
EE 3121
Differential Equations for Electrical Engineering

Lecture 6: Higher-Order Linear Ordinary Differential Equations



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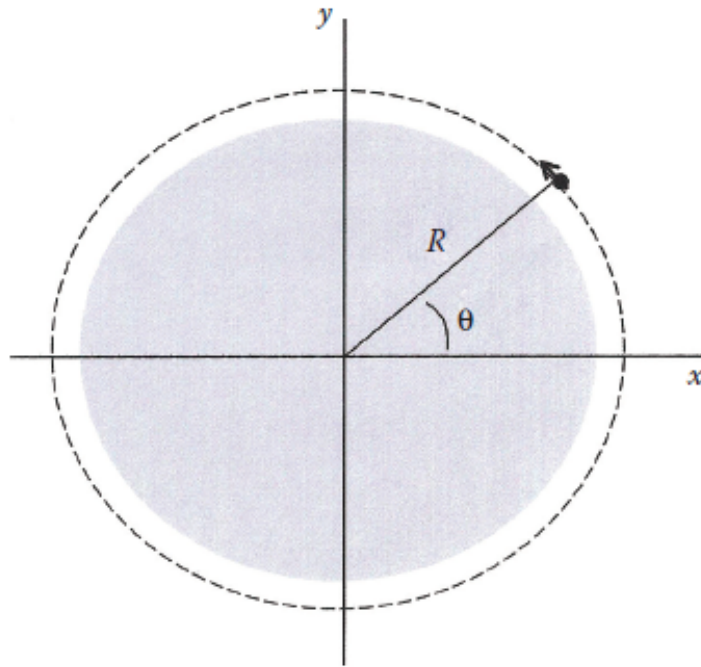
Framework of Lectures

- This chapter discusses linear ordinary differential equations (ODEs) of order higher than two.

$$\frac{d^n y}{dt^n} + a_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_{n-1}(t) \frac{dy}{dt} + a_n(t) y = g(t).$$

- We will find many similarities to second-order equations and so will summarize these results succinctly.

6.1 Example: Satellite Orbit Decay



■ Figure 6.1 Satellite in low Earth orbit.

6.1 Example: Satellite Orbit Decay

- The primary force on the satellite in [Figure 6.1](#) is Earth's gravity. However, although the atmosphere is very thin at these altitudes, the long-term effect of air resistance (drag) on the satellite cannot be ignored.
- The satellite's equations of motion are more easily written in Cartesian (x, y) coordinates and more easily solved in polar (R, θ) coordinates. From Newton's second law of motion $F = ma$, the equations of motion in Cartesian coordinates are:

6.1 Example: Satellite Orbit Decay

$$\begin{aligned}\ddot{x} &= -\frac{GM}{(x^2+y^2)^{3/2}}x - \frac{\rho SC_D}{2m}(\dot{x}^2 + \dot{y}^2)^{1/2}\dot{x} \\ \ddot{y} &= -\frac{GM}{(x^2+y^2)^{3/2}}y - \frac{\rho SC_D}{2m}(\dot{x}^2 + \dot{y}^2)^{1/2}\dot{y},\end{aligned}\tag{1}$$

where:

G is the gravitational constant

M is the mass of the earth

ρ is the air density

S is the satellites cross-sectional area

C_D is the satellites drag coefficient

m is the satellites mass

■ Polar coordinates:

$$x = R \cos \theta, \quad y = R \sin \theta.$$

6.1 Example: Satellite Orbit Decay

Transforming equation (1) into their polar form:

$$\begin{aligned}\ddot{R} &= -\frac{GM}{R^2} + R\dot{\theta}^2 - \frac{\rho SC_D \dot{R}}{2m} \sqrt{(R\dot{\theta})^2 + \dot{R}^2} \\ R\ddot{\theta} &= -2\dot{R}\dot{\theta} - \frac{\rho SC_D R\dot{\theta}}{2m} \sqrt{(R\dot{\theta})^2 + \dot{R}^2}.\end{aligned}\tag{2}$$

We assume that the air density ρ is small enough that the effects of drag can be considered a perturbation about a nominally circular zero-drag trajectory. Setting $\rho = 0$ in equation (2)

$$\begin{aligned}0 &= -\frac{GM}{R_0^2} + R_0\dot{\theta}_0^2 \\ 0 &= -2\dot{R}_0\dot{\theta}_0.\end{aligned}\tag{3}$$

6.1 Example: Satellite Orbit Decay

Then,

$$\dot{\theta}_0^2 = \frac{GM}{R_0^3} \quad (4)$$

$$\dot{R}_0 = 0.$$

For simplicity let $\dot{\theta} = \omega$. We consider small motions about the nominal trajectory:

$$\omega = \omega_0 + v \quad (5)$$

$$R = R_0 + r.$$

6.1 Example: Satellite Orbit Decay

Measurement and theory show that it varies roughly exponentially about a given altitude:

$$\rho(R) = \rho(R_0) \exp(-Z(R - R_0)) \quad (6)$$

Determining typical values for the key quantities

$$\rho_0 = \rho(R_0); \quad (7)$$

$$Z = -\frac{1}{\rho_0} \left(\frac{d\rho}{dR} \right)_0; \quad (8)$$

$$k = \frac{\rho_0 S C_D}{2m}. \quad (9)$$

6.1 Example: Satellite Orbit Decay

- Several steps to move quickly to the bottom line: (Details in CP6.2, 6.3 and 6.4)
 - We linearize equations (2), yielding two linear time-invariant equations in the variables r and v . (Recall equations (5).)
 - We use these two equations to eliminate v and obtain a third-order ODE for r , the variation in altitude, alone.
 - To transform the ODE into a homogeneous one, we define a new dependent variable $r_* = r - 1/Z$ where Z is as given in equation (8). For notational simplicity we next define a new independent variable $t_* = \omega_0 t$.

6.1 Example: Satellite Orbit Decay

The bottom line is

$$\ddot{r}_* + \dot{r}_* - \gamma r_* = 0, \quad (10)$$

where

$$\gamma = 2kZR_0^2, \quad (11)$$

The derivatives are with respect to the nondimensional time $t_* = \omega_0 t$.
Initial conditions are

$$r_*(0) = -1/Z$$

$$\dot{r}_*(0) = 0 \quad (12)$$

$$\ddot{r}_*(0) = 0.$$

6.1 Example: Satellite Orbit Decay

To solve equation (10), we try $r_*(t_*) = \exp(pt_*)$, then

$$p^3 + p - \gamma = 0, \quad (13)$$

because $\gamma \ll 1$, an approximate factorization of the cubic polynomial in equation (13) is

$$(p - \gamma)(p^2 + \gamma p + 1) = p^3 + (1 - \gamma^2)p - \gamma \cong p^3 + p - \gamma. \quad (14)$$

Hence the roots of the characteristic equation are approximately

$$p_1 = \gamma \quad p_2 = -\frac{\gamma}{2} + i \quad p_3 = -\frac{\gamma}{2} - i, \quad (15)$$

6.1 Example: Satellite Orbit Decay

The corresponding fundamental solutions are

$$\begin{aligned}r_{*1}(t_*) &= \exp(\gamma t_*) \\r_{*2}(t_*) &= \exp(-\gamma t_*/2) \cos t_* \\r_{*3}(t_*) &= \exp(-\gamma t_*/2) \sin t_*.\end{aligned}\tag{16}$$

Expressing

$$r_* = c_1 r_{*1} + c_2 r_{*2} + c_3 r_{*3}.\tag{17}$$

Selecting the coefficients c_1 , c_2 and c_3 to satisfy Equation (12):

$$r_*(t_*) = -\frac{1}{Z} \left(\exp(\gamma t_*) - \gamma \exp(-\gamma t_*/2) \sin t_* + 2\gamma^2 \exp(-\gamma t_*/2) \cos t_* \right)\tag{18}$$

6.1 Example: Satellite Orbit Decay

Returning to dimensional time, the satellites deviation in altitude from a circular orbit is approximated by

$$\begin{aligned} r(t) = & -\frac{1}{Z} (\exp(\gamma\omega_0 t) - 1) - \gamma \exp(-\gamma\omega_0 t/2) \sin \omega_0 t \\ & + 2\gamma^2 \exp(-\gamma\omega_0 t) \cos \omega_0 t, \end{aligned} \quad (19)$$

where

$$\begin{aligned} Z &= -\frac{1}{\rho_0} \left(\frac{d\rho}{dR} \right)_0 \\ \gamma &= 2kZ R_0^2 \end{aligned} \quad (20)$$

$$\omega_0 = \sqrt{GM/R^3}$$

$$k = \frac{\rho_0 S C_D}{2m}$$

6.1 Example: Satellite Orbit Decay

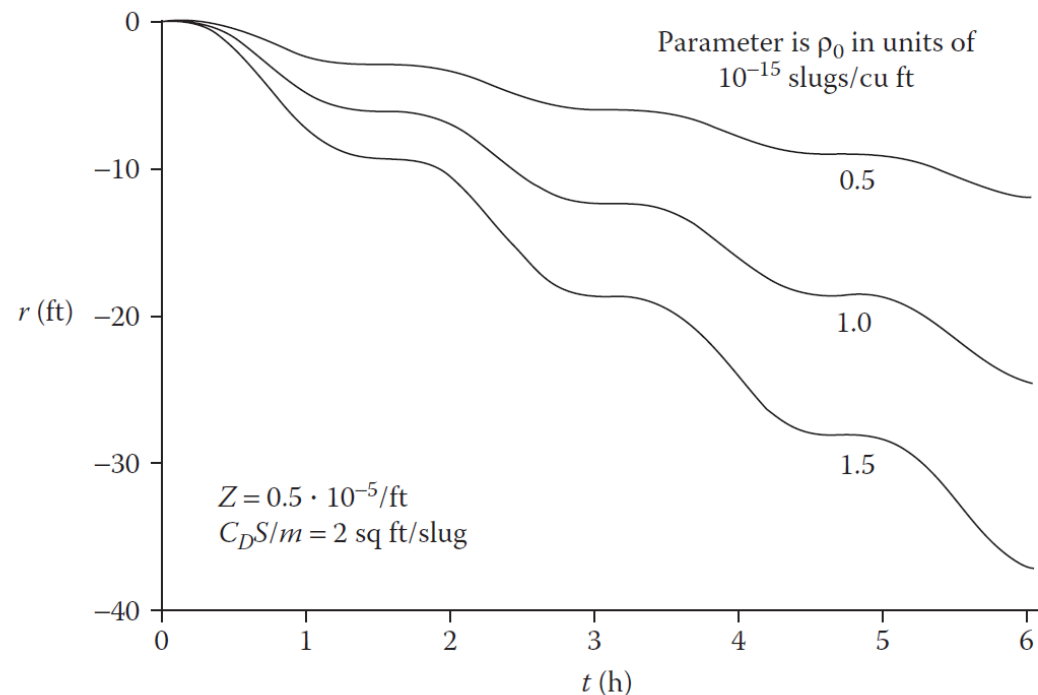
In short order, $\sin \omega_0 t$ and $\cos \omega_0 t$ satisfying initial conditions are transient and become negligible. The satellites deviation becomes:

$$r(t) = -\frac{1}{Z} (\exp(\gamma \omega_0 t) - 1) \quad (21)$$

Equation (21) shows that a low earth orbit is unstable.

Figure 6.2 considers the case of a typical satellite launched into an initial orbit of 300 statute miles above the Earth's surface.

6.1 Example: Satellite Orbit Decay



- Figure 6.2 Deviation in altitude from a circular orbit for a typical small satellite launched into an initial 300 statute mile orbit, with atmospheric density at the initial altitude as a parameter.

6.2 Higher-order Homogeneous Linear Equations

Consider the general n th order *homogeneous* linear ODE:

$$\frac{d^n y}{dt^n} + a_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n(t) y = 0 \quad (22)$$

If the functions $y_1(t), y_2(t), \dots, y_{n-1}(t), y_n(t)$ are solutions to equation (22) and their Wronskian

$$W(t) = \begin{vmatrix} y_1 & y_2 & \dots & y_{n-1} & y_n \\ \dot{y}_1 & \dot{y}_2 & \dots & \dot{y}_{n-1} & \dot{y}_n \\ \dots & \dots & \dots & \dots & \dots \\ y_1^{(n-2)} & y_2^{(n-2)} & \dots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_{n-1}^{(n-1)} & y_n^{(n-1)} \end{vmatrix} \quad (23)$$

6.2 Higher-order Homogeneous Linear Equations

is nonzero, then any solution to equation (22) has the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \dots + c_{n-1} y_{n-1}(t) + c_n y_n(t) \quad (24)$$

Consider the general n th order linear *time-invariant homogeneous* ODE

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \dots + a_n y = 0 \quad (25)$$

Trying the solution $y(t) = e^{rt}$ we derive the characteristic equation for equation (25):

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \dots + a_n = 0 \quad (26)$$

6.2 Higher-order Homogeneous Linear Equations

Roots to equation (26) can have four different patterns:

- Distinct real roots (appearing only once);
- Repeated real roots;
- Distinct conjugate pairs;
- Repeated conjugate pairs.

6.2 Higher-order Homogeneous Linear Equations

- If a real root r_1 appears m times, the solution

$$y(t) = (c_1 + c_2 t + \dots + c_m t^{m-1}) e^{r_1 t}$$

- If the conjugate pair $r = \lambda \pm i\omega$ appears only once

$$c_1 e^{\lambda t} \cos \omega t + c_2 e^{\lambda t} \sin \omega t$$

- If the conjugate pair $r = \lambda \pm i\omega$ appears m times

$$y(t) = (c_1 + c_2 t + \dots + c_m t^{m-1}) e^{\lambda t} \cos \omega t \\ + (c_{m+1} + c_{m+2} t + \dots + c_{2m} t^{m-1}) e^{\lambda t} \sin \omega t$$

6.2 Higher-order Homogeneous Linear Equations

For n linear equations in n unknowns, it has a unique solution because the determinant of the equations (the Wronskian) is nonzero:

$$c_1 y_1(t_0) + c_2 y_2(t_0) + \dots + c_n y_n(t_0) = y_0$$

$$c_1 \dot{y}_1(t_0) + c_2 \dot{y}_2(t_0) + \dots + c_n \dot{y}_n(t_0) = \dot{y}_0$$

$$c_1 \ddot{y}_1(t_0) + c_2 \ddot{y}_2(t_0) + \dots + c_n \ddot{y}_n(t_0) = \ddot{y}_0 \tag{27}$$

...

$$c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \dots + c_n y_n^{(n-1)}(t_0) = y_0^{(n-1)}$$

6.3 Higher-order Nonhomogeneous Linear Equations

- Three methods to solve nonhomogeneous higher-order linear time-invariant problems.
 - Variation of parameters (kernel) method;
 - Undetermined coefficients method;
 - The next lecture presents the third.

6.3.1 Variation of Parameters (Kernel) Method

Consider the general higher-order nonhomogeneous linear ODE ($g(t) \neq 0$):

$$\frac{d^n y}{dt^n} + a_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_{n-1}(t) \frac{dy}{dt} + a_n(t) y = g(t). \quad (28)$$

with initial conditions stated at $t = t_0$ as in Equation (27). The general solution to Equation (28) can be written as

$$y(t) = c_1 y_1 + c_2 y_2 + \dots + c_{n-1} y_{n-1} + c_n y_n + y_p$$

6.3.1 Variation of Parameters (Kernel) Method

where:

y_i are fundamental solutions to the homogeneous ODE equation (25)

c_i satisfy Equation (27)

y_p is a solution to Equation (28) with zero initial conditions

$y_p(t)$ can be written as

$$y_p(t) = \int_{t_0}^t K(t, \tau) g(\tau) d\tau \quad (29)$$

$$K(t, \tau) = \sum_{i=1}^n y_i(t) W_i(\tau) / W(\tau) \quad (30)$$

6.3.1 Variation of Parameters(Kernel) Method

$W(\tau)$ is the Wronskian and $W_i(\tau)$ is the Wronskian with i th column replaced with

$$\begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}$$

For linear time-invariant ODEs equation (29) has the form

$$y_p(t) = \int_{t_0}^t K(t - \tau)g(\tau)d\tau$$

6.3.1 Variation of Parameters(Kernel) Method

For the third-order time-invariant equation for satellite orbit decay (Equation (10)) and its three fundamental solutions

$$y_1(t) = \exp(\gamma\omega_0 t)$$

$$y_2(t) = \exp(-\gamma\omega_0 t/2) \cos \omega_0 t$$

$$y_3(t) = \exp(-\gamma\omega_0 t/2) \sin \omega_0 t,$$

the kernel is($\gamma \ll 1$)

$$K(t - \tau) = (y_1(t - \tau) - y_2(t - \tau) - (3\gamma/2)y_3(t - \tau))/\omega_0^2 \quad (31)$$

6.3.2 Undetermined Coefficients Method

- The undetermined coefficients method is a comparatively simple approach to try to solve higher-order equations.
- [Table 5.1](#) applies for higher-order equations as well as for second order.
- The limitations of the method expressed in Section 5.3.3.6 also apply to higher-order ODEs.

6.3.2 Undetermined Coefficients Method

TABLE 5.1

Trial Solutions for Given Input Functions Using the Undetermined Coefficients Method

Input	Trial Solution
e^{-kt}	Ce^{-kt}
t	$Ct + D$
te^{-kt}	$(Ct + D)e^{-kt}$
t^2	$Ct^2 + Dt + E$
$\sin \omega t$	$C \sin \omega t + D \cos \omega t$
$e^{-kt} \sin \omega t$	$(C \sin \omega t + D \cos \omega t)e^{-kt}$
$t \sin \omega t$	$(Ct + D) \sin \omega t + (Et + F) \cos \omega t$
$te^{-kt} \sin \omega t$	$((Ct + D) \sin \omega t + (Et + F) \cos \omega t)e^{-kt}$
$t^2 e^{-kt} \sin \omega t$	$((Ct^2 + Dt + E) \sin \omega t + (Ft^2 + Gt + H) \cos \omega t)e^{-kt}$

6.3.2 Undetermined Coefficients Method

Example: Find the steady-state solution to the general fourth-order linear time-invariant ODE with a sine-wave input:

$$\frac{d^4 y}{dt^4} + a_1 \frac{d^3 y}{dt^3} + a_2 \frac{d^2 y}{dt^2} + a_3 \frac{dy}{dt} + a_4 y = a_5 \sin \omega t \quad (32)$$

Following [Table 5.1](#), we try $y(t) = C \sin \omega t + D \cos \omega t$. Substituting directly and rearranging, we find

$$\begin{aligned} & \left[- (a_1 \omega^3 - a_3 \omega) C + (\omega^4 - a_2 \omega^2 + a_4) D \right] \cos \omega t \\ & + \left[(\omega^4 - a_2 \omega^2 + a_4) C + (a_1 \omega^3 - a_3 \omega) D \right] \sin \omega t = a_5 \sin \omega t \end{aligned}$$

6.3.2 Undetermined Coefficients Method

We must have

$$\begin{aligned} - (a_1\omega^3 - a_3\omega) C + (\omega^4 - a_2\omega^2 + a_4) D &= 0 \\ (\omega^4 - a_2\omega^2 + a_4) C + (a_1\omega^3 - a_3\omega) D &= a_5 \end{aligned}$$

This is two linear equations in two unknowns. The solutions are

$$C = \left(\frac{\omega^4 - a_2\omega^2 + a_4}{\Delta} \right) a_5$$

$$D = \left(\frac{a_1\omega^3 - a_3\omega}{\Delta} \right) a_5$$

$$\Delta = (\omega^4 - a_2\omega^2 + a_4)^2 + (a_1\omega^3 - a_3\omega)^2$$

6.3.2 Undetermined Coefficients Method

As we did for solutions to first- and second-order equations with a sine-wave input, we seek a steady-state solution to equation (32) in the form

$$y(t) = G \sin(\omega t - \theta) \quad (33)$$

Defining

$$\cos \theta = \frac{C}{\sqrt{C^2 + D^2}} \quad \sin \theta = \frac{-D}{\sqrt{C^2 + D^2}}$$

we determine that

$$G = \frac{a_5}{\sqrt{(\omega^4 - a_2\omega^2 + a_4)^2 + (a_1\omega^3 - a_3\omega)^2}} \quad (34)$$

$$\theta = \arctan(-a_1\omega^3 + a_3\omega, \omega^4 - a_2\omega^2 + a_4) \quad (35)$$

6.4 Existence and Uniqueness

Theorem: Let the functions $a_1(t), a_2(t), \dots, a_n(t), g(t)$ be continuous on the open interval I containing t_0 and let initial conditions be defined as in Equation (27). Then the ODE

$$\frac{d^n y}{dt^n} + a_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_{n-1}(t) \frac{dy}{dt} + a_n(t)y = g(t), \quad (36)$$

has exactly one solution and that solution exists throughout I .

Corollary: If the functions $a_1(t), a_2(t), \dots, a_n(t), g(t)$ are continuous everywhere then equation (36) has a unique solution at every instant of time. In particular, linear time-invariant equations have unique solutions at every instant of time.

6.5 Stability of Higher-order Systems

- For higher-order linear time-invariant systems, if any coefficient is negative then the system is unstable. However, a higher-order system may have all positive coefficients in the ODE but its solution is unstable.

Example:

$$(r + a)(r^2 - r + b) = r^3 + (a - 1)r^2 + (b - a)r + ab. \quad (37)$$

All coefficients in the cubic are positive if $b > a > 1$. But the quadratic factor has a negative coefficient, hence the system is unstable.

6.5 Stability of Higher-order Systems

- A linear time-invariant system is stable and damped if all the roots of its characteristic polynomial have negative real parts.
- *Rouths criteria* provide a means of determining whether a system is stable without solving for the roots, but with web-based root calculators available at our fingertips it is typically more practical to determine the roots.

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