

EE3210

Signals and Systems

LT3: System in Time Domain

Systems in Time Domain

Chapter Intended Learning Outcomes:

- (i) Understand basic systems properties: Memoryless, invertibility, linearity, time-invariance, causality, stability
- (ii) Understand the property of convolution and its relationship with linear time-invariant system
- (iii) Understand the relationship between differential equation, difference equation and linear time-invariant system
- (iv) Perform basic operations in systems

System Overview

It can be classified as **continuous-time** and **discrete-time**:

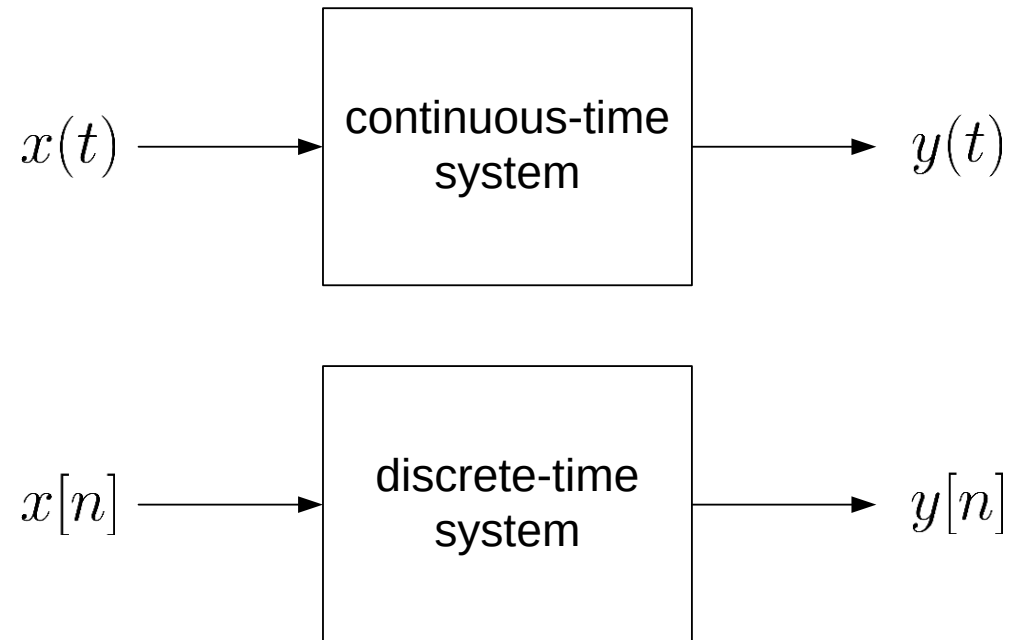


Fig. 3.1: Continuous-time and discrete-time systems

In a continuous-time (discrete-time) system, the input and output are continuous-time (discrete-time) signals.

A system is an operator $\mathcal{T}$ which maps input into output:

$$y(t) = \mathcal{T}\{x(t)\} \quad \text{or} \quad y[n] = \mathcal{T}\{x[n]\} \quad (3.1)$$

Systems can be connected/combined to form a bigger/overall system, e.g., break down a big task into several sub-tasks and each system handles one sub-task.

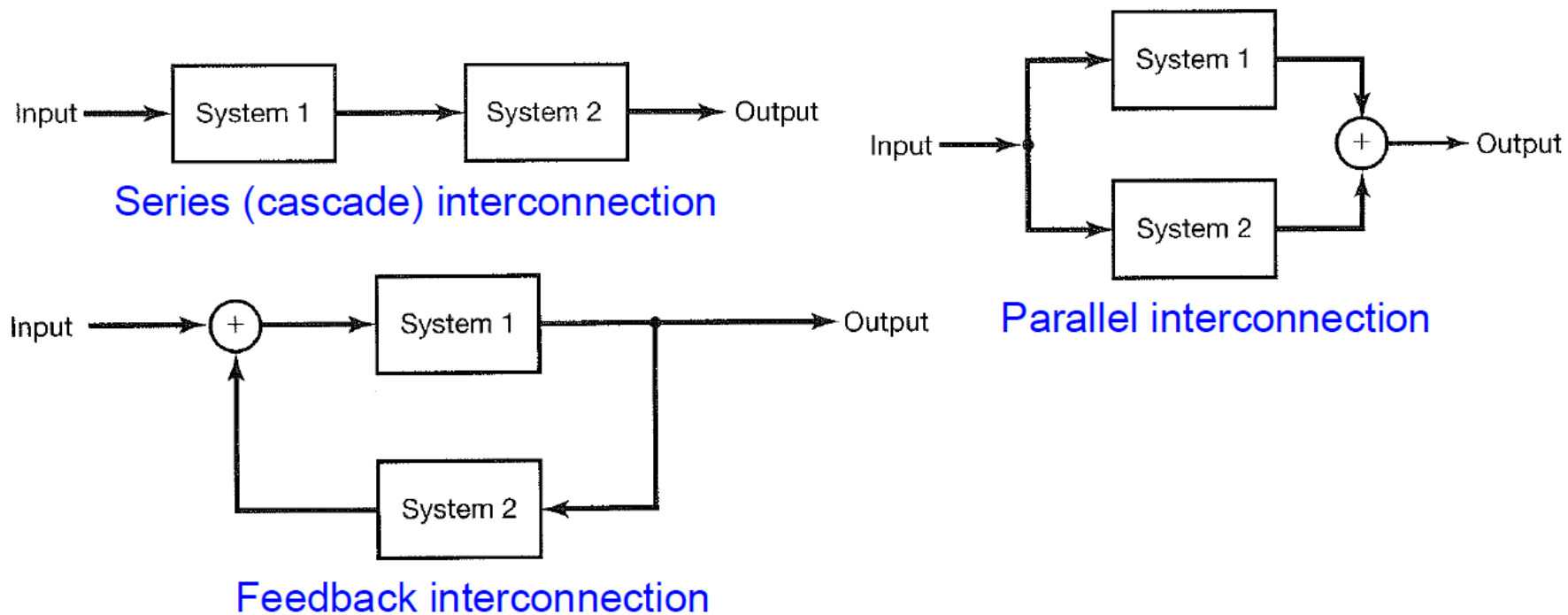


Fig. 3.2: Examples of system interconnections

Basic System Properties

Memoryless, invertibility, causality, stability, linearity, and time-invariance, are described as follows.

Memoryless

A system is memoryless if its output at a given time is dependent **only** on the input at that same time, i.e., $y(t)$ at time t depends **only** on $x(t)$ at time t ; $y[n]$ at time n depends **only** on $x[n]$ at time n .

A memoryless system does not have memory to store any input values because it just operates on the **current** input.

If a system is not memoryless, we can call it a system with memory.

Example 3.1

Determine if the following systems are memoryless or not

(a) $y(t) = x^2(t)$

(b) $y[n] = x[n] + x[n - 2]$

(a) The system is memoryless because the output at time t depends **only** on the input at time t .

(b) The system is not memoryless because $y[n]$ also depends **only** on $x[n - 2]$, which is a previous input, and thus it needs memory to store $x[n - 2]$ when processing the input at time n .

Invertibility

A system is invertible if distinct inputs lead to distinct outputs, or if an inverse system exists.

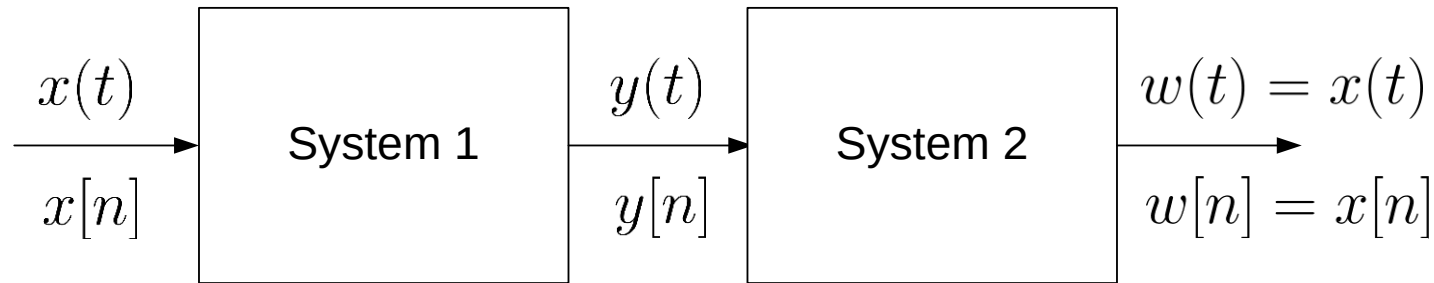


Fig. 3.3: Invertible system

That is, if we can get back the input $x(t)$ or $x[n]$ by passing the output $y(t)$ or $y[n]$ through another system, then the system is invertible, otherwise it is non-invertible.

Example 3.2

Determine if the following systems are invertible or not

(a) $y(t) = 2x(t)$

(b) $y(t) = x^2(t)$

(c) $y[n] = \sum_{k=-\infty}^n x[k]$

(d) $y[n] = 0$

(a) The system is invertible because we can pass $y(t)$ using another system to produce $w(t) = 0.5y(t) = x(t)$.

(b) The system is not invertible because the sign information is lost in the system output. Even employing the square root function, there are two possibilities: $w(t) = \sqrt{y(t)}$ or $w(t) = -\sqrt{y(t)}$.

(c)
$$y[n] = \sum_{k=-\infty}^n x[k] = [\cdots x[n-2] + x[n-1]] + x[n] = y[n-1] + x[n]$$

If we pass $y[n]$ using another system, $w[n] = y[n] - y[n-1] = x[n]$ can be obtained and hence the system is invertible.

(d) Any inputs will give the same output of zero and hence the system is not invertible.

Linearity

A system is linear if it obeys principle of **superposition**.

Given two pairs of inputs and outputs, linearity implies:

$$\mathcal{T}\{ax_1(t) + bx_2(t)\} = a\mathcal{T}\{x_1(t)\} + b\mathcal{T}\{x_2(t)\} = ay_1(t) + by_2(t) \quad (3.2)$$

and

$$\mathcal{T}\{ax_1[n] + bx_2[n]\} = a\mathcal{T}\{x_1[n]\} + b\mathcal{T}\{x_2[n]\} = ay_1[n] + by_2[n] \quad (3.3)$$

where $|a| < \infty$ and $|b| < \infty$.

If the system does not satisfy superposition, it is non-linear.

Example 3.3

Determine whether the following system with input $x[n]$ and output $y[n]$, is linear or not:

$$y[n] = \sum_{k=-\infty}^n x[k] = \cdots + x[n-1] + x[n]$$

A standard approach to determine the linearity of a system is given as follows. Let

$$y_i[n] = \mathcal{T}\{x_i[n]\}, \quad i = 1, 2, 3$$

with

$$x_3[n] = ax_1[n] + bx_2[n]$$

If $y_3[n] = ay_1[n] + by_2[n]$, then the system is linear. Otherwise, the system is non-linear. This also applies to continuous-time system.

Assigning $x_3[n] = ax_1[n] + bx_2[n]$, we have:

$$\begin{aligned}y_3[n] &= \sum_{k=-\infty}^n x_3[k] \\&= \sum_{k=-\infty}^n (ax_1[k] + bx_2[k]) \\&= a \sum_{k=-\infty}^n x_1[k] + b \sum_{k=-\infty}^n x_2[k] \\&= ay_1[n] + by_2[n]\end{aligned}$$

Note that the outputs for $x_1[n]$ and $x_2[n]$ are $y_1[n] = \sum_{k=-\infty}^n x_1[k]$ and $y_2[n] = \sum_{k=-\infty}^n x_2[k]$.

As a result, the system is linear.

Example 3.4

Determine whether the following system with input $x[n]$ and output $y[n]$, is linear or not.

$$y[n] = 3x^2[n] + 2x[n - 3]$$

The system outputs for $x_1[n]$ and $x_2[n]$ are $y_1[n] = 3x_1^2[n] + 2x_1[n - 3]$ and $y_2[n] = 3x_2^2[n] + 2x_2[n - 3]$. Assigning $x_3[n] = ax_1[n] + bx_2[n]$, its system output is then:

$$\begin{aligned} y_3[n] &= 3x_3^2[n] + 2x_3[n - 3] \\ &= 3(ax_1[n] + bx_2[n])^2 + 2ax_1[n - 3] + 2bx_2[n - 3] \\ &= 3(a^2x_1^2[n] + b^2x_2^2[n] + 2abx_1[n]x_2[n]) + 2ax_1[n - 3] + 2bx_2[n - 3] \\ &\neq a(3x_1^2[n] + 2x_1[n - 3]) + b(3x_2^2[n] + 2x_2[n - 3]) \\ &= ay_1[n] + by_2[n] \end{aligned}$$

As a result, the system is non-linear.

Time-Invariance

A system is time-invariant if a time-shift of input causes a corresponding shift in output:

$$\text{if } y(t) = \mathcal{T}\{x(t)\} \text{ then } y(t - t_0) = \mathcal{T}\{x(t - t_0)\} \quad (3.4)$$

and

$$\text{if } y[n] = \mathcal{T}\{x[n]\} \text{ then } y[n - n_0] = \mathcal{T}\{x[n - n_0]\} \quad (3.5)$$

That is, the system response is independent of time.

Example 3.5

Determine whether the following system with input $x[n]$ and output $y[n]$, is time-invariant or not.

$$y[n] = \sum_{k=-\infty}^n x[k]$$

A standard approach to determine the time-invariance of a system is given as follows.

Let $y_1[n] = \mathcal{T}\{x_1[n]\}$ where $x_1[n] = x[n - n_0]$. If $y_1[n] = y[n - n_0]$, then the system is time-invariant. Otherwise, the system is time-variant. This also applies to continuous-time system.

From the given input-output relationship, $y[n - n_0]$ is:

$$y[n - n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

Let $x_1[n] = x[n - n_0]$, its system output is:

$$\begin{aligned} y_1[n] &= \sum_{k=-\infty}^n x_1[k] = \sum_{k=-\infty}^n x[k - n_0] = \sum_{l=-\infty}^{n-n_0} x[l], \quad l = k - n_0 \\ &= y[n - n_0] \end{aligned}$$

As a result, the system is time-invariant.

Example 3.6

Determine whether the following system with input $x[n]$ and output $y[n]$, is time-invariant or not:

$$y[n] = 3x[3n]$$

From the given input-output relationship, $y[n - n_0]$ is of the form:

$$y[n - n_0] = 3x[3(n - n_0)] = 3x[3n - 3n_0]$$

Let $x_1[n] = x[n - n_0]$, its system output is:

$$y_1[n] = 3x_1[3n] = 3x[3n - n_0] \neq y[n - n_0]$$

As a result, the system is time-variant.

Causality

A system is causal if the output $y(t)$ (or $y[n]$) at time t (or n) depends on input $x(t)$ (or $x[n]$) **up to** time t (or n).

In casual system, output does not depend on **future** input.

On the other hand, in a non-causal system, the output depends on future input.

Example 3.7

Determine if the following systems are causal or not.

(a) $y(t) = x^2(t)$

(b) $y[n] = x[n] + x[n + 2]$

(c) $y[n] = \sum_{k=-\infty}^n x[k]$

- (a) The system is causal because it does not depend on future input.
- (b) The system is not causal because it depends on future input, namely, $x[n + 2]$.
- (c)
$$y[n] = \sum_{k=-\infty}^n x[k] = \cdots x[n - 2] + x[n - 1] + x[n]$$

We see that the output $y[n]$ at time n depends on input $x[n]$ up to time n . Hence the system is causal.

Stability

A system is stable if every bounded input $x(t)$ or $x[n]$ produces a bounded output $y(t)$ or $y[n]$ for all time t or n , or if the bounded-input bounded-output criterion is satisfied.

That is:

$$|y(t)| < B \quad \text{if} \quad |x(t)| < A, \quad |A| < \infty, \quad |B| < \infty \quad (3.6)$$

and

$$|y[n]| < B \quad \text{if} \quad |x[n]| < A, \quad |A| < \infty, \quad |B| < \infty \quad (3.7)$$

If a bounded input produces an unbounded output, then the system is unstable.

Example 3.8

Determine if the following systems are stable or not.

(a) $y(t) = x^2(t)$

(b) $y[n] = x[n] + x[n + 2]$

(c) $y[n] = \frac{1}{x[n]}$

(a) If $x(t)$ is bounded, say, $|x(t)| < A$ for all t , we easily get

$$|y(t)| < A^2$$

Hence the system is stable.

(b) The system is stable because:

$$|y[n]| = |x[n] + x[n+2]| \leq |x[n]| + |x[n+2]| < 2A$$

for a bounded input with $|x[n]| < A$ for all n .

(c) The system is not stable. It is because for a bounded input, namely, $x[n] = 0$, the output is unbounded.

Linear Time-Invariant System Characterization

In this course, we will mainly study systems which are **both linear and time-invariant**.

Apart from being fundamental, many practical applications correspond to linear time-invariant (LTI) system.

Impulse Response

The impulse response ($h(t)$ or $h[n]$) is the **output** of a LTI system when the input is the unit impulse ($\delta(t)$ or $\delta[n]$):

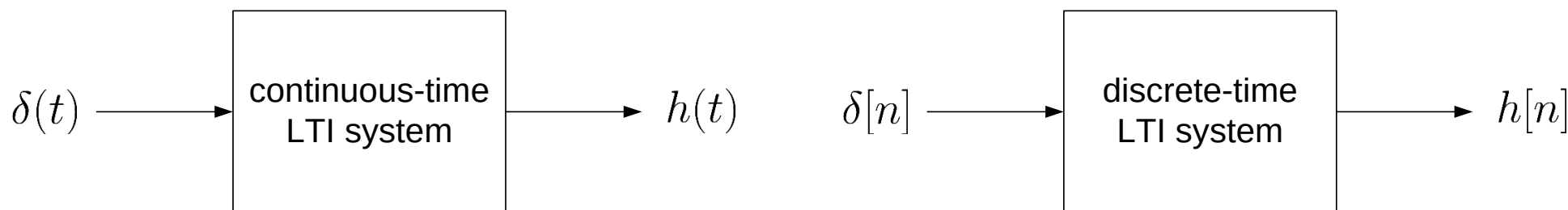


Fig. 3.4: Impulse response

For a **continuous-time** system, the impulse response is also **continuous-time** signal.

For a **discrete-time** system, the impulse response is also **discrete-time** signal.

A LTI system can be characterized by its impulse response, which indicates the system **functionality**.

Convolution

Convolution is used to describe the relationship between **input**, **output** and **impulse response** of a LTI in **time domain**.

We start with considering the discrete-time impulse response $h[n] = \mathcal{T}\{\delta[n]\}$ of a LTI system.

Recall (2.35) that a discrete-time signal is a **linear combination** of impulses with different time-shifts:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \quad (3.8)$$

Consider $x[n]$ as the system input with $y[n]$ being the output:

$$\begin{aligned} y[n] &= \mathcal{T}\{x[n]\} = \mathcal{T}\left\{\sum_{m=-\infty}^{\infty} x[m]\delta[n-m]\right\} \\ &= \sum_{m=-\infty}^{\infty} x[m]\mathcal{T}\{\delta[n-m]\} \end{aligned} \quad (3.9)$$

due to the **linearity** property of (3.3).

Furthermore, using **time-invariance** property yields:

$$h[n - m] = \mathcal{T}\{\delta[n - m]\} \quad (3.10)$$

Substituting (3.10) into (3.9), we obtain:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n - m] = x[n] \otimes h[n] \quad (3.11)$$

which is called the **convolution** of $x[n]$ and $h[n]$, and $\otimes$ denotes the convolution operator.

According to (3.11), $h[n]$ completely characterizes the LTI system because for any input $x[n]$, the output $y[n]$ can be computed with the use of $h[n]$ via convolution where only **multiplication** and **addition** are involved.

There are three properties in convolution:

- **Commutative**

$$\begin{aligned} x[n] \otimes h[n] &= h[n] \otimes x[n] \\ &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \end{aligned} \quad (3.12)$$

- **Associative**

$$x[n] \otimes (h_1[n] \otimes h_2[n]) = (x[n] \otimes h_1[n]) \otimes h_2[n] \quad (3.13)$$

Combining (3.12) and (3.13) yields:

$$\begin{aligned} y[n] &= x[n] \otimes h_1[n] \otimes h_2[n] \\ &= x[n] \otimes h_2[n] \otimes h_1[n] \\ &= x[n] \otimes (h_1[n] \otimes h_2[n]) \end{aligned} \quad (3.14)$$

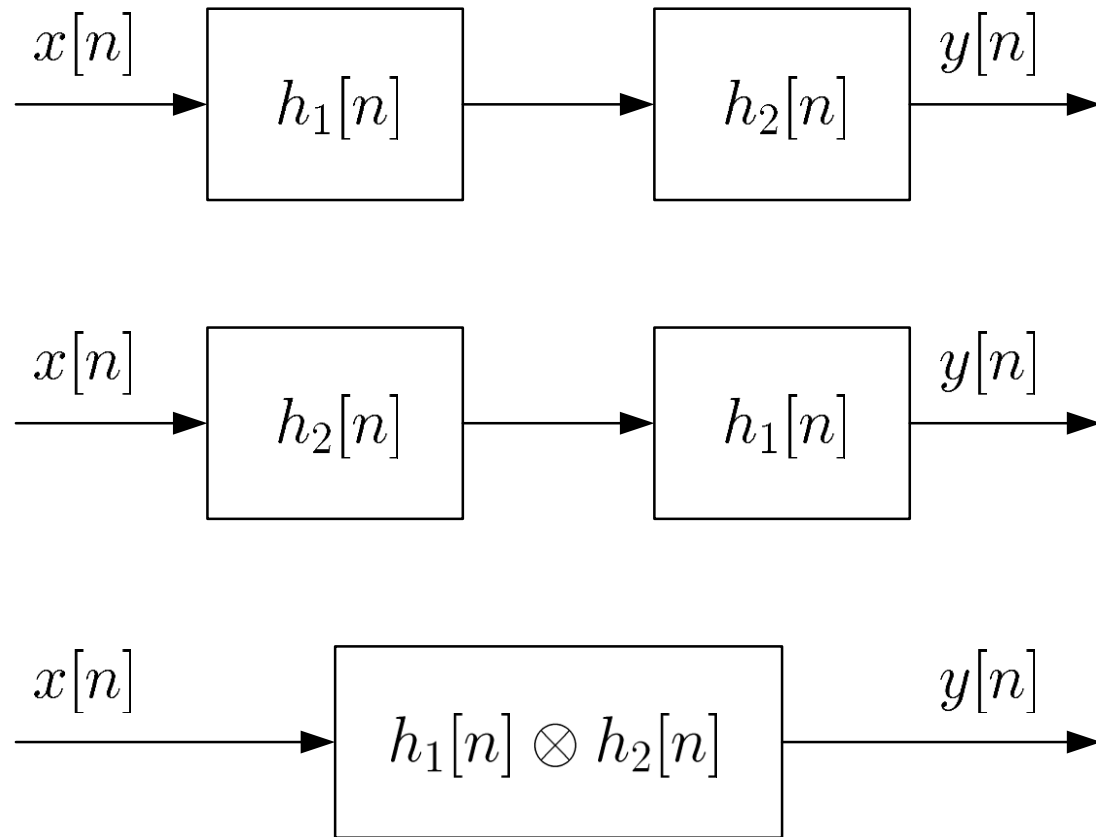


Fig. 3.5: Cascade interconnection and convolution

- Distributive

$$y[n] = x[n] \otimes (h_1[n] + h_2[n]) = x[n] \otimes h_1[n] + x[n] \otimes h_2[n] \quad (3.15)$$

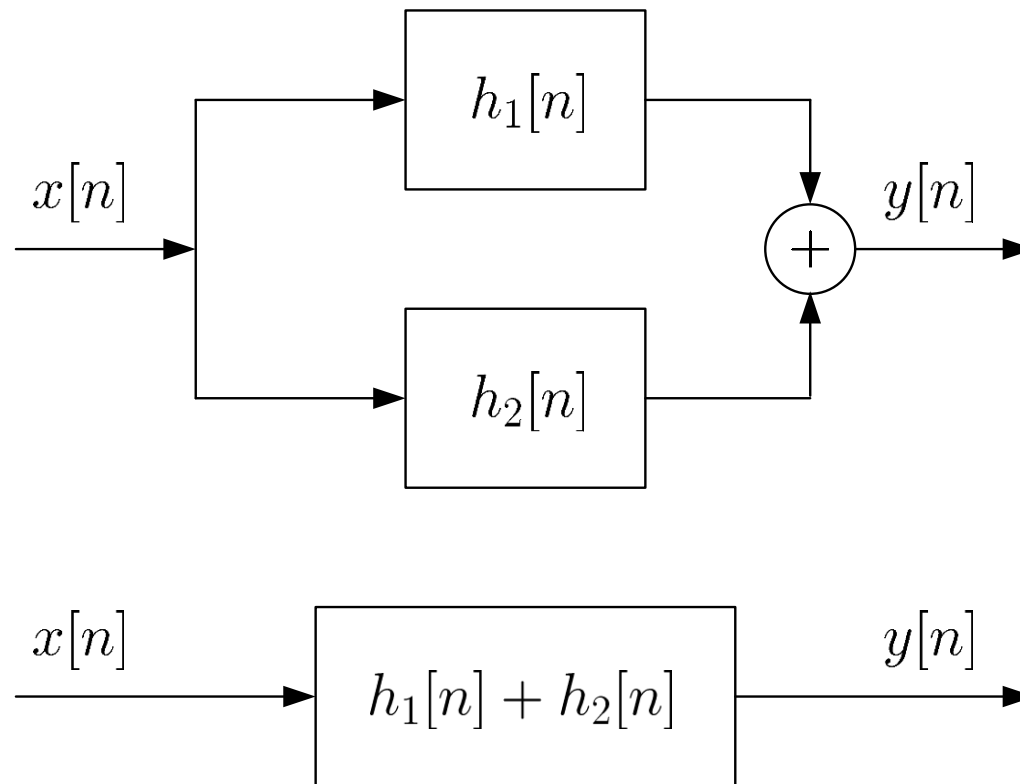


Fig. 3.6: Parallel interconnection and convolution

Example 3.9

Determine the function of a LTI discrete-time system if its impulse response is $h[n] = 0.5\delta[n] + 0.5\delta[n - 1]$.

Using (3.11) and (3.8), we have:

$$\begin{aligned} y[n] &= x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n - m] \\ &= \sum_{m=-\infty}^{\infty} x[m] (0.5\delta[n - m] + 0.5\delta[n - 1 - m]) \\ &= 0.5 \sum_{m=-\infty}^{\infty} x[m]\delta[n - m] + 0.5 \sum_{m=-\infty}^{\infty} x[m]\delta[n - 1 - m] \\ &= 0.5(x[n] + x[n - 1]) \end{aligned}$$

The system computes the mean value of two input samples, i.e., current value and past value.

Similarly, for the **continuous-time** case, we start with considering $h(t) = \mathcal{T}\{\delta(t)\}$ of a LTI system.

Recall (2.21) that a continuous-time signal is considered as a **linear combination** of impulses with different time-shifts:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau \quad (3.16)$$

Analogous to the development in (3.9)-(3.11), we get:

$$\begin{aligned} y(t) &= \mathcal{T}\{x(t)\} = \mathcal{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau\right\} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathcal{T}\{\delta(t - \tau)\}d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = x(t) \otimes h(t) \end{aligned} \quad (3.17)$$

Equation (3.17) is the **convolution** for the continuous-time case. However, the computation is more complicated than the discrete-time convolution because **integration** is needed.

Again, $h(t)$ characterizes the input-output relationship of LTI system.

Same as the discrete-time case, $h(t)$ specifies the system functionality and satisfies the commutative, associative as well as distributive properties.

Example 3.10

Determine the function of a LTI continuous-time system if its impulse response is $h(t) = \delta(t) + \delta(t - 1)$.

Using (3.17) and (2.19)-(2.20), we obtain:

$$\begin{aligned}y(t) &= x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \\&= \int_{-\infty}^{\infty} x(\tau)[\delta(t - \tau) + \delta(t - 1 - \tau)]d\tau \\&= \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau + \int_{-\infty}^{\infty} x(\tau)\delta(t - 1 - \tau)d\tau \\&= \int_{-\infty}^{\infty} x(t)\delta(t - \tau)d\tau + \int_{-\infty}^{\infty} x(t - 1)\delta(t - 1 - \tau)d\tau \\&= x(t) \int_{-\infty}^{\infty} \delta(t - \tau)d\tau + x(t - 1) \int_{-\infty}^{\infty} \delta(t - 1 - \tau)d\tau \\&= x(t) + x(t - 1)\end{aligned}$$

The system computes **sum of inputs** at two time instants: one at current time and the other at current time minus 1.

Example 3.11

Determine the function of a LTI continuous-time system if its impulse response is $h(t) = 0.1[u(t) - u(t - 10)]$.

Using (3.17) and the commutative property, we get:

$$\begin{aligned} y(t) &= h(t) \otimes x(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \\ &= 0.1 \int_{-\infty}^{\infty} [u(\tau) - u(\tau - 10)]x(t - \tau)d\tau \\ &= \frac{1}{10} \int_0^{10} x(t - \tau)d\tau \end{aligned}$$

Note that $[u(\tau) - u(\tau - 10)]$ is a **rectangular pulse** for $\tau \in (0, 10)$.

The system computes **average input** value from the current time minus 10 to current time.

For LTI systems, we can also use the impulse response to check the system causality and stability.

A LTI system is **causal** if its impulse response satisfies:

$$h(t) = 0, \quad t < 0 \quad (3.18)$$

$$h[n] = 0, \quad n < 0 \quad (3.19)$$

A LTI system is **stable** if its impulse response satisfies:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \quad (3.20)$$

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \quad (3.21)$$

Example 3.12

Show that for a LTI discrete-time system, the causality definition in (3.19) is identical to the universal definition, i.e., $y[n]$ at time n depends on $x[n]$ up to time n .

Expanding the convolution formula in (3.12):

$$\begin{aligned} y[n] = x[n] \otimes h[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\ &= \cdots h[-2]x[n+2] + h[-1]x[n+1] + \\ &\quad h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \cdots \end{aligned}$$

If $y[n]$ does not depend on future inputs $x[n+1], x[n+2], \dots$, we must have $h[-1] = h[-2] = \cdots = 0$ or $h[n] = 0$ for $n < 0$.

Hence the two definitions regarding causality are identical.

Example 3.13

Compute the output $y[n]$ if the input is $x[n] = u[n]$ and the LTI system impulse response is $h[n] = \delta[n] + 0.5\delta[n - 1]$. Discuss the stability and causality of system.

Using (3.11), we have:

$$\begin{aligned} y[n] &= x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= \sum_{m=-\infty}^{\infty} u[m] (\delta[n-m] + 0.5\delta[n-1-m]) \\ &= \sum_{m=0}^{\infty} (\delta[n-m] + 0.5\delta[n-1-m]) \\ &= \sum_{m=0}^{\infty} \delta[n-m] + 0.5 \sum_{m=0}^{\infty} \delta[n-1-m] = u[n] + 0.5u[n-1] \end{aligned}$$

Alternatively, we can first establish the general relationship between $y[n]$ and $x[n]$ with the specific $h[n]$ as in Example 3.9:

$$\begin{aligned} y[n] &= x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= \sum_{m=-\infty}^{\infty} x[m] (\delta[n-m] + 0.5\delta[n-1-m]) \\ &= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] + 0.5 \sum_{m=-\infty}^{\infty} x[m]\delta[n-1-m] \\ &= x[n] + 0.5x[n-1] \end{aligned}$$

Substituting $x[n] = u[n]$ yields the same $y[n]$.

Since $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^1 |h[n]| = 1.5 < \infty$ and $h[n] = 0$ for $n < 0$ the system is stable and causal.

Example 3.14

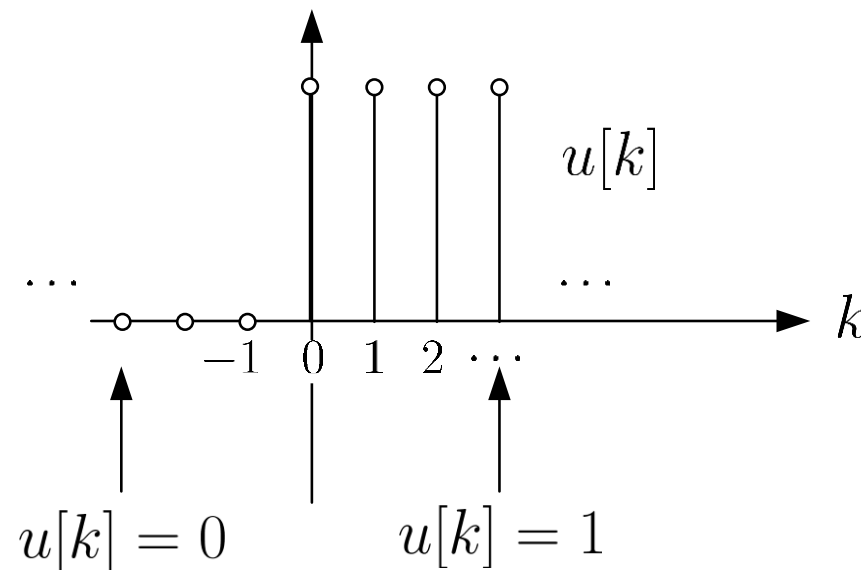
Compute the output $y[n]$ if the input is $x[n] = a^n u[n]$ and the LTI system impulse response is $h[n] = u[n] - u[n - 10]$. Discuss the stability and causality of system.

Using (3.11), we have:

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} x[m] h[n - m] \\ &= \sum_{m=-\infty}^{\infty} a^m u[m] (u[n - m] - u[n - 10 - m]) \\ &= \sum_{m=0}^{\infty} a^m (u[n - m] - u[n - 10 - m]) \\ &= \sum_{m=0}^{\infty} a^m u[n - m] - \sum_{m=0}^{\infty} a^m u[n - 10 - m] \end{aligned}$$

Let $y_1[n] = \sum_{m=0}^{\infty} a^m u[n-m]$ and $y_2[n] = \sum_{m=0}^{\infty} a^m u[n-10-m]$ such that $y[n] = y_1[n] - y_2[n]$. By employing a change of variable, $y_1[n]$ is expressed as

$$\begin{aligned} y_1[n] &= \sum_{m=0}^{\infty} a^m u[n-m] = \sum_{k=n}^{-\infty} a^{n-k} u[k], \quad k = n - m \\ &= \sum_{k=-\infty}^n a^{n-k} u[k] \end{aligned}$$



For $n < 0$, $y_1[n] = 0$ because $u[k] = 0$ for $k < 0$. For $n \geq 0$, $y_1[n]$ is:

$$y_1[n] = \sum_{k=0}^n a^{n-k} = 1 + a + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

where the geometric sum formula is applied:

$$\alpha + \alpha r + \cdots + \alpha r^{N-1} = \alpha \frac{1 - r^N}{1 - r}$$

That is,

$$y_1[n] = \frac{1 - a^{n+1}}{1 - a} u[n]$$

Similarly, $y_2[n]$ is:

$$\begin{aligned} y_2[n] &= \sum_{m=0}^{\infty} a^m u[n-10-m] \\ &= \sum_{k=-\infty}^{n-10} a^{n-10-k} u[k], \quad k = n-10-m \end{aligned}$$

Since $u[k] = 0$ for $k < 0$, $y_2[n] = 0$ for $n < 10$. For $n \geq 10$, $y_2[n]$ is:

$$y_2[n] = \sum_{k=0}^{n-10} a^{n-10-k} = 1 + a + \dots + a^{n-10} = \frac{1 - a^{n-9}}{1 - a}$$

That is,

$$y_2[n] = \frac{1 - a^{n-9}}{1 - a} u[n - 10]$$

Combining the results, we have:

$$y[n] = \frac{1 - a^{n+1}}{1 - a} u[n] - \frac{1 - a^{n-9}}{1 - a} u[n - 10]$$

or

$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1 - a^{n+1}}{1 - a}, & 0 \leq n < 10 \\ \frac{a^{n-9} (1 - a^{10})}{1 - a}, & 10 \leq n \end{cases}$$

Since $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^9 |h[n]| = 10 < \infty$, the system is stable.

Moreover, the system is causal because $h[n] = 0$ for $n < 0$.

Example 3.15

Determine $y[n] = x[n] \otimes h[n]$ where $x[n]$ and $h[n]$ are

$$x[n] = \begin{cases} n^2 + 1, & 0 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h[n] = \begin{cases} n + 1, & 0 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Here, the lengths of both $x[n]$ and $h[n]$ are finite. More precisely, $x[0] = 1$, $x[1] = 2$, $x[2] = 5$, $x[3] = 10$, $h[0] = 1$, $h[1] = 2$, $h[2] = 3$ and $h[3] = 4$ while all other $x[n]$ and $h[n]$ have zero values.

We still use (3.11) but now it reduces to a finite summation:

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3] \end{aligned}$$

By considering the non-zero values of $h[n]$, we obtain:

$$y[n] = \begin{cases} 1, & n = 0 \\ 4, & n = 1 \\ 12, & n = 2 \\ 30, & n = 3 \\ 43, & n = 4 \\ 50, & n = 5 \\ 40, & n = 6 \\ 0, & \text{otherwise} \end{cases}$$

Alternatively, for finite-length discrete-time signals, we can use the MATLAB command `conv` to compute the convolution of **finite-length** sequences:

```
n=0:3;
x=n.^2+1;
h=n+1;
```

$$y = \text{conv}(x, h)$$

The results are

$$y = 1 \quad 4 \quad 12 \quad 30 \quad 43 \quad 50 \quad 40$$

As the default starting time indices in both h and x are 1, we need to determine the appropriate time index for y .

The correct index can be obtained by computing one value of $y[n]$ using (3.11). For simplicity, we may compute $y[0]$:

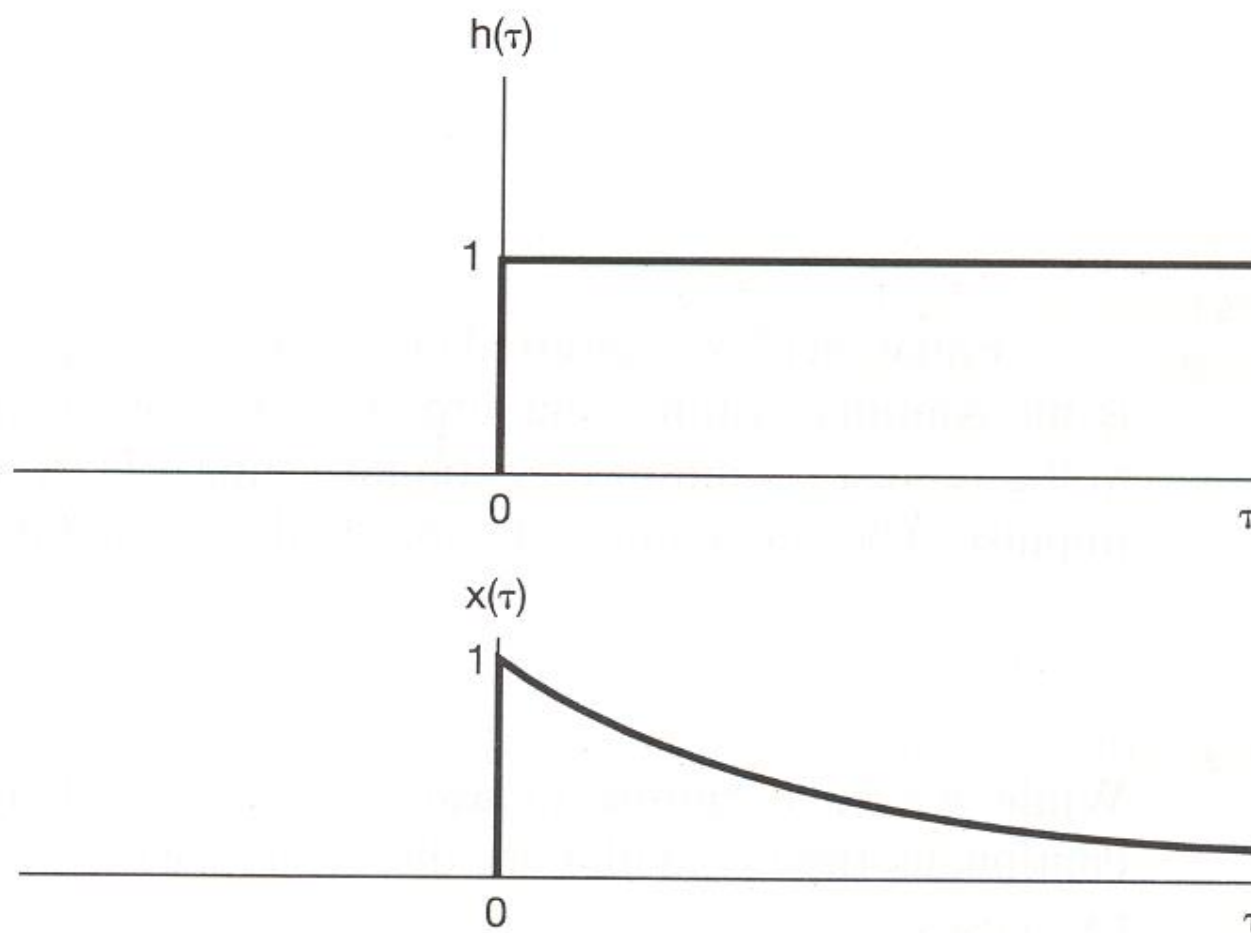
$$\begin{aligned}
 y[0] &= \sum_{m=-\infty}^{\infty} x[m]h[-m] \\
 &= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + \cdots \\
 &= x[0]h[0] \\
 &= 1
 \end{aligned}$$

In general, if the lengths of $x[n]$ and $h[n]$ are M and N , respectively, the length of $y[n] = x[n] \otimes h[n]$ is $(M + N - 1)$.

Example 3.16

Compute the output $y(t)$ if the input is $x(t) = e^{-at}u(t)$ with $a > 0$ and the LTI system impulse response is $h(t) = u(t)$. Discuss

the stability and causality of system.



Using (3.17), we have:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t - \tau)d\tau \\
&= \int_0^{\infty} e^{-a\tau}u(t - \tau)d\tau, \quad \lambda = t - \tau \\
&= \int_t^{-\infty} e^{-a(t-\lambda)}u(\lambda) \cdot -d\lambda \\
&= e^{-at} \int_{-\infty}^t e^{a\lambda}u(\lambda)d\lambda
\end{aligned}$$

Similar to Example 3.14, when $t < 0$, the integral will only involve the zero part of $u(\lambda)$ because $u(\lambda) = 0$ for $\lambda < 0$. Hence

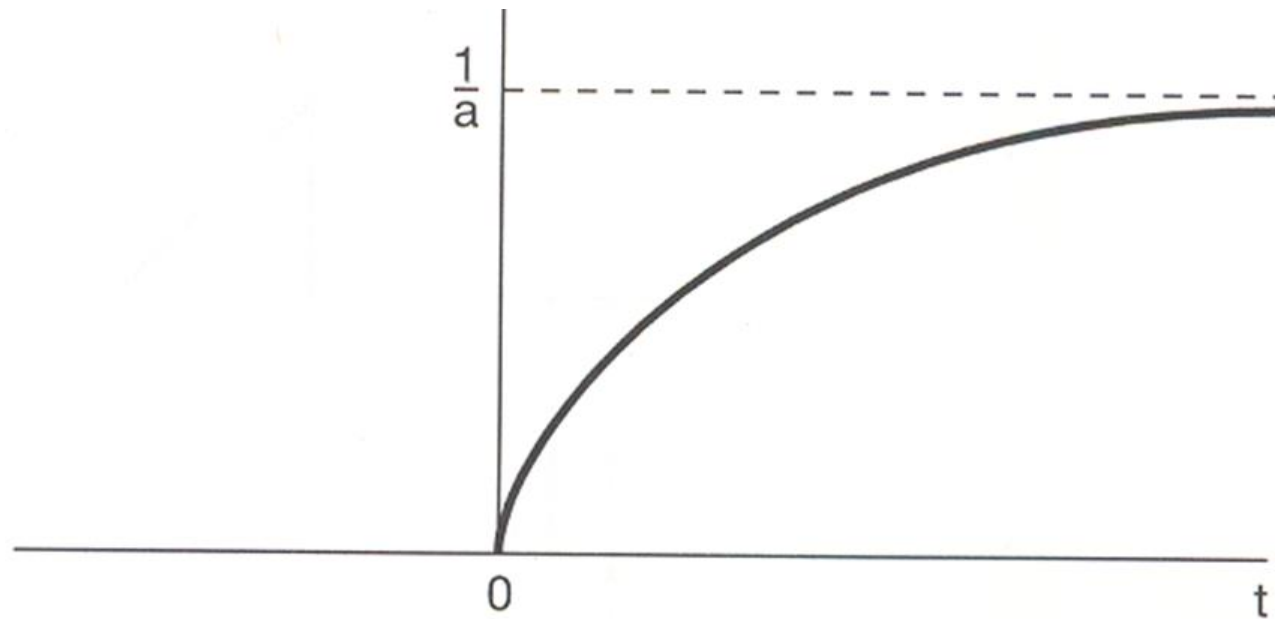
$$y(t) = e^{-at} \int_{-\infty}^t e^{a\lambda}u(\lambda)d\lambda = 0, \quad t < 0$$

When $t > 0$, the integral will involve the non-zero part of $u(\lambda)$ because $u(\lambda) = 1$ for $0 < \lambda \leq t$. The output is then:

$$\begin{aligned}
 y(t) &= e^{-at} \int_{-\infty}^t e^{a\lambda} u(\lambda) d\lambda = e^{-at} \int_0^t e^{a\lambda} d\lambda \\
 &= e^{-at} \cdot \frac{1}{a} e^{a\lambda} \Big|_0^t = e^{-at} \cdot \frac{1}{a} (e^{at} - 1) = \frac{1}{a} (1 - e^{-at})
 \end{aligned}$$

We can combine the results for $t < 0$ and $t > 0$ to yield:

$$y(t) = \frac{1}{a} (1 - e^{-at}) u(t)$$



Linear Constant Coefficient Difference Equation

For a LTI **discrete-time** system, its input $x[n]$ and output $y[n]$ are related via a N th-order linear constant coefficient **difference equation**:

$$\sum_{k=0}^N a_k y[n - k] = \sum_{k=0}^M b_k x[n - k] \quad (3.22)$$

which is useful to check whether a system is **both** linear and time-invariant or not.

Example 3.17

Determine if the following input-output relationships correspond to LTI systems.

- (a) $y[n] = 0.1y[n - 1] + x[n] + x[n - 1]$
- (b) $y[n] = x[n + 1] + x[n]$
- (c) $y[n] = 1/x[n]$

(a) It corresponds to a LTI system with $N = M = 1$, $a_0 = 1$, $a_1 = -0.1$ and $b_0 = b_1 = 1$

(b) We reorganize the equation as:

$$y[n] = x[n+1] + x[n] \Rightarrow y[n-1] = x[n] + x[n-1]$$

which agrees with (3.22) when $N = M = 1$, $a_0 = 0$ and $a_1 = b_0 = b_1 = 1$. Hence it also corresponds to a LTI system.

(c) It does not correspond to a LTI system because $x[n]$ and $y[n]$ are not linear in the equation.

Note that if a system cannot be fitted into (3.22), there are three possibilities: linear and time-variant; non-linear and time-invariant; or non-linear and time-variant.

Example 3.18

Compute the impulse response $h[n]$ for a LTI system which is characterized by the following difference equation:

$$y[n] = x[n] - x[n - 1]$$

Using (3.12), we have:

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n - m] \\ &= \cdots + h[-1]x[n + 1] + h[0]x[n] + h[1]x[n - 1] + \cdots \end{aligned}$$

we can easily deduce that only $h[0]$ and $h[1]$ are nonzero. That is, the impulse response is:

$$h[n] = \delta[n] - \delta[n - 1]$$

The difference equation can be used to generate the system output and even the system input.

Assuming that $a_0 \neq 0$, $y[n]$ is computed as:

$$y[n] = \frac{1}{a_0} \left(- \sum_{k=1}^N a_k y[n-k] + \sum_{k=0}^M b_k x[n-k] \right) \quad (3.23)$$

Assuming that $b_0 \neq 0$, $x[n]$ can be obtained from:

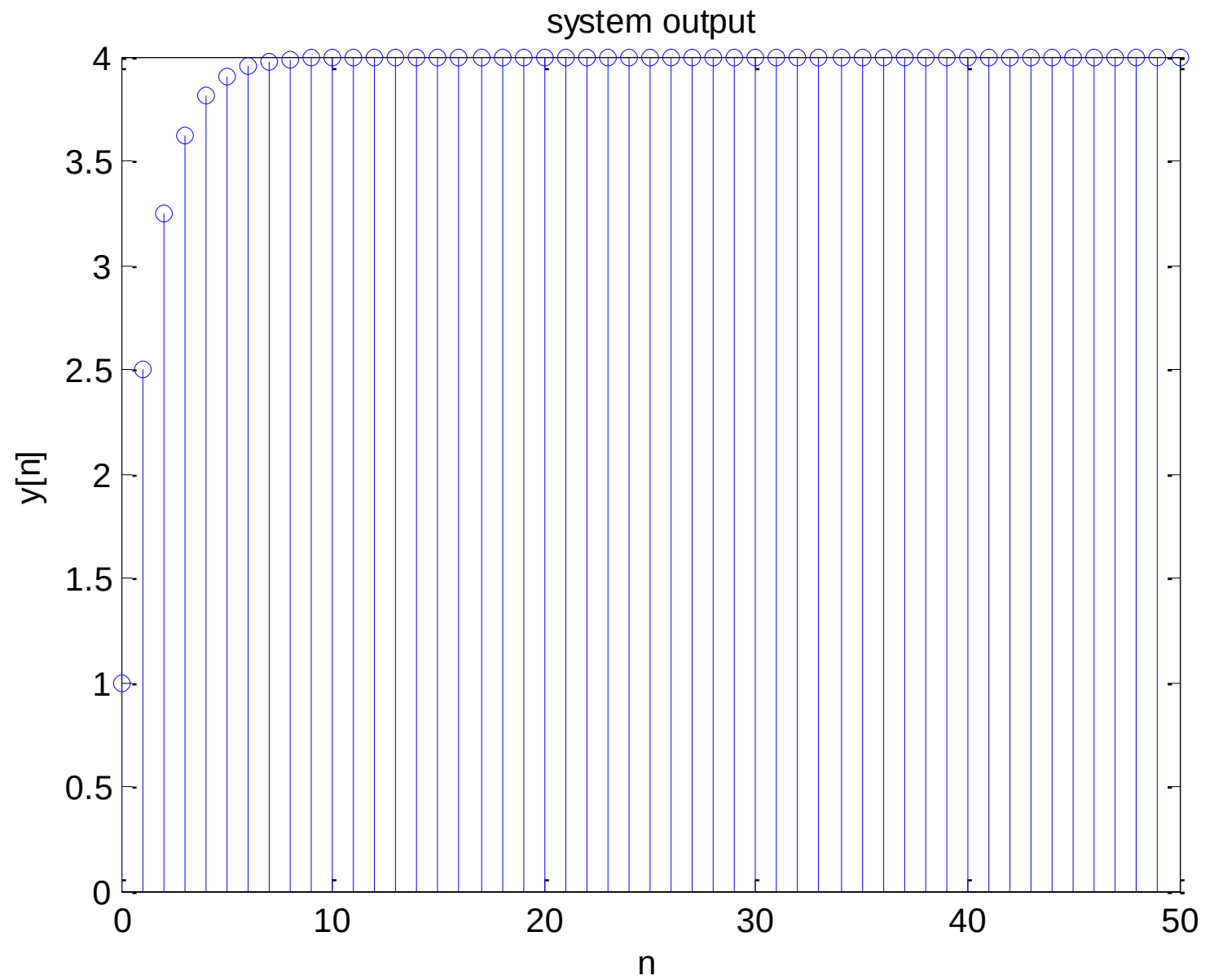
$$x[n] = \frac{1}{b_0} \left(\sum_{k=0}^N a_k y[n-k] - \sum_{k=1}^M b_k x[n-k] \right) \quad (3.24)$$

Example 3.19

Given a LTI system described by difference equation of $y[n] = 0.5y[n-1] + x[n] + x[n-1]$, compute the system output $y[n]$ for $0 \leq n \leq 50$ with an input of $x[n] = u[n]$. It is assumed that $y[-1] = 0$.

The MATLAB code is:

```
N=50;           %data length is N+1
y(1)=1;         %compute y[0], only x[n] is nonzero
for n=2:N+1
y(n)=0.5*y(n-1)+2; %compute y[1],y[2],...,y[50]
                  %x[n]=x[n-1]=1 for n>=1
end
n=[0:N];        %set time axis
stem(n,y);
```



Alternatively, we can use the MATLAB command `filter` by rewriting the equation as:

$$y[n] - 0.5y[n - 1] = x[n] + x[n - 1]$$

The corresponding MATLAB code is:

```
x=ones(1,51);           %define input
a=[1,-0.5];             %define vector of a_k
b=[1,1];                 %define vector of b_k
y=filter(b,a,x);         %produce output
stem(0:length(y)-1,y)
```

The `x` is the input which has a value of 1 for $0 \leq n \leq 50$, while `a` and `b` are vectors which contain $\{a_k\}$ and $\{b_k\}$, respectively.

The MATLAB programs for this example are provided as `ex3_19.m` and `ex3_19_2.m`.

Linear Constant Coefficient Differential Equation

For a **continuous-time** LTI system, its input $x(t)$ and output $y(t)$ are related via a N th-order linear constant coefficient **differential equation**:

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k} \quad (3.25)$$

which is useful to check whether a system is **both** linear and time-invariant or not.

Analogous to the discrete-time case, we can use (3.25) to compute system input, output and impulse response.