
EE 3121

Differential Equations for Electrical Engineering

Tutorial 8: Systems of First-Order ODE: State Space Description



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Summary

- This lecture has dealt with systems of first-order ODEs, also known as the state space representation. The general form is

$$\begin{aligned}\frac{d\underline{x}}{dt} &= \underline{f}(t, \underline{x}), \\ \underline{x}(t_0) &= \underline{x}_0.\end{aligned}$$

- The lecture presented an existence and uniqueness theorem very similar to the one for scalar first-order ODEs, showing that **systems of equations in state space form have at least a local solution provided $\underline{f}(t, \underline{x})$ is sufficiently smooth.**
- An extensive variety of computer programs to numerically integrate ODEs in state space form are used by practicing engineers and we were introduced to several of them, including the *Runge-Kutta* and *Adams-Moulton algorithms*.

Summary

- For homogeneous LTI systems in the form

$$\begin{aligned}\dot{\underline{x}} &= \underline{A} \underline{x}, \\ \underline{x}(0) &= \underline{x}_0.\end{aligned}$$

finding fundamental solutions begins with trying $\underline{x} = \underline{\xi} e^{rt}$, where for each eigenvalue r_i satisfying

$$\det(\underline{A} - r\underline{I}) = 0.$$

- There is an eigenvector $\underline{\xi}^{(i)}$ satisfying

$$(\underline{A} - r_i \underline{I}) \underline{\xi} = 0.$$

- When an eigenvalue is complex $r = \lambda + i\omega$ its eigenvector is also ($\underline{\xi} = \underline{a} + i\underline{b}$) and the real fundamental solutions are

$$\begin{aligned}\underline{x}^{(1)}(t) &= e^{\lambda t}(\underline{a} \cos \omega t - \underline{b} \sin \omega t), \\ \underline{x}^{(2)}(t) &= e^{\lambda t}(\underline{a} \sin \omega t + \underline{b} \cos \omega t).\end{aligned}$$

Summary

- If the n by n matrix $\underline{A}$ is symmetric, it will have n independent eigenvectors even if it has repeated eigenvalues and then all fundamental solutions will be in the aforementioned forms.
- If $\underline{A}$ is not symmetric and it has repeated eigenvalues it may not have n independent eigenvectors, in which case some of its fundamental solutions will have more complicated forms involving $t^k e^{rt}$, including where r is complex.
- An LTI system is stable and damped if the real part of every eigenvalue is negative. It is unstable if it has one or more eigenvalues with a positive real part.
- We have a choice of state variables in writing the state space description of a system and our choice can sometimes ease our solution of the ODEs or enhance our understanding of the dynamics of the system.

Summary

- The chapter determined that the solution of the nonhomogeneous linear time-varying system

$$\dot{\underline{x}} = \underline{A}(t)\underline{x} + \underline{B}(t)\underline{u}(t), \quad \underline{x}(t_0) = \underline{x}_0.$$

can be written

$$\underline{x}(t) = \underline{\Phi}(t, t_0)\underline{x}(t_0) + \int_{t_0}^t \underline{\Phi}(t, \tau)\underline{B}(\tau)(\underline{u}\tau)d\tau.$$

- The state transition matrix $\underline{\Phi}(t, \tau)$ is given by

$$\underline{\Phi}(t, \tau) = \underline{\Psi}(t)\underline{\Psi}^{-1}(\tau),$$

- and the columns of $\underline{\Psi}(t)$ are the fundamental solutions of

$$\dot{\underline{x}} = \underline{A}(t)\underline{x}.$$

Summary

- We learned three methods to solve the nonhomogenous LTI system

$$\dot{\underline{x}} = \underline{A}(t)\underline{x} + \underline{B}(t)\underline{u}(t), \quad \underline{x}(t_0) = \underline{x}_0.$$

- **The matrix kernel method:**

$$\underline{x}(t) = \underline{\Phi}(t - t_0)\underline{x}(t_0) + \int_{t_0}^t \underline{\Phi}(t - \tau)\underline{B}(\tau)(\underline{u}\tau)d\tau.$$

- **The matrix Laplace transform method:** which can be formally written as

$$\begin{aligned}\underline{x}(t) &= L^{-1} \{ \underline{x}(s) \}, \\ \underline{x}(s) &= (s\underline{I} - \underline{A})^{-1}(\underline{x}_0 + \underline{B} \underline{u}(s)), \\ \underline{u}(s) &= L \{ \underline{u}(t) \} .\end{aligned}$$

- **The matrix undetermined coefficients method:** which can be formally written as

Summary

$$\underline{x}(t) = \underline{\Phi}(t)(\underline{x}_0 - \underline{x}_p(0)) + \underline{x}_p(t),$$
$$\underline{x}_p(t) = \sum_{i=1}^m \underline{c}^{(i)} f_i(t).$$

where the scalar functions $f_i(t)$ are suggested by the scalar functions found in $\underline{u}(t)$ and each $\underline{c}^{(i)}$ is a vector of constants that one solves for by substituting the trial solution in the matrix ODE.

- This lecture explored *nonlinear systems* briefly and compared and contrasted their behavior with that of linear systems.
- Nonlinear systems can have multiple equilibrium points where linear systems have at most one.

Summary

- Under certain not very stringent conditions a nonlinear system

$$\dot{\underline{w}} = \underline{g}(t, \underline{w}), \quad \underline{w}(t_0) = \underline{w}_0.$$

is asymptotically stable or unstable at an equilibrium point $\underline{w}_0$ depending on whether the **linearized system**

$$\dot{\underline{x}} = \underline{A} \underline{x}, \quad A_{ij} = \frac{\partial g_i}{\partial w_j}(\underline{w}_0)$$

is asymptotically stable (damped) or unstable.

Summary

- **Nonlinearities impose limits where linear systems do not.** And finally, nonlinear systems can exhibit behavior such as harmonic distortion, limit cycles and virtual unpredictability (chaos) that linear systems do not.
- Engineers have developed effective methods for designing and analyzing nonlinear systems but they tend to be ad hoc.
- This lecture gave one example, the concept of equivalent gain, which enables engineers to employ on nonlinear systems the powerful methods available for design and analysis of linear systems.

Summary-Solved Problems

- **SP8.1** Use the Euler method to compute to five decimal places the first three time steps ($t = 0.025, 0.05, 0.075$) in the numerical integration of

$$\begin{aligned}\frac{dz}{dt} &= \ln(vz) \cos(4t), & z(0) &= 6, \\ \frac{dv}{dt} &= -\left(\frac{z}{v}\right) (1 + 8t)^2, & v(0) &= 4.\end{aligned}$$

- *Solution:*

- The Euler method applied to this problem yields

$$\begin{aligned}t_{n+1} &= t_n + \Delta t, \\ z_{n+1} &= z_n + \ln(v_n z_n) \cos(4t_n) \cdot \Delta t, \\ v_{n+1} &= v_n - \left(\frac{z_n}{v_n}\right) (1 + 8t_n)^2 \cdot \Delta t.\end{aligned}$$

where $\Delta t = 0.025$.

Summary-Solved Problems

■ SP8.1 *Solution:*

■ For $n = 0$:

$$t_1 = 0 + 0.025 = 0.025$$

$$z_1 = 6 + \ln(6 \cdot 4) \cos(4(0))(0.025)$$

$$z_1 = 6 + (3.178)(1)(0.025) = 6.07945$$

$$v_1 = 4 - \left(\frac{6}{4}\right) (1 + 8(0))^2(0.025)$$

$$v_1 = 4 - (1.5)(1)(0.025) = 3.96250$$

■ For $n = 1$:

$$t_2 = 0.025 + 0.025 = 0.05$$

$$z_2 = 6.07945 + \ln(6.07945)(3.9625) \cos(4(0.025))(0.025)$$

$$z_2 = 6.07945 + (3.18179)(0.9950)(0.025) = 6.15860$$

$$v_2 = 3.9625 - \left(\frac{6.07945}{3.9625}\right) (1 + 8(0.025))^2(0.025)$$

$$v_2 = 3.9625 - (1.5342)(1.440)(0.025) = 3.90727$$

Summary-Solved Problems

■ SP8.1 *Solution:*

■ For $n = 2$:

$$t_3 = 0.05 + 0.025 = 0.075$$

$$z_3 = 6.15860 + \ln(6.15860) \cdot (3.90727) \cos(4(0.05))(0.025)$$

$$z_3 = 6.15860 + (3.18069)(0.98007)(0.025) = 6.23653$$

$$v_3 = 3.90727 - \left(\frac{6.15860}{3.90727}\right) (1 + 8(0.025))^2(0.025)$$

$$v_3 = 3.90727 - (1.57619)(1.96)(0.025) = 3.83004$$

Summary-Solved Problems

- **SP8.2** Use the Euler method to compute to four decimal places the first three time steps ($t = 0.05, 0.10, 0.15$) in the numerical integration of

$$\begin{aligned}\frac{dz}{dt} &= \frac{z}{v^2}, & z(0) &= 2, \\ \frac{dv}{dt} &= -\frac{\sqrt{v(t+0.1)}}{z^2}, & v(0) &= 4.\end{aligned}$$

- *Solution:*

- The Euler method applied to this problem yields

$$\begin{aligned}t_{n+1} &= t_n + \Delta t, \\ z_{n+1} &= z_n + (z_n/v_n^2) \cdot \Delta t, \\ v_{n+1} &= v_n + (\sqrt{v_n(t_n + 0.1)}/z_n^2) \cdot \Delta t.\end{aligned}$$

where $\Delta t = 0.05$.

Summary-Solved Problems

■ SP8.2 Solution:

■ For $n = 0$:

$$t_1 = 0 + 0.05 = 0.05$$

$$z_1 = 2 + (2/4^2)(0.05)$$

$$z_1 = 2 + (0.125)(0.05) = 2.0063$$

$$v_1 = 2 + (\sqrt{4(0 + 0.1)}/2^2)(0.05)$$

$$v_1 = 4 + (0.1581)(0.025) = 4.0079$$

■ For $n = 1$:

$$t_2 = 0.05 + 0.05 = 0.1$$

$$z_2 = 2.0063 + (2.0063/(4.0079)^2)(0.05)$$

$$z_2 = 2.0063 + (0.1250)(0.05) = 2.0125$$

$$v_2 = 4.0079 + (\sqrt{(4.0079)(0.05 + 0.1)}(2.0063)^2)(0.05)$$

$$v_2 = 4.0079 + (0.1926)(0.05) = 4.0175$$

Summary-Solved Problems

■ SP8.2 *Solution:*

■ For $n = 2$:

$$t_3 = 0.1 + 0.05 = 0.15$$

$$z_3 = 2.0125 + (2.0125/(4.0175)^2)(0.05)$$

$$z_3 = 2.0125 + (0.1247)(0.05) = 2.0187$$

$$v_3 = 4.0175 + (\sqrt{(4.0175)(0.1 + 0.1)}/(2.0125)^2)(0.05)$$

$$v_3 = 4.0175 + (0.2213)(0.05) = 4.0286$$

Summary-Solved Problems

■ SP8.3 Solve

$$\frac{d\underline{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \underline{x}, \quad \underline{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (1)$$

■ *Solution:*

■ Try $\underline{x} = \underline{\xi}e^{rt}$ Equation (1),

$$r\underline{\xi}e^{rt} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \underline{\xi}e^{rt}$$

$$(\underline{A} - r\underline{I})\underline{\xi} = \begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} \underline{\xi} = 0. \quad (2)$$

■ The first condition for Equation (2) to be satisfied for nonzero $\underline{\xi}$ is

Summary-Solved Problems

■ SP8.3 *Solution:*

$$\begin{aligned}\det \begin{pmatrix} 1-r & 1 \\ 4 & -2-r \end{pmatrix} &= 0, \\ (1-r)(-2-r) - (4)(1) &= 0, \\ r^2 + r - 6 &= 0. \end{aligned} \tag{3}$$

■ The quadratic in Equation (3) factors into

$$(r + 3)(r - 2) = 0.$$

■ Hence the eigenvalues for Equation (1) are

$$r_1 = -3, \quad r_2 = +2$$

Summary-Solved Problems

■ SP8.3 *Solution:*

- Substituting $r_1 = -3$ into Equation (2),

$$\begin{pmatrix} r_1 - 1 & -1 \\ -4 & r_1 + 2 \end{pmatrix} \underline{\xi}^{(1)} = 0$$

$$\begin{pmatrix} -3 - 1 & -1 \\ -4 & -3 + 2 \end{pmatrix} \underline{\xi}^{(1)} = 0$$

$$\begin{pmatrix} -4 & -1 \\ -4 & -1 \end{pmatrix} \underline{\xi}^{(1)} = 0. \quad (4)$$

- The matrix equation (4) is not two independent equations. It cannot be; its determinant is zero. It is one equation:

$$-4\underline{\xi}_1^{(1)} - \underline{\xi}_2^{(1)} = 0. \quad (5)$$

Summary-Solved Problems

■ SP8.3 *Solution:*

- To proceed we set a value for one or the other of the unknowns. It is simplest to choose the value one.
- We set $\underline{\xi}_1^{(1)} = 1$ and then, by Equation (5), $\underline{\xi}_2^{(1)} = -4$, and so the eigenvector associated with the first eigenvalue, $r_1 = -3$, is

$$\underline{\xi}^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \quad (6)$$

- Next we return to Equation (2), substitute the second eigenvalue $r_2 = 2$ and solve for the eigenvector $\underline{\xi}_2$ associated with it.

$$\begin{pmatrix} r_2 - 1 & -1 \\ -4 & r_2 + 2 \end{pmatrix} \underline{\xi}^{(2)} = 0.$$

Summary-Solved Problems

■ SP8.3 *Solution:*

$$\begin{pmatrix} 2-1 & -1 \\ -4 & 2+2 \end{pmatrix} \underline{\xi}^{(2)} = 0.$$

■ We have

$$\begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \underline{\xi}^{(2)} = 0. \quad (7)$$

■ The two equations in Equation (7) are not exactly the same, as in Equation (4), but they are not independent. (Again, they cannot be.) They are both equivalent to

$$\underline{\xi}_1^{(2)} - \underline{\xi}_2^{(2)} = 0. \quad (8)$$

■ Again we have one equation and two unknowns. We set $\underline{\xi}_1^{(2)} = 1$ and then, by Equation (8), $\underline{\xi}_2^{(2)} = 1$, and so the eigenvector associated with the first eigenvalue, $r_2 = 2$, is

$$\underline{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (9)$$

Summary-Solved Problems

■ SP8.

■ *Solution:*

- Thus the general solution to Equation (1) has the form

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}. \quad (10)$$

- We determine the constants c_1 and c_2 from the initial condition

$$\underline{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- Setting $t = 0$ in Equation (10) and applying the initial condition we have

$$\underline{x}(0) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3(0)} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2(0)} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \quad (11)$$

Summary-Solved Problems

■ SP8.3 *Solution:*

- The vector Equation 8.184 represents two scalar equations:

$$\begin{aligned}c_1 \cdot 1 + c_2 \cdot 1 &= 1 \\c_1 \cdot (-4) + c_2 \cdot 1 &= 2\end{aligned}$$

or

$$\begin{aligned}c_1 + c_2 &= 1 \\-4c_1 + c_2 &= 2\end{aligned}$$

which has solution

$$\begin{aligned}c_1 &= -\frac{1}{5}, \\c_2 &= \frac{6}{5}.\end{aligned}\tag{12}$$

- From Equations (10) and (12), the solution to Equation (1) is

$$\underline{x}(t) = (-1/5) \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + (6/5) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}\tag{13}$$

Summary-Solved Problems

■ SP8.4 Solve

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -5 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}. \quad (14)$$

■ *Solution:*

- As always, for LTI homogeneous matrix ODEs such as this, we try

$$\underline{x} = \underline{\xi} e^{rt}. \quad (15)$$

- Substituting Equation (15) into the matrix ODE in Equations (14) leads in short order to

$$(\underline{A} - r\underline{I})\underline{\xi} = \begin{pmatrix} -5 - r & -1 \\ 5 & -3 - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (16)$$

which, for nonzero $\underline{\xi}$, implies that r must satisfy

Summary-Solved Problems

■ SP8.4 Solution:

$$\det(\underline{A} - r\underline{I}) = \det \begin{pmatrix} -5 - r & -1 \\ 5 & -3 - r \end{pmatrix} = (-5 - r)(-3 - r) - (5)(-1) = r^2 + 8r + 20 = 0. \quad (17)$$

■ Completing the square:

$$r^2 + 8r + 20 = (r + 4)^2 - 4^2 + 20 = (r + 4)^2 + 4 = 0. \quad (18)$$

■ Hence the two eigenvalues of the system in Equation (10) are

$$\begin{aligned} r_1 &= -4 + 2i, \\ r_2 &= -4 - 2i. \end{aligned} \quad (19)$$

where $i = \sqrt{-1}$.

Summary-Solved Problems

■ SP8.4 Solution:

- We must next find the eigenvector $\underline{\xi}^{(1)}$ associated with the eigenvalue r_1 . Substituting $r_1 = -4 + 2i$ into Equation (16):

$$\begin{aligned}(r_1 \underline{I} - \underline{A}) \underline{\xi}^{(1)} &= \begin{pmatrix} r_1 + 5 & 1 \\ -5 & r_1 + 3 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} \\ &= \begin{pmatrix} -4 + 2i + 5 & 1 \\ -5 & -4 + 2i + 3 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \\ \begin{pmatrix} 1 + 2i & 1 \\ -5 & -1 + 2i \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{20}\end{aligned}$$

- The matrix Equation (20) is equivalent to two scalar equations:

$$\begin{aligned}(1 + 2i)\xi_1^{(1)} + \xi_2^{(1)} &= 0, \\ (-5)\xi_1^{(1)} + (-1 + 2i)\xi_2^{(1)} &= 0.\end{aligned} \tag{21}$$

Summary-Solved Problems

■ SP8.4 *Solution:*

- Although it may not be apparent looking at Equations (21), the two scalar equations are not independent. This is as it must be, since the determinant is zero. It is really one equation in two unknowns.
- Therefore we are free to set the value of one of the unknowns. We choose to set $\xi_1^{(1)} = 1$. Then, $\xi_2^{(1)} = -1 - 2i$, and so the eigenvector associated with the first eigenvalue, $r_1^2 = -4 + 2i$, is

$$\underline{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 - 2i \end{pmatrix}$$

This can be written as

$$\underline{\xi}^{(1)} = \underline{a} + \underline{b}i = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} i. \quad (22)$$

Summary-Solved Problems

■ SP8.4 *Solution:*

- From Section 8.5.1.2 we learned that if the first eigenvalue is $r_1 = -\lambda + i\omega$ and its associated eigenvector is $\underline{\xi}^{(1)} = \underline{a} + \underline{b}i$ then the two real fundamental solutions associated with r_1 and its complex conjugate are

$$\begin{aligned}\underline{x}^{(1)}(t) &= e^{rt}(\underline{a} \cos \omega t - \underline{b} \sin \omega t), \\ \underline{x}^{(2)}(t) &= e^{rt}(\underline{a} \sin \omega t + \underline{b} \cos \omega t).\end{aligned}$$

- Here, from Equation (19), $\lambda = -4$ and $\omega = 2$, and from Equation (22),

$$\underline{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

Summary-Solved Problems

■ SP8.4 Solution:

■ Then

$$\begin{aligned}\underline{x}^{(1)}(t) &= e^{-4t} \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t \right), \\ \underline{x}^{(2)}(t) &= e^{-4t} \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t \right).\end{aligned}\tag{23}$$

■ Any solution of Equations (14) must be of the form

$$\underline{x}(t) = c_1 \underline{x}^{(1)}(t) + c_2 \underline{x}^{(2)}(t).\tag{24}$$

■ The coefficients c_1, c_2 are determined by the initial conditions. From Equations (23) and (24) and the second part of Equations (14) we have

$$\underline{x}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}\tag{25}$$

Summary-Solved Problems

■ SP8.4 *Solution:*

- Equation (25) is equivalent to the two scalar equations

$$\begin{aligned}c_1 \cdot (1) + c_2 \cdot (0) &= 3, \\c_1 \cdot (-1) + c_2 \cdot (-2) &= -4.\end{aligned}$$

or

$$\begin{aligned}c_1 &= 3, \\-c_1 - 2c_2 &= -4,\end{aligned}$$

which have solution $c_1 = 3$ and $c_2 = 1/2$.

Summary-Solved Problems

■ SP8.4 *Solution:*

■ Then, from Equations (23) and (24),

$$\underline{x}(t) = 3e^{-4t} \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t \right) + \left(\frac{1}{2} \right) e^{-4t} \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t \right),$$

which can be simplified to

$$\underline{x}(t) = e^{-4t} \left(\begin{pmatrix} 3 \\ -4 \end{pmatrix} \cos 2t - \begin{pmatrix} 1/2 \\ 11/2 \end{pmatrix} \sin 2t \right). \quad (26)$$

Summary-Solved Problems

- **SP8.5** This problem is from Section 8.5.1.4. Solve

$$\frac{d\underline{x}}{dt} = \underline{A}\underline{x}, \quad (27)$$

where:

$$\underline{x}(0) = \begin{pmatrix} v_s \\ 0 \\ 0 \end{pmatrix} \quad (28)$$

and

$$\underline{A} = \frac{-1}{4RC} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \quad (29)$$

- *Solution:*

- Trying $\underline{x}(t) = \underline{\xi}e^{rt}$, we find as before that Equation (27) implies $r\underline{\xi}e^{rt} = \underline{A}\underline{\xi}e^{rt}$, which in turn requires

$$\det(\underline{A} - r\underline{I}) = 0. \quad (30)$$

Summary-Solved Problems

■ SP8.5 Solution:

- For simplicity define

$$r_* = (4RC)r. \quad (31)$$

- Then substituting Equations (29) and (31) into Equation (30), we arrive at

$$\det \begin{pmatrix} 2 + r_* & 1 & 1 \\ 1 & 2 + r_* & 1 \\ 1 & 1 & 2 + r_* \end{pmatrix} = 0. \quad (32)$$

- Equation (32) becomes

$$\begin{aligned} (2 + r_*)[(2 + r_*)^2 - 1] - [(2 + r_*) - 1] + [1 - (2 + r_*)] &= \\ (2 + r_*)(2 + r_*)(2 + r_*) - 3(2 + r_*) + 2 &= \\ r_*^3 + 6r_*^2 + 9r_* + 4 &= 0. \end{aligned} \quad (33)$$

Summary-Solved Problems

■ SP8.5 Solution:

- Using the hand-calculation method we applied in Lecture 6, or using a web-based root calculator, we find that the cubic in Equation (33) factors into

$$(r_* + 4)(r_* + 1)^2 = 0. \quad (34)$$

- Hence the nondimensional eigenvalues for the system represented by Equations (27), (28), and (29) are

$$\begin{aligned} r_{*1} &= -4, \\ r_{*2} &= r_{*3} = -1. \end{aligned}$$

- The eigenvectors for this system are given for $i = 1, 2, 3$ by

$$\begin{pmatrix} 2 + r_* & 1 & 1 \\ 1 & 2 + r_* & 1 \\ 1 & 1 & 2 + r_* \end{pmatrix} \begin{pmatrix} \xi_1^{(i)} \\ \xi_2^{(i)} \\ \xi_3^{(i)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (35)$$

Summary-Solved Problems

■ SP8.5 Solution:

- We choose $\underline{\xi}_1^{(i)}$ as the undetermined component and set $\underline{\xi}_1^{(i)} = 1$. For the first eigenvalue $r_{*1} = -4$ the first and second rows of Equation (31) become

$$\begin{aligned}\xi_2^{(1)} + \xi_3^{(1)} &= 2, \\ -2\xi_2^{(1)} + \xi_3^{(1)} &= -1.\end{aligned}\tag{36}$$

- The solutions to Equation (36) are

$$\begin{aligned}\xi_2^{(1)} &= 1, \\ \xi_3^{(1)} &= 1.\end{aligned}$$

- Hence the eigenvector associated with eigenvalue $r_{*1} = -4$ is

$$\underline{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.\tag{37}$$

Summary-Solved Problems

■ SP8.5 Solution:

- For the eigenvalues $r_{*2} = -1$, $r_{*3} = -1$, the first and second rows of Equation (35) become, for $i = 1, 2$

$$\begin{aligned}\xi_2^{(i)} + \xi_3^{(i)} &= -1, \\ \xi_2^{(i)} + \xi_3^{(i)} &= -1.\end{aligned}\tag{38}$$

- The equations are identical. We can form two independent eigenvalues from them.
- In each we arbitrarily set one of the components equal to zero. Then together with the first component the eigenvectors for the repeated eigenvalues $r_{*2} = r_{*3} = -1$ are

$$\underline{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \underline{\xi}^{(3)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\tag{39}$$

Summary-Solved Problems

■ SP8.5 Solution:

- Summarizing our results so far, returning to dimensional variables, the general solution to Equations (27) and (29) is

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t/RC} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t/4RC} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-t/4RC} \quad (40)$$

- We must determine the coefficients c_i from initial conditions, Equation (28) Setting $t = 0$ in Equation (40) and applying initial conditions we have

$$\underline{x}(0) = \begin{pmatrix} v_s \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

Summary-Solved Problems

■ SP8.5 Solution:

or

$$\begin{aligned}c_1 + c_2 + c_3 &= v_s, \\c_1 - c_3 &= 0, \\c_1 - c_2 &= 0.\end{aligned}\tag{41}$$

■ The solution to Equations (41) is

$$c_1 = c_2 = c_3 = \frac{v_s}{3}.\tag{42}$$

■ Then the solution to Equations (27) through (29) is

$$\underline{x}(t) = \frac{v_s}{3} \begin{pmatrix} e^{-t/RC} + 2e^{-t/4RC} \\ e^{-t/RC} - e^{-t/4RC} \\ e^{-t/RC} - e^{-t/4RC} \end{pmatrix}.\tag{43}$$

Summary-Solved Problems

■ SP8.6 Solve

$$\dot{\underline{x}} = \begin{pmatrix} 1 & -5 \\ 5 & -9 \end{pmatrix} \underline{x}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (44)$$

■ *Solution:*

- As before we try the solution $\underline{x}(t) = \underline{\xi}e^{rt}$ and arrive again at the equations

$$\det(\underline{A} - r\underline{I}) = 0, \quad (45)$$

$$(\underline{A} - r\underline{I})\underline{\xi} = \underline{0}. \quad (46)$$

- From Equations (44) and (45)

$$\det \begin{pmatrix} 1-r & -5 \\ 5 & -9-r \end{pmatrix} = (1-r)(-9-r) + 25 = r^2 + 8r + 16 = (r+4)^2 = 0,$$

$$r_1 = r_2 = -4. \quad (47)$$

Summary-Solved Problems

■ SP8.6 *Solution:*

- Substituting Equation (47) into Equation(46)

$$5\xi_1^{(1)} - 5\xi_2^{(1)} = 0. \quad (48)$$

- We choose $\xi_1^{(1)} = 1$ and then from Equation (48) $\xi_2^{(1)} = 1$. Hence

$$\underline{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (49)$$

- This provides us

$$\underline{x}^{(1)}(t) = \xi^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}. \quad (50)$$

- But what of $\underline{x}^{(2)}(t)$?

Summary-Solved Problems

■ SP8.6 *Solution:*

- Recall from Section 5.1.2 that when repeated roots occur in second-order systems the second solution has the form te^{rt} . Extrapolating that result to the state-space case, we might expect the second solution

$$\underline{x}^{(2)}(t) = \underline{\gamma}te^{rt}. \quad (51)$$

- Trying Equation (51) here:

$$\underline{x}^{(2)}(t) = \underline{\gamma}e^{rt} + r\underline{\gamma}te^{rt},$$

$$\underline{x}^{(2)} - \underline{A}\underline{x} = \underline{\gamma}e^{rt} + r\underline{\gamma}te^{rt} - \underline{A}(\underline{\gamma}te^{rt}) = \underline{\gamma}e^{rt} - te^{rt}(\underline{A} - r\underline{I})\underline{\gamma}. \quad (52)$$

- Equation (52) is supposed to equal $\underline{0}$. We can make the vector $(\underline{A} - r\underline{I})\underline{\gamma}$ equal $\underline{0}$ by setting $\underline{\gamma} = \underline{\xi}^{(1)}$. (Recall Equations (46) and (49).)

Summary-Solved Problems

■ SP8.6 *Solution:*

- But then the first term on the right-hand side in Equation (48), $\underline{\gamma}e^{rt}$, is nonzero. So our first trial solution has failed.
- Given the nature of the failure we next try

$$\underline{x}^{(2)}(t) = \underline{\gamma}te^{rt} + \underline{\eta}e^{rt}. \quad (53)$$

- Substituting Equation (53) into

$$\frac{d\underline{x}}{dt} = \underline{A}\underline{x},$$

we arrive at

$$\underline{\gamma}e^{rt} + r\underline{\gamma}te^{rt} + r\underline{\eta}e^{rt} = \underline{A}(\underline{\gamma}te^{rt} + \underline{\eta}e^{rt}). \quad (54)$$

Summary-Solved Problems

■ SP8.6 *Solution:*

- Equating coefficients of t^n in Equation (54) with $n = 0, 1$ we find

$$(\underline{A} - r\underline{I})\underline{\gamma} = 0, \quad (55)$$

$$(\underline{A} - r\underline{I})\underline{\eta} = \underline{\gamma}. \quad (56)$$

- Now Equation (55) is the same as Equation (46) and we found that the only solution to Equation (46) is

$$\underline{\gamma} = \underline{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (57)$$

- Hence Equation (56) becomes

$$(\underline{A} - r\underline{I})\underline{\eta} = \underline{\xi}^{(1)}. \quad (58)$$

Summary-Solved Problems

■ SP8.6 Solution:

- Using $r = -4$ we find that Equation (58) reduces to

$$5\eta_1 - 5\eta_2 = 1. \quad (59)$$

- We have

$$\eta_2 = \frac{-1}{5} + \eta_1. \quad (60)$$

- Then, using Equations (53), (57), and (60), the second solution can be written as

$$\underline{x}^{(2)}(t) = te^{-4t}\underline{\xi}^{(1)} + \begin{pmatrix} 0 \\ -1/5 \end{pmatrix} e^{-4t} + \eta_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}, \quad (61)$$

where η_1 is an undetermined constant.

- But, recalling Equation (50), the last term in Equation (61) duplicates the first solution $\underline{x}^{(1)}(t)$, so we can set $\eta_1 = 0$ and then Equation (61) simplifies to

$$\underline{x}^{(2)}(t) = te^{-4t}\underline{\xi}^{(1)} + \begin{pmatrix} 0 \\ -1/5 \end{pmatrix} e^{-4t}. \quad (62)$$

Summary-Solved Problems

■ SP8.6 *Solution:*

- Comparing (53) and (62) we see that

$$\underline{\eta} = \begin{pmatrix} 0 \\ -1/5 \end{pmatrix}. \quad (63)$$

- Next, we write

$$\underline{x}(t) = c_1 \underline{x}^{(1)}(t) + c_2 \underline{x}^{(2)}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t} + c_2 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-4t} + \begin{pmatrix} 0 \\ -1/5 \end{pmatrix} e^{-4t} \right),$$

and solve for the coefficients via initial conditions. We must have

$$\underline{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1/5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Summary-Solved Problems

■ SP8.6 *Solution:*

- In scalar form, this is

$$\begin{aligned}c_1 &= 2, \\c_1 - \frac{c_2}{5} &= -1.\end{aligned}$$

- The solutions are

$$\begin{aligned}c_1 &= 2, \\c_2 &= 15.\end{aligned}$$

- So, the solution to Equations (44) is

$$\underline{x}(t) = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t} + 15 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-4t} + \begin{pmatrix} 0 \\ -1/5 \end{pmatrix} e^{-4t} \right),$$

which can be simplified to

$$\underline{x}(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-4t} + 15 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-4t}.$$

Summary-Solved Problems

- **SP8.7** Find the state transition matrix $\underline{\Phi}(t)$ for the system

$$\dot{\underline{x}} = \begin{pmatrix} -8 & 4 \\ -5 & -4 \end{pmatrix} \underline{x}. \quad (64)$$

- *Solution:*

- Step 1 is to find the two fundamental solutions. We do this by trying $\underline{x} = \underline{\xi}e^{rt}$, which leads to

$$(\underline{A} - r\underline{I})\underline{\xi} = \begin{pmatrix} -8 - r & 4 \\ -5 & -4 - r \end{pmatrix} \underline{\xi} = \underline{0}. \quad (65)$$

which in turn implies

$$\det(\underline{A} - r\underline{I}) = \det \begin{pmatrix} -8 - r & 4 \\ -5 & -4 - r \end{pmatrix} = 0,$$

Summary-Solved Problems

■ SP8.7 Solution:

$$(-8 - r)(-4 - r) - (4)(-5) = r^2 + 12r + 52 = 0.$$

■ Completing the square:

$$r^2 + 12r + 52 = (r + 6)^2 + 6^2 + 52 = (r + 6)^2 + 4^2 = 0.$$

■ Hence, the eigenvalues of the system are $r_{1,2} = -6 \pm 4i$. Substituting r_1 into Equation (65) results in

$$\begin{aligned} (r_1 \underline{I} - \underline{A}) \underline{\xi}^{(1)} &= \begin{pmatrix} -6 + 4i + 8 & -4 \\ 5 & -6 + 4i + 4 \end{pmatrix} \underline{\xi}^{(1)} \\ &= \begin{pmatrix} 2 + 4i & -4 \\ 5 & -2 + 4i \end{pmatrix} \underline{\xi}^{(1)} = \underline{0}. \end{aligned} \tag{66}$$

■ Setting $\underline{\xi}_1^{(1)} = 1$ in Equation (66) yields $\underline{\xi}_2^{(1)} = 1/2 + i$.

Summary-Solved Problems

■ SP8.7 Solution:

- Writing $\underline{\xi} = \underline{a} + i\underline{b}$ we determine that

$$\underline{a} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (67)$$

- From Section 8.5.1.2, we learned that if the first eigenvalue is $r_1 = -\lambda + i\omega$ and its associated eigenvector is $\underline{\xi}^{(1)} = \underline{a} + i\underline{b}$ then the two real fundamental solutions associated with r_1 and its complex conjugate are $\underline{x}^{(1)}(t) = e^{\lambda t} (\underline{a} \cos \omega t - \underline{b} \sin \omega t)$ and $\underline{x}^{(2)}(t) = e^{\lambda t} (\underline{a} \cos \omega t + \underline{b} \sin \omega t)$. Using that here:

$$\begin{aligned} \underline{x}^{(1)}(t) &= e^{-6t} \left(\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \cos 4t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 4t \right), \\ \underline{x}^{(2)}(t) &= e^{-6t} \left(\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \sin 4t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 4t \right). \end{aligned} \quad (68)$$

Summary-Solved Problems

■ SP8.7 Solution:

- Step 2 is to form the fundamental matrix $\underline{\Psi}(t)$, whose columns consist of the fundamental solutions:

$$\underline{\Psi}(t) = \begin{pmatrix} e^{-6t} \cos 4t & e^{-6t} \sin 4t \\ e^{-6t}((1/2) \cos 4t - \sin 4t) & e^{-6t}((1/2) \sin 4t + \cos 4t) \end{pmatrix}. \quad (69)$$

- Step 3 is to determine $\underline{\Psi}^{-1}(0)$, which is defined by $\underline{\Psi}(0)\underline{\Psi}^{-1}(0) = \underline{I}$. Letting

$$\underline{\Psi}^{-1}(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have

$$\begin{aligned} \underline{\Psi}(0)\underline{\Psi}^{-1}(0) &= \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 1 \cdot a + (0) \cdot c &= 1, & 1 \cdot b + (0) \cdot d &= 0 \\ (1/2) \cdot a + 1 \cdot c &= 1, & (1/2) \cdot b + 1 \cdot d &= 1 \end{aligned}$$

Summary-Solved Problems

■ SP8.7 Solution:

which has solution $a = 1$, $b = 0$, $c = -1/2$, $d = 1$, so that

$$\underline{\Psi}^{-1}(0) = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}. \quad (70)$$

■ Step 4 is to compute the state transition matrix $\underline{\Phi}(t)$ from

$$\underline{\Phi}(t) = \underline{\Psi}(t)\underline{\Psi}^{-1}(0). \quad (71)$$

■ We have

$$\underline{\Phi}(t) = \begin{pmatrix} e^{-6t} \cos 4t & e^{-6t} \sin 4t \\ e^{-6t}((1/2) \cos 4t - \sin 4t) & e^{-6t}((1/2) \sin 4t + \cos 4t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix},$$

$$\underline{\Phi}(t) = \begin{pmatrix} e^{-6t}(\cos 4t - (1/2) \sin 4t) & e^{-6t} \sin 4t \\ -e^{-6t}(5/4) \sin 4t & e^{-6t}(\cos 4t + (1/2) \sin 4t) \end{pmatrix}.$$

■ We can (partially) check our calculations by finding out if $\underline{\Phi}(0) = \underline{I}$. It does.

Summary-Solved Problems

- **SP8.8** Using matrix Laplace transform methods, solve

$$\dot{\underline{x}} = \begin{pmatrix} -5 & 4 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cos 2t, \quad (72)$$

$$\underline{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \quad (73)$$

- *Solution:*

- We take the Laplace transform of Equation (72) and call $L\{\underline{x}(t)\} = \underline{x}(s)$, Now

$$L\{\dot{\underline{x}}(t)\} = sL\{\underline{x}(t)\} - \underline{x}(0) = s\underline{x}(s) - \underline{x}(0),$$

$$L\{\cos 2t\} = \frac{s}{s^2 + 2^2}.$$

Summary-Solved Problems

■ SP8.8 Solution:

and there results, using Equations (72) and (73),

$$(s\underline{I} - \underline{A})\underline{x}(s) = \underline{x}(0) + \underline{b}u(s),$$
$$\begin{pmatrix} s+5 & 4 \\ -5 & s+1 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} \frac{s}{s^2+4}.$$

■ Calling $\underline{x}(s) = \underline{x}$ for brevity, we have the scalar equations

$$(s+5)x_1 + 4x_2 = 2 + \frac{s}{(s^2+4)},$$
$$-5x_1 + (s+1)x_2 = -1 + \frac{3s}{(s^2+4)}.$$

■ Multiplying the second equation by 4 and using the first to eliminate x_2 we have

$$4x_2 = -(s+5)x_1 + 2 + \frac{s}{(s^2+4)}, \quad (74)$$
$$-20x_1 + (s+1) \left[-(s+5)x_1 + 2 + \frac{s}{(s^2+4)} \right] = -4 + \frac{12s}{(s^2+4)}.$$

Summary-Solved Problems

■ SP8.8 Solution:

$$\begin{aligned} -((s+1)(s+5)+20)x_1 &= -2(s+1) - \frac{s(s+1)}{(s^2+4)} - 4 + \frac{12s}{(s^2+4)}, \\ x_1 &= \frac{2s+6}{s^2+6s+25} + \frac{s^2-11s}{(s^2+6s+25)(s^2+4)} = \frac{2s^3+7s^2-3s+24}{(s^2+6s+25)(s^2+4)}. \end{aligned} \quad (75)$$

■ Using a partial fraction expansion:

$$x_1 = \frac{c_1s + c_2}{s^2 + 6s + 25} + \frac{c_3s + c_4}{s^2 + 4}l, \quad (76)$$

$$x_1 = \frac{(c_1 + c_3)s^3 + (c_2 + 6c_3 + c_4)s^2 + (4c_1 + 25c_3 + 6c_4)s + (4c_2 + 25c_4))}{(s^2 + 6s + 25)(s^2 + 4)}. \quad (77)$$

Summary-Solved Problems

■ SP8.8 *Solution:*

- Equating the coefficients of like powers of s in the numerators of Equations (75) and (77), we have four simultaneous linear equations in four unknowns that can be written in matrix form as

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 6 & 1 \\ 4 & 0 & 25 & 6 \\ 0 & 4 & 0 & 25 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -3 \\ 24 \end{pmatrix}. \quad (78)$$

- Using an online calculator on the Internet we are given the result

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 2.3538 \\ 9.7179 \\ -0.3538 \\ -0.5949 \end{pmatrix}. \quad (79)$$

Summary-Solved Problems

■ SP8.8 *Solution:*

- We can quickly check (79) by substitution into Equation (78) and determine with a hand calculator to be correct.
- From Equations (76) and (80)

$$x_1 = \frac{2.3538s + 9.7179}{s^2 + 6s + 25} + \frac{(0.3538s + 0.5949)}{s^2 + 4}. \quad (80)$$

- We must manipulate Equation (80) to match the entries in our short table of Laplace transforms (Table 7.2):

$$\begin{aligned} x_1 &= \frac{2.3538(s+3) - 2.3538(3) + 9.7179}{(s+3)^2 + 4^2} - \frac{0.3538s}{s^2 + 2^2} - \frac{0.5949}{s^2 + 2^2}, \\ x_1 &= 2.3538 \frac{(s+3)}{(s+3)^2 + 4^2} + \left(\frac{2.6565}{4}\right) \frac{4}{(s+3)^2 + 4^2} - 0.3538 \frac{s}{s^2 + 2^2} - \left(\frac{0.5949}{2}\right) \frac{2}{s^2 + 2^2}, \\ x_1(s) &= 2.3538 \frac{(s+3)}{(s+3)^2 + 4^2} + 0.6641 \frac{4}{(s+3)^2 + 4^2} - 0.3538 \frac{s}{s^2 + 2^2} - 0.2974 \frac{2}{s^2 + 2^2}. \end{aligned}$$

Summary-Solved Problems

■ SP8.8 Solution:

$$x_1(s) = 2.3538 \frac{(s+3)}{(s+3)^2 + 4^2} + 0.6641 \frac{4}{(s+3)^2 + 4^2} - 0.3538 \frac{s}{s^2 + 2^2} - 0.2974 \frac{2}{s^2 + 2^2}. \quad (81)$$

■ From Equation (81) and Table 7.2,

$$\begin{aligned} x_1(t) &= L^{-1} \{x_1(s)\} \\ &= 2.3538 e^{-3t} \cos 4t + 0.6641 e^{-3t} \sin 4t - 0.3538 \cos 2t - 0.2974 \sin 2t \end{aligned} \quad (82)$$

■ From Equations (74) and (75),

$$4x_2(s) = -(s+5) \left[\frac{2s^3 + 7s^2 - 3s + 24}{(s^2 + 6s + 25)(s^2 + 4)} \right] + 2 + \frac{s}{s^2 + 4}. \quad (83)$$

Summary-Solved Problems

■ SP8.8 Solution:

■ After some algebra,

$$x_2(s) = \frac{-s^3 + 8s^2 + 16s + 20}{(s^2 + 6s + 25)(s^2 + 4)}. \quad (84)$$

■ Again using a partial fraction expansion,

$$x_2 = \frac{c_5 s + c_6}{s^2 + 6s + 25} + \frac{c_7 s + c_8}{s^2 + 4}, \quad (85)$$

$$x_2 = \frac{(c_5 + c_7)s^3 + (c_6 + 6c_7 + c_8)s^2 + (4c_5 + 25c_7 + 6c_8)s + (4c_6 + 25c_8)}{(s^2 + 6s + 25)(s^2 + 4)}. \quad (86)$$

Summary-Solved Problems

■ SP8.8 *Solution:*

- Again equating coefficients of like powers of s in the numerators in Equations (84) and (86), we arrive at

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 6 & 1 \\ 4 & 0 & 25 & 6 \\ 0 & 4 & 0 & 25 \end{pmatrix} \begin{pmatrix} c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ 16 \\ 20 \end{pmatrix}. \quad (87)$$

- Using an online calculator on the Internet we are given the result

$$\begin{pmatrix} c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix} = \begin{pmatrix} -1.8410 \\ 2.5641 \\ 0.8410 \\ 0.3897 \end{pmatrix}. \quad (88)$$

Summary-Solved Problems

■ SP8.8 Solution:

■ From Equations (85) and (88),

$$x_2(s) = \frac{-1.8410s + 2.5641}{s^2 + 6s + 25} + \frac{0.8410s + 0.3897}{s^2 + 4}. \quad (89)$$

■ Manipulating Equation (89) into a form matching entries in Table 7.2, we arrive at

$$x_2(s) = -1.8410 \frac{(s+3)}{(s+3)^2 + 4^2} + 2.0218 \frac{4}{(s+3)^2 + 4^2} + 0.8410 \frac{s}{s^2 + 2^2} + 0.1949 \frac{2}{s^2 + 2^2}.$$

and then

$$\begin{aligned} x_2(t) &= L^{-1} \{x_2(s)\} \\ &= -1.8410e^{-3t} \cos 4t + 2.0218e^{-3t} \sin 4t + 0.8410 \cos 2t + 0.1949 \sin 2t \end{aligned} \quad (90)$$

Summary-Solved Problems

■ SP8.8 *Solution:*

- In summary, from Equations (82) and (90), the solution to Equations (72) and (73) is

$$\begin{aligned} \underline{x}(t) = & \begin{pmatrix} 2.3538 \\ -1.8410 \end{pmatrix} e^{-3t} \cos 4t + \begin{pmatrix} 0.6641 \\ 2.0218 \end{pmatrix} e^{-3t} \sin 4t \\ & + \begin{pmatrix} -0.3538 \\ 0.8410 \end{pmatrix} \cos 2t + \begin{pmatrix} -0.2974 \\ 0.1949 \end{pmatrix} \sin 2t. \end{aligned} \quad (91)$$

Summary-Solved Problems

- **SP8.9** Find the steady-state solution to

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} t \sin 2t. \quad (92)$$

- *Solution:*

- **Step 0** is to ensure that the system is stable and damped, so that the systems fundamental solutions are not part of the steady-state response. The characteristic equation is

$$r^2 + r + 1 = 0.$$

- The roots of this have negative real parts, so the systems fundamental solutions die away, leaving only the solution we are about to compute.
- Equation (92) is of the form

$$\underline{\dot{x}} = \underline{A} \underline{x} + \underline{b} t \sin \omega t. \quad (93)$$

Summary-Solved Problems

■ SP8.9 *Solution:*

- We will develop a strategy for solution using this general form. Then, once we have a strategy, we will employ the given values and execute it.
- **Step 1:** Based on Table 8.5, we try the solution

$$\underline{x}(t) = (\underline{c}t + \underline{d}) \sin \omega t + (\underline{e}t + \underline{f}) \cos \omega t, \quad (94)$$

in Equation (93).

- Now

$$\dot{\underline{x}}(t) = \underline{c} \sin \omega t + \omega(\underline{c}t + \underline{d}) \cos \omega t + \underline{e} \cos \omega t + \omega(\underline{e}t + \underline{f}) \sin \omega t. \quad (95)$$

- Substituting Equations (94) and (95) into Equation (93) and combining terms we have

$$\begin{aligned} &(\omega \underline{e} + \underline{A} \underline{c} + \underline{b})t \sin \omega t + (\omega \underline{f} + \underline{A} \underline{d} - \underline{c}) \sin \omega t + \\ &(-\omega \underline{c} + \underline{A} \underline{e})t \cos \omega t + (-\omega \underline{d} + \underline{A} \underline{f} - \underline{e}) \cos \omega t = 0. \end{aligned} \quad (96)$$

Summary-Solved Problems

■ SP8.9 *Solution:*

- For Equation (96) to be true at all instants of time we must have

$$\begin{aligned}\omega \underline{e} + \underline{A} \underline{c} + \underline{b} &= 0 \\ \omega \underline{f} + \underline{A} \underline{d} - \underline{c} &= 0 \\ -\omega \underline{c} + \underline{A} \underline{e} &= 0 \\ -\omega \underline{d} + \underline{A} \underline{f} - \underline{e} &= 0\end{aligned}\tag{97}$$

- Since each line in Equations (97) is two scalar equations this represents eight equations in eight unknowns, which on the surface of it is a difficult computational challenge. But it is not as formidable as it seems.
- Our solution strategy begins with the first and third lines, from which we derive

$$(\omega^2 \underline{I} + \underline{A}^2) = -\omega \underline{b}\tag{98}$$

which we can solve for $\underline{e}$.

Summary-Solved Problems

■ SP8.9 *Solution:*

- Then from the third line we have

$$\underline{c} = \underline{Ae}/\omega \quad (99)$$

- Now $\underline{e}$ and $\underline{c}$ are known and we are left with the second and fourth lines, which we can write as:

$$\omega \underline{f} + \underline{A} \underline{d} = \underline{c}, \quad (100)$$

$$\underline{A} \underline{f} - \omega \underline{d} = \underline{e}. \quad (101)$$

- From Equation (100),

$$\omega \underline{f} = \underline{c} - \underline{A} \underline{d}, \quad (102)$$

- From Equation (101),

$$\underline{A}(\omega \underline{f}) = \omega \underline{e} + \omega^2 \underline{d}. \quad (103)$$

Summary-Solved Problems

■ SP8.9 *Solution:*

- Substituting Equation (102) into Equation (103) and rearranging we find

$$(\omega^2 \underline{I} + \underline{A}^2) = \underline{A}\underline{c} - \omega \underline{e}. \quad (104)$$

which we can solve for $\underline{d}$.

- Then, from Equation (102),

$$\underline{f} = (\underline{c} - \underline{A}\underline{d})/\omega \quad (105)$$

- Equations (98), (99), (104), and (105) represent a solution strategy, so we are now prepared to tackle the problem with its specific, given values.

Summary-Solved Problems

■ SP8.9 Solution:

■ **Step 2:** Here $\omega = 2$, $\underline{A} = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, so

$$\begin{aligned} \underline{A}^2 &= \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 1 + (-1) \cdot 3 & 1 \cdot (-1) + (-1) \cdot (-2) \\ 3 \cdot 1 + (-2) \cdot 3 & 3 \cdot (-1) + (-2) \cdot (-2) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}, \end{aligned}$$

$$\omega^2 \underline{I} + \underline{A}^2 = \begin{pmatrix} 4 - 2 & 1 \\ -3 & 4 + 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix}. \quad (106)$$

■ From Equation (98),

$$\begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}, \quad (107)$$

Summary-Solved Problems

- **SP8.9** *Solution:*
which has solution

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} -22/13 \\ -8/13 \end{pmatrix}. \quad (108)$$

- From Equation (99),

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -22/13 \\ -8/13 \end{pmatrix} = \begin{pmatrix} -7/13 \\ -25/13 \end{pmatrix} \quad (109)$$

- From Equations (104) and (106),

$$\begin{pmatrix} 2 & 1 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -7/13 \\ -25/13 \end{pmatrix} - 2 \begin{pmatrix} -22/13 \\ -8/13 \end{pmatrix} = \begin{pmatrix} 62/13 \\ 45/13 \end{pmatrix} \quad (110)$$

which has solution

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 265/169 \\ 276/169 \end{pmatrix}. \quad (111)$$

Summary-Solved Problems

■ SP8.9 Solution:

■ From Equation (102)

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} -7/13 \\ -25/13 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 265/169 \\ 276/169 \end{pmatrix} = \begin{pmatrix} -40/169 \\ -284/169 \end{pmatrix} \quad (112)$$

■ Therefore the steady-state solution to Equation (92) is

$$\underline{x}(t) = (\underline{c}t + \underline{d}) \sin \omega t + (\underline{e}t + \underline{f}) \cos \omega t,$$

where $\underline{c}$, $\underline{d}$, $\underline{e}$, $\underline{f}$ are given by Equations (108), (109), (111) and (112).

End