

MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 3 **Polynomials and Rational Functions**

Polynomials

A **polynomial function** of **degree n** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the a_i 's are real numbers with $a_n \neq 0$ and n is a non-negative integer. The constants a_i 's are called **coefficients** of the corresponding x^i terms.

Two commonly used polynomials include:

- $f(x) = ax + b$ ($a \neq 0$) is called a **linear** function (i.e. polynomial of degree 1).
- $f(x) = ax^2 + bx + c$ ($a \neq 0$) is called a **quadratic** function (i.e. polynomial of degree 2)

E.g. The constant function $f(x) = k$, where $k \in \mathbb{R}$, is a polynomial of degree 0.

E.g. $3x^5 + \frac{2}{3}x^3 - \sqrt{6}x^2 + 2$ is a polynomial of degree 5.

E.g. $x^{\frac{1}{2}}$, x^{-1} and $x^{\cos x}$ are not polynomials.

Note: The **largest possible domain** of any polynomial is $\mathbb{R}$.

Quadratic Functions

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$ is the coefficient of x^2 in the quadratic function.

By completing the square, we can rewrite the quadratic function as

$$\begin{aligned} f(x) = ax^2 + bx + c &= a \left(x^2 + \frac{b}{a}x \right) + c = a \left[\left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right] + c \\ &= a \left[x - \underbrace{\left(-\frac{b}{2a} \right)}_{=h} \right]^2 + \underbrace{\left(c - \frac{b^2}{4a} \right)}_{=k} \end{aligned}$$

Putting $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$, the quadratic function $f(x)$ can be written as

$$f(x) = a(x \ominus h)^2 \oplus k,$$

which is called the standard form of a quadratic function.

$y = a(x-h)^2 + k$
 vertex at (h, k)
 $a > 0$ or $a < 0$

Let $y = a(x - h)^2 + k$. Rearranging $y = a(x - h)^2 + k$ gives

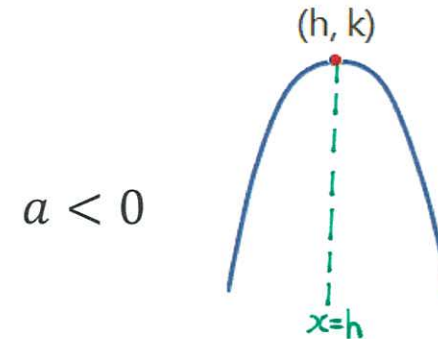
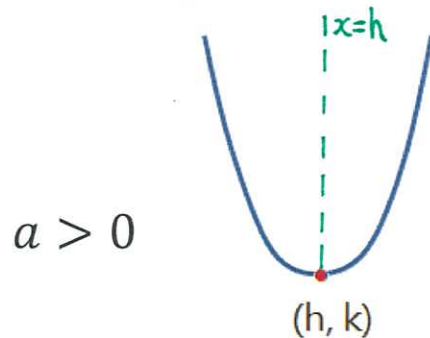
$$\begin{aligned} y = a(x - h)^2 + k &\Rightarrow y - k = a(x - h)^2 \Rightarrow (x - h)^2 = \frac{1}{a}(y - k) \\ &\Rightarrow (x - h)^2 = 4\left(\frac{1}{4a}\right)(y - k). \end{aligned}$$

Putting $p = \frac{1}{4a}$, a quadratic function can be written in the form $(x - h)^2 = 4p(y - k)$, which represents a **parabola** (see Chapter 1).

Properties of the graph of a quadratic function $f(x) = a(x - h)^2 + k$:

- If $a > 0$, the parabola **opens upward** (U-shaped).
If $a < 0$, the parabola **opens downward** (∩-shaped).
- The **vertex** of the parabola is at (h, k) .
 - If **$a > 0$** , the **minimum** value of $f(x)$ is $f(x) = k$ $\left(= c - \frac{b^2}{4a}\right)$ and it is attained at $x = h$ $\left(= -\frac{b}{2a}\right)$. The largest possible range of $f(x)$ is $[k, \infty)$.

- If $a < 0$, the **maximum** value of $f(x)$ is $f(x) = k \left(= c - \frac{b^2}{4a} \right)$ and it is attained at $x = h \left(= -\frac{b}{2a} \right)$. The largest possible range of $f(x)$ is $(-\infty, k]$.



- The parabola is symmetric about the vertical line $x = h$, which is the **axis of symmetry** of this parabola.
- Intersection of parabola with the y -axis (i.e. the y -intercept):**

Consider the parabola $y = ax^2 + bx + c$.

To find where its graph cuts the y -axis (i.e. the vertical line $x = 0$), we put $x = 0$ into $y = ax^2 + bx + c$ to get $y = c$.

$\therefore$ The graph of $y = ax^2 + bx + c$ cuts the y -axis at **$(0, c)$** .

- **Intersection of parabola with the x -axis (i.e. the x -intercept):**

Consider the parabola $y = ax^2 + bx + c = a \left[x - \left(-\frac{b}{2a} \right) \right]^2 + \left(c - \frac{b^2}{4a} \right)$, where $a \neq 0$. To find where its graph cuts the x -axis (i.e. the horizontal line $y = 0$), we put $y = 0$ into

$y = a \left[x - \left(-\frac{b}{2a} \right) \right]^2 + \left(c - \frac{b^2}{4a} \right)$ and solve the equation

$$a \left[x - \left(-\frac{b}{2a} \right) \right]^2 + \left(c - \frac{b^2}{4a} \right) = 0 \Rightarrow \left[x - \left(-\frac{b}{2a} \right) \right]^2 = \frac{1}{a} \left(\frac{b^2}{4a} - c \right)$$

$$\Rightarrow \left[x - \left(-\frac{b}{2a} \right) \right]^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x - \left(-\frac{b}{2a} \right) = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\Rightarrow x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

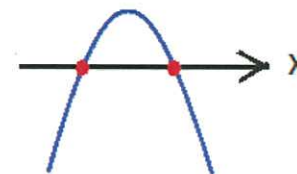
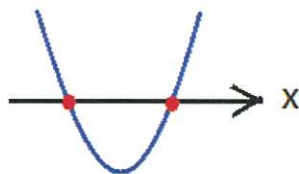
$$\Rightarrow \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This is called the **quadratic equation formula**.

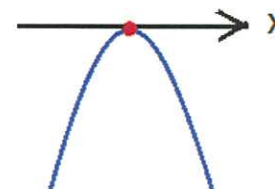
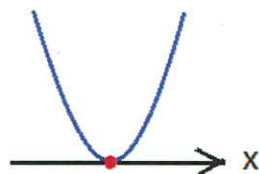
The quantity $\Delta = b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

Depending on the sign of the discriminant, there are three possibilities:

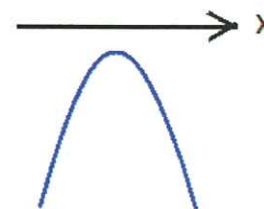
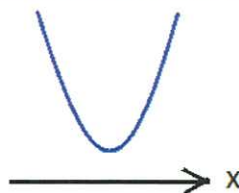
- If $b^2 - 4ac > 0$, the quadratic equation $ax^2 + bx + c = 0$ has two distinct real solutions (or roots), i.e. the graph of $y = ax^2 + bx + c$ **cuts (or intersects) the x -axis at two distinct points**.



- If $b^2 - 4ac = 0$, the quadratic equation $ax^2 + bx + c = 0$ has one real solution (or root), i.e. the graph of $y = ax^2 + bx + c$ **touches the x -axis at one point**.



- If $b^2 - 4ac < 0$, the quadratic equation $ax^2 + bx + c = 0$ has no real solution, i.e. the graph of $y = ax^2 + bx + c$ **does not cut the x -axis**.



Example 1

Given the function $f(x) = 3x^2 - 12x + 9$.

- (a) Express it in the standard form of quadratic function.
- (b) Find the coordinates of the vertex and determine whether this is the maximum or minimum point of the function.
- (c) Find the points where the graph of the function intersects the y -axis and the x -axis.
- (d) Sketch its graph.
- (e) Determine the largest possible domain and largest possible range of $f(x)$.

Solution

(a) $f(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x) + 9 = 3[(x - 2)^2 - 2^2] + 9 = 3(x - \mathbf{2})^2 - \mathbf{3}$.

(b) The vertex is at $(\mathbf{2}, \mathbf{-3})$. Since the coefficient of x^2 is 3 (> 0), the parabola opens upward. Therefore, the vertex $(2, -3)$ is the minimum point of the parabola.

(c) $f(0) = 3(0)^2 - 12(0) + 9 = 9$

$\therefore$ The parabola intersects the y -axis at $(0, 9)$.

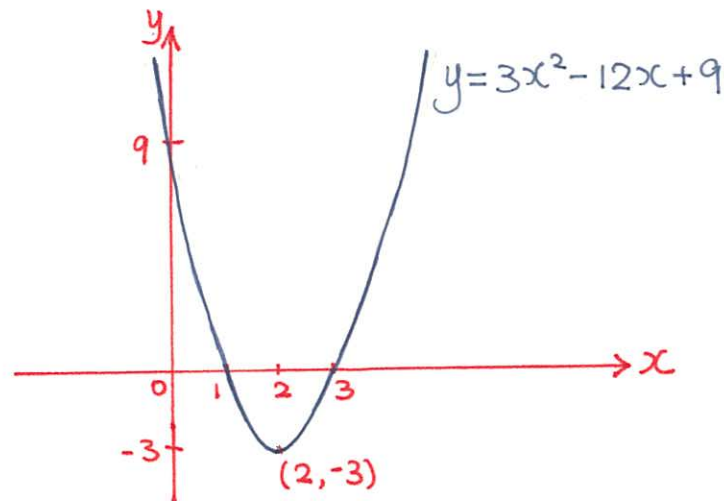
To find the points (if any) where the parabola intersects the x -axis, we solve the equation

$\underbrace{3}_{=a} x^2 - \underbrace{12}_{=b} x + \underbrace{9}_{=c} = 0$. By the quadratic equation formula, we obtain

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(9)}}{2(3)} = \frac{12 \pm \sqrt{36}}{6} = \frac{12 \pm 6}{6} = 1, 3$$

$\therefore$ The parabola intersects the x -axis at $(1, 0)$ and $(3, 0)$.

(d)



(e) The largest possible domain of $f(x)$ is $\text{Dom}(f) = \mathbb{R}$.

The largest possible range of $f(x)$ is $\text{Ran}(f) = [-3, \infty)$.

Example 2

Given the function $g(x) = -2x^2 - 4x - 4$.

- (a) Find the coordinates of the vertex.
- (b) Find the points (if any) where the parabola cuts the y -axis and the x -axis.
- (c) Sketch its graph.
- (d) Determine the largest possible domain and largest possible range of $g(x)$.

Solution

$$\begin{aligned} \text{(a)} \quad g(x) &= -2x^2 - 4x - 4 = -2(x^2 + 2x) - 4 = -2[(x + 1)^2 - 1^2] - 4 \\ &= -2(x + 1)^2 - 2 = -2[x - (-1)]^2 - 2 \end{aligned}$$

$\therefore$ The vertex is at $(-1, -2)$.

- (b) Since $g(0) = -4$, the parabola cuts the y -axis at $(0, -4)$.

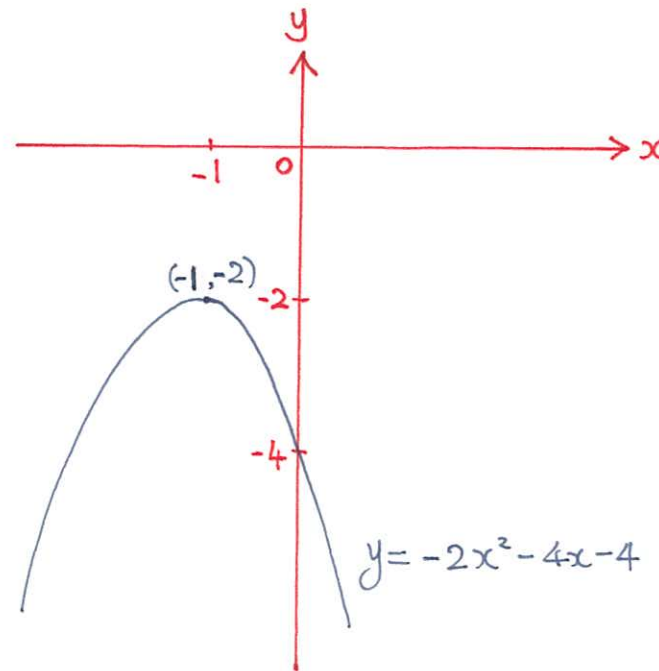
By the quadratic equation formula,

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(-2)(-4)}}{2(-2)} = \frac{4 \pm \sqrt{-16}}{-4}$$

which has no real solution. Hence the parabola does not cut the x -axis.

(c) Since the coefficient of x^2 is -2 (< 0), the parabola opens downward.

The function attains its maximum at $x = -1$ and maximum value of the function is $g(-1) = -2$.



(d) $\text{Dom}(g) = \mathbb{R}$,

$\text{Ran}(g) = (-\infty, -2]$.

Example 3 (Already discussed in Ch.2)

Recall that in **Chapter 2 Example 9**, we have determined the largest possible domains for the following functions:

(a) $f(x) = \sqrt{x^2 - 3x + 2}$

(b) $f(x) = \sqrt{3 + 2x - x^2}$

(c) $f(x) = \frac{9}{x^2 + 4x - 5}$

(d) $f(x) = \sqrt{\frac{x+1}{x+2}}$

Now we find the largest possible range for each of the above functions.

Solution

(a) $f(x) = \sqrt{x^2 - 3x + 2}$

Recall from **Example 9 of Chapter 2** that the function $f(x)$ is well-defined when $x^2 - 3x + 2 \geq 0$, i.e. when $x \leq 1$ or $x \geq 2$.

Thus, $\text{Dom}(f) = (-\infty, 1] \cup [2, \infty)$.

To find the range of $f(x)$, since it is difficult to sketch the graph of $f(x)$, we would consider the quadratic function inside the square root first.

Consider $x^2 - 3x + 2$:

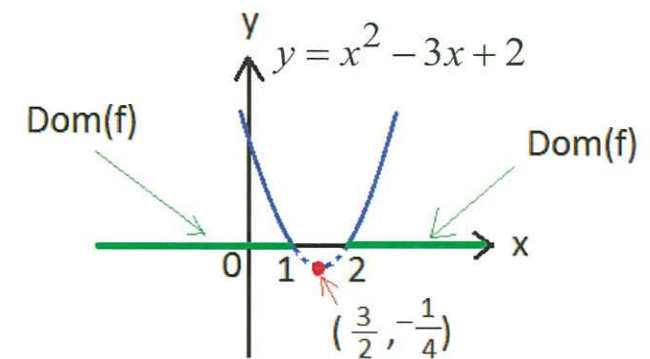
$$x^2 - 3x + 2 = (x - 2)(x - 1).$$

$\therefore$ The parabola passes through the x -axis at $x = 1$ and $x = 2$.

Coefficient of x^2 is $1 > 0$

$\therefore$ The parabola opens upward.

A sketch of the graph of $y = x^2 - 3x + 2$ is shown on the right:



For any $x \in \text{Dom}(f) = (-\infty, 1] \cup [2, \infty)$, we observe that $x^2 - 3x + 2 \geq 0 \Rightarrow \underbrace{\sqrt{x^2 - 3x + 2}}_{=f(x)} \geq 0$.

Hence, the largest possible range of $f(x)$ is $\text{Ran}(f) = [0, \infty)$.

(b) $f(x) = \sqrt{3 + 2x - x^2}$

Recall from **Example 9 of Chapter 2** that the function $f(x)$ is well-defined when $3 + 2x - x^2 \geq 0$, i.e. when $-1 \leq x \leq 3$. Thus, $\text{Dom}(f) = [-1, 3]$.

Consider $3 + 2x - x^2$:

$$3 + 2x - x^2 = (3 - x)(1 + x).$$

$\therefore$ The parabola passes through the x -axis at $x = -1$ and $x = 3$.

Coefficient of x^2 is $-1 < 0$

$\therefore$ The parabola opens downward.

By completing the square,

$$3 + 2x - x^2 = -(x^2 - 2x) + 3 = -[(x - 1)^2 - 1^2] + 3 = 4 - (x - 1)^2.$$

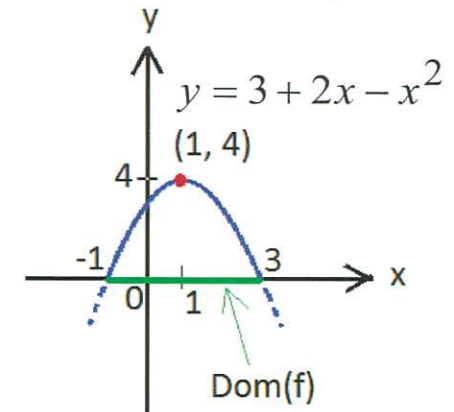
$\therefore$ The vertex is at $(1, 4)$.

A sketch of the graph of $y = 3 + 2x - x^2$ is shown on the right.

For any $x \in \text{Dom}(f) = [-1, 3]$, observe that

$$0 \leq 3 + 2x - x^2 \leq 4 \Rightarrow 0 \leq \underbrace{\sqrt{3 + 2x - x^2}}_{=f(x)} \leq \sqrt{4} = 2.$$

$\therefore$ The largest possible range of $f(x)$ is $\text{Ran}(f) = [0, 2]$.



(c) $f(x) = \frac{9}{x^2 + 4x - 5}$

Recall from **Example 9 of Chapter 2** that the function $f(x)$ is well-defined when $x^2 + 4x - 5 \neq 0$, i.e. when $x \neq -5, 1$. Thus, $\text{Dom}(f) = \mathbb{R} \setminus \{-5, 1\}$.

To find the range, we let $y = \frac{9}{x^2 + 4x - 5}$.

First note that $y \neq 0$.

(If $y = 0$, then $0 = \frac{9}{x^2 + 4x - 5} \Rightarrow 0 = 9$ which is impossible.)

$$y = f(x) \Rightarrow \dots \Rightarrow x = g(y)$$

$$\text{Then } y = \frac{9}{x^2+4x-5} \Rightarrow (x^2 + 4x - 5)y = 9$$

$$\Rightarrow \underbrace{y}_{=a} x^2 + \underbrace{(4y)}_{=b} x + \underbrace{(-5y-9)}_{=c} = 0 \dots (*)$$

$$ax^2+bx+c=0$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

which is a quadratic equation provided $y \neq 0$.

(*) has real roots if and only if the discriminant $b^2 - 4ac \geq 0$, that is,

$$(4y)^2 - 4y(-5y - 9) \geq 0 \Rightarrow 16y^2 + 20y^2 + 36y \geq 0 \Rightarrow 36(y^2 + y) \geq 0$$

$$\Rightarrow y(y+1) \geq 0$$

	$y < -1$	$y = -1$	$-1 < y < 0$	$y = 0$	$y > 0$
Sign of y	-	-	-		+
Sign of $y + 1$	-	0	+		+
Sign of $y(y+1)$	+	0	-		+

$\therefore$ The largest possible range of $f(x)$ is $\text{Ran}(f) = (-\infty, -1] \cup (0, \infty)$.

$$(d) \quad f(x) = \sqrt{\frac{x+1}{x+2}}$$

Recall from **Example 9 of Chapter 2** that the function $f(x)$ is well-defined when $\frac{x+1}{x+2} \geq 0$ and $x+2 \neq 0$. Thus, $\text{Dom}(f) = (-\infty, -2) \cup [-1, \infty)$.

To find the range, let $y = \sqrt{\frac{x+1}{x+2}}$.

First note that $y \geq 0 \dots (1)$, since the RHS ≥ 0 for all $x \in \text{Dom}(f)$.

$$y = \sqrt{\frac{x+1}{x+2}} \Rightarrow y^2 = \frac{x+1}{x+2} \Rightarrow y^2(x+2) = x+1 \Rightarrow x(y^2-1) = 1-2y^2$$

$$\Rightarrow x = \frac{1-2y^2}{y^2-1} \text{ which has real solutions iff } y^2-1 \neq 0 \Rightarrow y^2 \neq 1 \Rightarrow y \neq \pm 1 \dots (2)$$

Combining conditions (1) and (2) gives

$$\text{Ran}(f) = [0, \infty) \setminus \{1\}.$$

Behavior of a polynomial function as x tends to ∞ or $-\infty$

Recall that a **polynomial function** of **degree n** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \dots, a_n \in \mathbb{R}$ with $a_n \neq 0$ and n is a non-negative integer. The number a_n (i.e. the coefficient of x^n) is called the **leading coefficient**.

Graphs of polynomial functions are **smooth** (i.e. contain no sharp corners) and **continuous** (i.e. have no break). By observing the sign of the leading coefficient, we can deduce the behavior of a polynomial function as x tends to ∞ or $-\infty$.

We use the notation “ $\rightarrow$ ” to denote “tends to”.

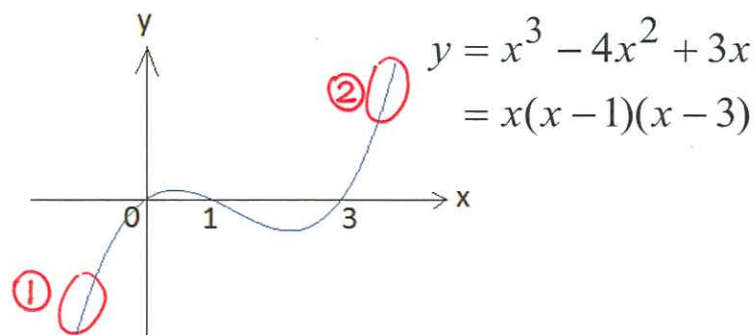
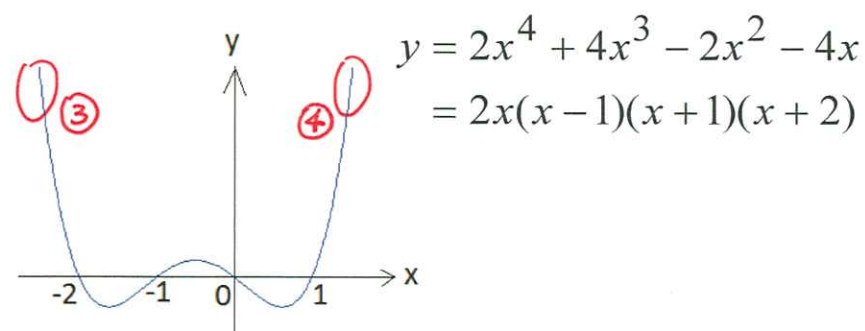
$$x \rightarrow \infty$$

$$x \rightarrow -\infty$$

Case I: $a_n > 0$ dominant term $\rightarrow a_n x^n + a_{n-1} x^{n-1} + \dots$

- If n is odd and $a_n > 0$, then ^① $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$; $(x^n \rightarrow -\infty \therefore a_n x^n \rightarrow -\infty)$
and ^② $f(x) \rightarrow \infty$ as $x \rightarrow \infty$. $(x^n \rightarrow \infty \therefore a_n x^n \rightarrow \infty)$
- If n is even and $a_n > 0$, then ^③ $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$; $(x^n \rightarrow \infty \therefore a_n x^n \rightarrow \infty)$
and ^④ $f(x) \rightarrow \infty$ as $x \rightarrow \infty$. $(x^n \rightarrow \infty \therefore a_n x^n \rightarrow \infty)$

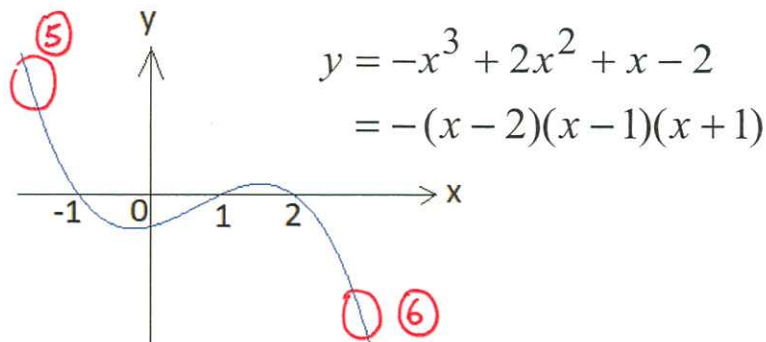
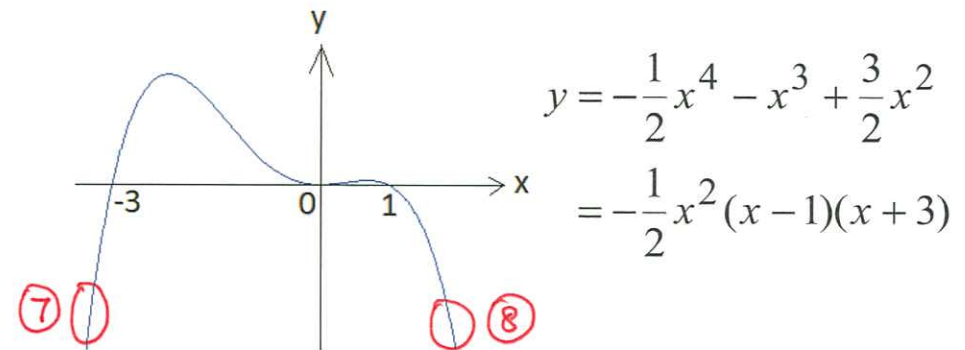
E.g.

 $a_3 = 1 > 0$ and n is odd $a_4 = 2 > 0$ and n is even

Case II: $a_n < 0$

- If n is odd and $a_n < 0$, then ^⑤ $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$; ($x^n \rightarrow -\infty \therefore a_n x^n \rightarrow \infty$)
and ^⑥ $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$. ($x^n \rightarrow \infty \therefore a_n x^n \rightarrow -\infty$)
- If n is even and $a_n < 0$, then ^⑦ $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$; ($x^n \rightarrow \infty \therefore a_n x^n \rightarrow -\infty$)
and ^⑧ $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$. ($x^n \rightarrow \infty \therefore a_n x^n \rightarrow -\infty$)

E.g.

 $a_3 = -1 < 0$ and n is odd $a_4 = -\frac{1}{2} < 0$ and n is even

Division of polynomials

Recall that when a number (called “dividend”) is divided by a smaller non-zero number (called “divisor”), we obtain:

$$\begin{aligned} \text{Dividend} &= \text{Quotient} \times \text{Divisor} + \text{Remainder} \\ \Rightarrow \frac{\text{Dividend}}{\text{Divisor}} &= \frac{\text{Quotient} \times \text{Divisor} + \text{Remainder}}{\text{Divisor}} \\ \Rightarrow \frac{\text{Dividend}}{\text{Divisor}} &= \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}} \end{aligned}$$

E.g. We divide 53 by 8 using **long division** to obtain the quotient (=6) and remainder (=5).

$$\begin{array}{rcl} \text{Divisor} \rightarrow & \begin{array}{r} 6 \\ 8 \overline{)53} \\ -) 48 \\ \hline 5 \end{array} & \begin{array}{l} \leftarrow \text{Quotient} \\ \leftarrow \text{Dividend} \\ \leftarrow \text{Remainder} \end{array} \\ \therefore 53 &= 6 \times 8 + 5 & \\ \frac{53}{8} &= 6 + \frac{5}{8} & \end{array}$$

E.g. We divide 56 by 7 using **long division** to obtain the quotient (=8) and remainder (=0).

$$\begin{array}{rcl} \text{Divisor} \rightarrow & \begin{array}{r} 8 \\ 7 \overline{)56} \\ -) 56 \\ \hline 0 \end{array} & \begin{array}{l} \leftarrow \text{Quotient} \\ \leftarrow \text{Dividend} \\ \leftarrow \text{Remainder (=0)} \end{array} \\ \therefore 56 &= 8 \times 7 + 0 & \end{array}$$

Similarly, when a polynomial $p(x)$ is divided by a polynomial $d(x)$ of lower degree, we have the result

$$p(x) = q(x) d(x) + r(x),$$

i.e.

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

where the quotient $q(x)$ and remainder $r(x)$ can be found using **long division**.

$$\begin{array}{rcl} \text{Divisor} \rightarrow & d(x) \overline{) p(x)} & \begin{array}{l} q(x) \leftarrow \text{Quotient} \\ \leftarrow \text{Dividend} \end{array} \\ & \underline{-) \quad q(x)d(x)} & \\ & r(x) & \leftarrow \text{Remainder} \end{array}$$

$$r(x) = p(x) - q(x)d(x)$$

Note that:

- degree of $q(x) = \text{degree of } p(x) - \text{degree of } d(x)$;
- $0 \leq \text{degree of } r(x) < \text{degree of } d(x)$.

An alternative method called synthetic division can also be used to divide polynomials if the divisor is of the form $x - c$. Consider the following examples:

(linear term &
coeff. of x is 1)

it cannot be used if the divisor has $\deg. \geq 2$

Example 4

Find the quotient and remainder when the polynomial $4x^3 + 5x^2 - x + 7$ is divided by $x + 2$.

Solution

Method 1: By **long division**:

$$\begin{array}{r}
 \text{Divisor} \rightarrow x + 2 \overline{) 4x^3 + 5x^2 - x + 7} \quad \leftarrow \text{Dividend} \\
 \underline{- (4x^3 + 8x^2)} \\
 -3x^2 - x \\
 \underline{- (-3x^2 - 6x)} \\
 5x + 7 \\
 \underline{- (5x + 10)} \\
 -3 \quad \leftarrow \text{Remainder}
 \end{array}$$

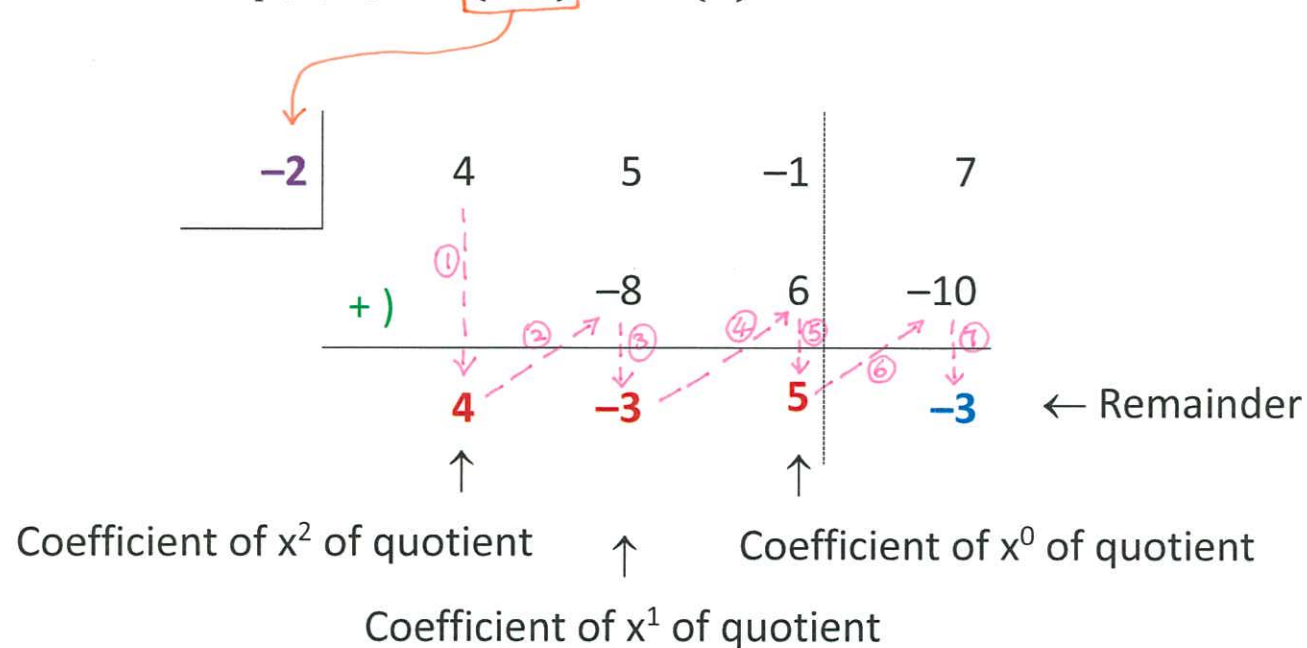
$4x^2 - 3x + 5 \quad \leftarrow \text{Quotient}$

$$\therefore 4x^3 + 5x^2 - x + 7 = (4x^2 - 3x + 5)(x + 2) - 3$$

The quotient is $4x^2 - 3x + 5$ and the remainder is -3 .

Method 2: By **synthetic division**:

$$4x^3 + 5x^2 - x + 7 = q(x) (x - (-2)) + r(x).$$



$\therefore$ The quotient is $4x^2 - 3x + 5$ and the remainder is -3 .

$$\begin{aligned} 4x^3 + 5x^2 - x + 7 &= (4x^2 - 3x + 5)(x - (-2)) - 3 \\ &= (4x^2 - 3x + 5)(x + 2) - 3. \end{aligned}$$

Example 5

Find the quotient and remainder when the polynomial $2x^3 + x^2 + 8$ is divided by $2x - 3$.

Solution

Method 1: By **long division**,

$$\begin{array}{r}
 \text{Divisor} \rightarrow 2x - 3 \overline{) 2x^3 + x^2 + 8} \quad \begin{array}{l} x^2 + 2x + 3 \leftarrow \text{Quotient} \\ \leftarrow \text{Dividend} \end{array} \\
 \underline{2x^3 - 3x^2} \\
 4x^2 \\
 \underline{4x^2 - 6x} \\
 6x + 8 \\
 \underline{6x - 9} \\
 17 \leftarrow \text{Remainder}
 \end{array}$$

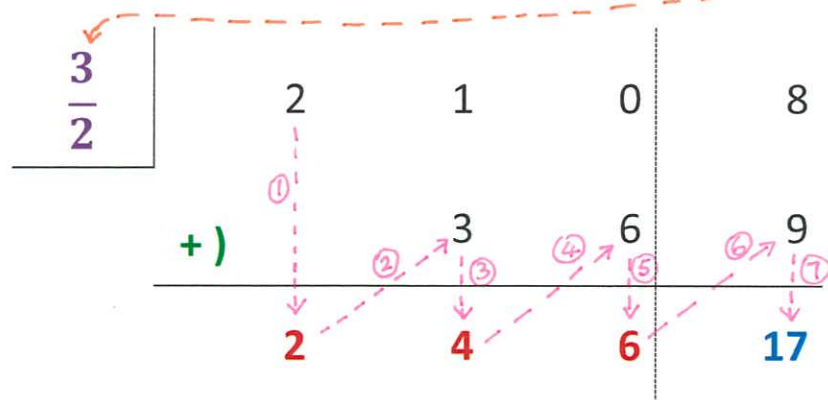
$$\therefore 2x^3 + x^2 + 8 = (x^2 + 2x + 3)(2x - 3) + 17$$

The quotient is $x^2 + 2x + 3$ and the remainder is 17.

Method 2: By **synthetic division**,

$$2x^3 + x^2 + 8 = q(x) \cdot \underline{(2x - 3)} + r(x) = 2q(x) \cdot \left(x - \boxed{\frac{3}{2}}\right) + r(x)$$

Not of the form $x - c$



$$\therefore 2x^3 + x^2 + 8 = (2x^2 + 4x + 6) \left(x - \frac{3}{2}\right) + 17 = (x^2 + 2x + 3)(2x - 3) + 17$$

That is, the quotient is $x^2 + 2x + 3$ and the remainder is 17.

Note that when a polynomial $p(x)$ of degree n is divided by a linear divisor $ax - b$ ($a \neq 0$), the degree of the quotient $q(x)$ is $n - 1$ and the degree of the remainder r is 0 (i.e. r is a constant).

Remainder Theorem

When a polynomial $f(x)$ is divided by $ax - b$ (where $a \neq 0$), the remainder is $f\left(\frac{b}{a}\right)$.

Proof:

We have $f(x) = q(x) \cdot (ax - b) + r$. Putting $x = \frac{b}{a}$ into this expression, we obtain

$$f\left(\frac{b}{a}\right) = q\left(\frac{b}{a}\right) \cdot \underbrace{\left[a\left(\frac{b}{a}\right) - b\right]}_{=0} + r = r.$$

Example 6

By using the **Remainder Theorem**,

- (a) find the remainder when $f(x) = 4x^3 + 5x^2 - x + 7$ is divided by $x + 2$. (Example 4)
- (b) find the remainder when $f(x) = 2x^3 + x^2 + 8$ is divided by $2x - 3$. (Example 5)

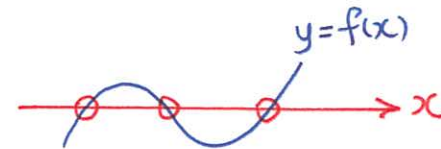
Solution

- (a) $a = 1$, $b = -2$. By the **Remainder Theorem**, the remainder is

$$f\left(\frac{-2}{1}\right) = 4\left(\frac{-2}{1}\right)^3 + 5\left(\frac{-2}{1}\right)^2 - \left(\frac{-2}{1}\right) + 7 = -3.$$

(b) $a = 2$, $b = 3$. By the **Remainder Theorem**, the remainder is

$$f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^2 + 8 = 17.$$



Zeros of polynomial functions

The values of x for which the polynomial function $f(x) = 0$ are called the **zeros**. They are also called the **roots** (or **solutions**) of the equation $f(x) = 0$. The **real roots** are the x -coordinates of the points where the graph of $y = f(x)$ meets the x -axis, and these points are called the x -intercepts.

The following theorem is useful in finding the linear factors of a polynomial function:

Factor Theorem

The linear term $ax - b$ (where $a \neq 0$) is a factor of a polynomial $f(x)$ if and only if $f\left(\frac{b}{a}\right) = 0$.

$$f(x) = (ax - b)(\dots)$$

Example 7

Determine whether $2x - 1$ and $x + 1$ are factors of $f(x) = 2x^3 + 3x^2 - 8x + 3$. Then factorize $f(x) = 2x^3 + 3x^2 - 8x + 3$.

Solution

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 8\left(\frac{1}{2}\right) + 3 = 0$$

$\therefore$ By the **Factor Theorem**, $2x - 1$ is a factor of $f(x)$.

$$f\left(\frac{-1}{1}\right) = f(-1) = 2(-1)^3 + 3(-1)^2 - 8(-1) + 3 = 12 \neq 0$$

$\therefore$ By the **Factor Theorem**, $x + 1$ is **NOT** a factor of $f(x)$.

Since $2x - 1$ is a factor of $f(x) = 2x^3 + 3x^2 - 8x + 3$, we use

long division to divide $f(x)$ by $2x - 1$.

$$\therefore f(x) = 2x^3 + 3x^2 - 8x + 3$$

$$= (2x - 1)(x^2 + 2x - 3)$$

$$= (2x - 1)(x + 3)(x - 1)$$

$$\begin{array}{r} x^2 + 2x - 3 \\ 2x - 1 \overline{) 2x^3 + 3x^2 - 8x + 3} \\ \underline{2x^3 - x^2} \end{array}$$

$$4x^2 - 8x$$

$$\underline{4x^2 - 2x}$$

$$-6x + 3$$

$$\underline{-6x + 3}$$

remainder must
be zero if $2x - 1$
is a factor of
 $2x^3 + 3x^2 - 8x + 3$



Example 8

Factorize $f(x) = 2x^6 - 8x^4 - 2x^2 + 8$.

Solution

Since the polynomial $f(x) = 2x^6 - 8x^4 - 2x^2 + 8$ only consists of even powers of x , we let $u = x^2$ and the function becomes $g(u) = 2u^3 - 8u^2 - 2u + 8$. This polynomial is simpler and has lower degree than $f(x)$, so it should be easier to factorize.

Consider the **constant term** (which is 8) in $g(u)$.

Factors of 8 are ± 1 , ± 2 , ± 4 and ± 8 .

We use the “trial and error” method to find some roots first.

$$g(1) = 2(1)^3 - 8(1)^2 - 2(1) + 8 = 0 \quad \checkmark \quad \Rightarrow (u - 1) \text{ is a factor of } g(u).$$

$$g(2) = 2(2)^3 - 8(2)^2 - 2(2) + 8 = -12 (\neq 0)$$

$$g(-1) = 2(-1)^3 - 8(-1)^2 - 2(-1) + 8 = 0 \quad \checkmark \quad \Rightarrow (u + 1) \text{ is a factor of } g(u).$$

Since $(u - 1)$ and $(u + 1)$ are factors of $g(u)$, then $g(u)$ must be divisible by $(u - 1)(u + 1) = u^2 - 1$.

By long division, $g(u) = 2u^3 - 8u^2 - 2u + 8$

$$= (u^2 - 1)(2u - 8)$$

$$\boxed{a^2 - b^2 = (a - b)(a + b)}$$

$$= 2(u - 1)(u + 1)(u - 4)$$

$$\begin{array}{r} 2u - 8 \\ u^2 - 1 \overline{) 2u^3 - 8u^2 - 2u + 8} \\ \underline{2u^3} - 2u \\ - 8u^2 + 8 \\ \underline{- 8u^2} + 8 \\ 0 \end{array}$$

Replacing $u = x^2$, we get $f(x) = 2(x^2 - 1)(x^2 + 1)(x^2 - 4)$

$$= 2(x - 1)(x + 1)(x - 2)(x + 2)(x^2 + 1).$$

Note: The quadratic term $x^2 + 1$ cannot be further factorized.

$\therefore x^2 + 1 = 0$ has no real solutions
(or discriminant < 0)