

MA1300 Assignment # 4, due September 27, 2012
Solutions

(first hand-in assignment)

1. (P92, #46) Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3. \end{cases}$$

Solution. For making f continuous everywhere, sufficient and necessary conditions are

$$\begin{cases} \lim_{x \rightarrow 2^+} ax^2 - bx + 3 = \lim_{x \rightarrow 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2^-} x + 2, \\ \lim_{x \rightarrow 3^-} ax^2 - bx + 3 = \lim_{x \rightarrow 3^+} 2x - a + b. \end{cases}$$

Equivalently,

$$\begin{cases} 4 = 4a - 2b + 3, \\ 9a - 3b + 3 = 6 - a + b. \end{cases}$$

Solve the equations to give $a = b = \frac{1}{2}$.

2. (P92, #51, 53) Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

(a) $x^4 + x - 3 = 0, \quad (1, 2)$

(b) $\cos x = x, \quad (0, 1)$

Solution.

- (a) Let $f(x) = x^4 + x - 3$ to give $f(1) = -1$, $f(2) = 15$. By the Intermediate Value Theorem, there exists some $c \in (1, 2)$ s.t. $f(c) = 0$, a root.

- (b) Let $f(x) = \cos x - x$ to give $f(0) = 1$, $f(1) = \cos 1 - 1 < 0$. By the Intermediate Value Theorem, there exists some $c \in (0, 1)$, such that $f(c) = 0$. Hence the root exists.

3. (P93, #65) Is there a number that is exactly 1 more than its cube?

Solution. Let $f(x) = x - x^3 - 1$ to give $f(0) = -1$ and $f(-2) = 5$. Therefore by the Intermediate Value Theorem, there exists a number $c \in (-2, 0)$ such that $f(c) = 0$, that is, $c = c^3 + 1$, or c is exactly 1 more than its cube.

4. (P93, #66) If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

Solution. Since $x^3 + 2x^2 - 1 = (x + 1)\left(x + \frac{\sqrt{5}+1}{2}\right)\left(x - \frac{\sqrt{5}-1}{2}\right)$, and $x^3 + x - 2 = (x - 1)(x^2 + x + 2)$, we have $\lim_{x \rightarrow \left(\frac{\sqrt{5}-1}{2}\right)^+} \frac{a}{x^3 + 2x^2 - 1} = \infty$, $\lim_{x \rightarrow \left(\frac{\sqrt{5}-1}{2}\right)^+} \frac{b}{x^3 + x - 2}$ exists and is finite, $\lim_{x \rightarrow 1^-} \frac{a}{x^3 + 2x^2 - 1} = \frac{a}{2}$ is finite, and $\lim_{x \rightarrow 1^-} \frac{b}{x^3 + x - 2} = -\infty$. Let $f(x) = \frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2}$, we have that f is continuous on $\left(\frac{\sqrt{5}-1}{2}, 1\right)$, and $\lim_{x \rightarrow \left(\frac{\sqrt{5}-1}{2}\right)^+} f(x) = \infty$, $\lim_{x \rightarrow 1^-} f(x) = -\infty$. Therefore by the Intermediate Value Theorem, there exists some $c \in \left(\frac{\sqrt{5}-1}{2}, 1\right) \subset (-1, 1)$ such that $f(c) = 0$.

5. (P96, #26, 28, 32, 34, 37) Find the limit.

$$\begin{aligned} \lim_{x \rightarrow -3} \frac{x^2 - 9}{x^2 + 2x - 3} \quad \lim_{x \rightarrow 1^+} \frac{x^2 - 9}{x^2 + 2x - 3} \quad \lim_{v \rightarrow 4^+} \frac{4 - v}{|4 - v|} \\ \lim_{x \rightarrow 3} \frac{\sqrt{x+6} - x}{x^3 - 3x^2} \quad \lim_{x \rightarrow 0} \frac{1 - \sqrt{1 - x^2}}{x} \end{aligned}$$

Solutions. Since $\frac{x^2-9}{x^2+2x-3} = \frac{(x+3)(x-3)}{(x-1)(x+3)}$, we have

$$\lim_{x \rightarrow -3} \frac{x^2 - 9}{x^2 + 2x - 3} = \frac{-6}{-4} = \frac{3}{2}, \quad \lim_{x \rightarrow 1^+} \frac{x^2 - 9}{x^2 + 2x - 3} = -\infty.$$

For the other limits,

$$\begin{aligned} \lim_{v \rightarrow 4^+} \frac{4 - v}{|4 - v|} &= \lim_{v \rightarrow 4^+} \frac{4 - v}{v - 4} = -1, \\ \lim_{x \rightarrow 3} \frac{\sqrt{x+6} - x}{x^3 - 3x^2} &= \lim_{x \rightarrow 3} \frac{x + 6 - x^2}{x^2(x-3)(\sqrt{x+6} + x)} = \lim_{x \rightarrow 3} \frac{-(x+2)(x-3)}{x^2(x-3)(\sqrt{x+6} + x)} = \frac{-5}{54}, \\ \lim_{x \rightarrow 0} \frac{1 - \sqrt{1 - x^2}}{x} &= \lim_{x \rightarrow 0} \frac{1 - 1 + x^2}{x(1 + \sqrt{1 - x^2})} = 0. \end{aligned}$$

6. (P96, #40) Prove that

$$\lim_{x \rightarrow 0} x^2 \cos \frac{1}{x^2} = 0.$$

Proof. Since $-x^2 \leq x^2 \cos \frac{1}{x^2} \leq x^2$ for any $x \neq 0$, and $\lim_{x \rightarrow 0} (-x^2) = \lim_{x \rightarrow 0} x^2 = 0$, the Squeeze Theorem implies that

$$\lim_{x \rightarrow 0} x^2 \cos \frac{1}{x^2} = 0.$$

7. (P96, #45) Let

$$f(x) = \begin{cases} \sqrt{-x}, & \text{if } x < 0, \\ 3 - x, & \text{if } 0 \leq x \leq 3, \\ (x - 3)^2, & \text{if } x > 3. \end{cases}$$

(a) Evaluate each limit, if it exists.

$$\begin{aligned} \text{(i)} \quad \lim_{x \rightarrow 0^+} f(x) \quad \text{(ii)} \quad \lim_{x \rightarrow 0^-} f(x) \quad \text{(iii)} \quad \lim_{x \rightarrow 0} f(x) \\ \text{(iv)} \quad \lim_{x \rightarrow 3^-} f(x) \quad \text{(v)} \quad \lim_{x \rightarrow 3^+} f(x) \quad \text{(vi)} \quad \lim_{x \rightarrow 3} f(x) \end{aligned}$$

(b) Where is f discontinuous?

(c) Sketch the graph of f .

Solution.

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 3 - x = 3$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \sqrt{-x} = 0$$

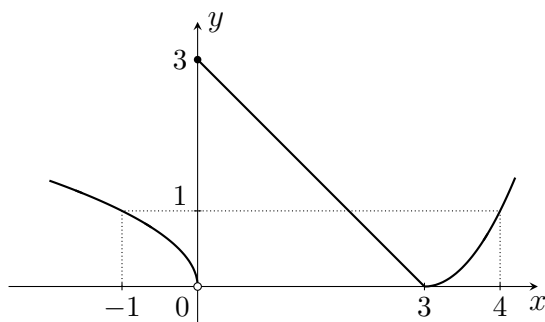
$$\lim_{x \rightarrow 0} f(x) \text{ does not exist, since } \lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x).$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} 3 - x = 0$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (x - 3)^2 = 0$$

$$\lim_{x \rightarrow 3} f(x) = 0.$$

f is discontinuous at $x = 0$. Here is a graph of the function f .



8. (P96, #46) Let

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leq x \leq 2 \\ 2 - x & \text{if } 2 < x \leq 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \geq 4 \end{cases}$$

(a) For each of the numbers 2, 3, and 4, discover whether g is continuous from the left, continuous from the right, or continuous at the number.

(b) Sketch the graph of g .

Solution. We have

$$\begin{array}{lll} \lim_{x \rightarrow 2^-} g(x) = 0 & \lim_{x \rightarrow 2^+} g(x) = 0 & \lim_{x \rightarrow 2} g(x) = 0 = g(2) \\ \lim_{x \rightarrow 3^-} g(x) = -1 & \lim_{x \rightarrow 3^+} g(x) = -1 & \lim_{x \rightarrow 3} g(x) = -1 = g(3) \\ \lim_{x \rightarrow 4^-} g(x) = 0 & \lim_{x \rightarrow 4^+} g(x) = \pi & \lim_{x \rightarrow 4} g(x) \text{ does not exist} \end{array}$$

Therefore, g is continuous at 2, 3, and continuous from the right at 4. Here is a graph of the function g .

