.

8.2 During the lecture, we designed the minimum probability error detector for binary source $\{\mathbf{s}_0, \mathbf{s}_1\}$. The designed detector is $T(\mathbf{x}) = \mathbf{x}^T (\mathbf{s}_1 - \mathbf{s}_0) - \frac{\xi_1 - \xi_0}{2} \underset{0}{\geq} 0$ and based on this we calculated the error probability

$$P_e = Q\left(\sqrt{\frac{\|\mathbf{s}_1 - \mathbf{s}_0\|^2}{4\sigma^2}}\right).$$

Given $\xi_0 = \xi_1 = \xi$, what are the optimal $\mathbf{s}_1$ and $\mathbf{s}_0$ that minimize P_e ?

Sol. 8.1 To maximize P_D , 1°

$$\begin{aligned}
\mathbf{s} &= \underset{\mathbf{s}, \|\mathbf{s}\|=\xi}{\operatorname{argmax}} Q \left(Q^{-1}(\alpha) - \sqrt{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}} \right) \\
&= \underset{\mathbf{s}, \|\mathbf{s}\|=\xi}{\operatorname{argmin}} Q^{-1}(\alpha) - \sqrt{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}} \\
&= \underset{\mathbf{s}, \|\mathbf{s}\|=\xi}{\operatorname{argmax}} \mathbf{s}^T \mathbf{C}^{-1} \mathbf{s} \\
&\quad \text{substitute } \mathbf{v} = \mathbf{s}/\xi \\
&= \xi^2 \underset{\mathbf{v}, \|\mathbf{v}\|=1}{\operatorname{argmax}} \mathbf{v}^T \mathbf{C}^{-1} \mathbf{v} \\
&= \xi^2 \underset{\mathbf{v}, \|\mathbf{v}\|=1}{\operatorname{argmax}} \sum_{i=1}^N \lambda_i |\mathbf{u}_i^T \mathbf{v}|^2 = \lambda_1
\end{aligned}$$

This is achieved when $\mathbf{v} = \mathbf{u}_1$ and thus, $\mathbf{s} = \xi \mathbf{u}_1$.

Point signal in the direction of the N -dimensional space with the least noise.

Sol. 8.2 From the property of Q -function, the $P_e = Q \left(\sqrt{\frac{\|\mathbf{s}_1 - \mathbf{s}_0\|^2}{4\sigma^2}} \right)$ is minimized if $\frac{\|\mathbf{s}_1 - \mathbf{s}_0\|^2}{4\sigma^2}$ is maximized, i.e.,

$$\begin{aligned}
\{\mathbf{s}_0, \mathbf{s}_1\}^* &= \underset{\mathbf{s}_0, \mathbf{s}_1}{\operatorname{argmax}} \|\mathbf{s}_1 - \mathbf{s}_0\|^2 \\
&= \underset{\mathbf{s}_0, \mathbf{s}_1}{\operatorname{argmax}} 2\xi - 2\mathbf{s}_1^T \mathbf{s}_0 \\
&= 2\xi \underset{\mathbf{s}_0, \mathbf{s}_1}{\operatorname{argmax}} \left(1 - \frac{\mathbf{s}_1^T}{\sqrt{\xi}} \frac{\mathbf{s}_0}{\sqrt{\xi}} \right)
\end{aligned}$$

Since $\frac{\mathbf{s}_1}{\sqrt{\xi}}$ and $\frac{\mathbf{s}_0}{\sqrt{\xi}}$ are unit norm, it is obvious that $\left(1 - \frac{\mathbf{s}_1^T}{\sqrt{\xi}} \frac{\mathbf{s}_0}{\sqrt{\xi}} \right) \leq 2$. This is achieved when $\mathbf{s}_1 = -\mathbf{s}_0$.

We call the source vectors with $\mathbf{s}_1 = -\mathbf{s}_0$ property antipodal sources.