

# **EE3121**

# **Differential Equations for Engineers**

## **Lecture 3: First-Order Nonlinear Ordinary Differential Equations**



**Instructor: Prof. Jie Chen**

**DEPARTMENT OF ELECTRICAL ENGINEERING**

# Framework of Lectures

- Three techniques for solving nonlinear ordinary differential equations (ODEs)
  - Separating variables.
  - Transforming the nonlinear ODE into a linear one.
  - Successive approximations.

# 3.1 Nonlinear Separable Ordinary Differential Equations

- Applying only to first-order separable equations

$$\frac{dy}{dt} = f(t)g(y), \quad y(t_0) = y_0. \quad (1)$$

To solve (1), we separate variables by dividing through by  $g(y)$ :

$$\frac{1}{g(y)} \frac{dy}{dt} = f(t).$$

Then we employ the shortcut procedure introduced in Lecture 2:

$$\int_{y(t_0)}^y \frac{dy}{g(y)} = \int_{t_0}^t f(\tau) d\tau. \quad (2)$$

## 3.1 Nonlinear Separable Ordinary Differential Equations

Sometimes we can perform the integration on both side of equation (2) and arrive at an implicit equation

$$G(y) = F(t). \quad (3)$$

In rare cicumstances we are able to solve equation (3).

## 3.1.1 Example: Sounding Rocket Trajectory

- There are three fundamental questions about the sounding rocket.
  - What is its maximum speed going up?
  - What altitude does it reach?
  - What is its maximum speed coming down?

### 3.1.1.1 Maximum Speed Going Up

- The equation of motion is

$$m \frac{dV}{dt} + \frac{dm}{dt} c = 0,$$

or

$$\frac{dV}{dt} = -\frac{1}{m} \frac{dm}{dt} c, \quad (4)$$

where

$V$  is the speed of the vehicle,

$m$  is the mass,

$c$  is the speed of the exit gases relative to the rocket.

### 3.1.1.1 Maximum Speed Going Up

Eliminate time by writing

$$\frac{dV}{dt} = \frac{dV}{dm} \frac{dm}{dt},$$

then cancelling  $\frac{dm}{dt}$ , equation (4) becomes the separable equation

$$\frac{dV}{dm} = -\frac{c}{m}.$$

Following the procedure outlined in equation (1) through (3),

$$\begin{aligned} dV &= -c \frac{dm}{m} \\ \int_0^{V(m)} dV &= -c \int_{m_0}^m \frac{dm}{m} \\ V(m) &= c \ln \left( \frac{m_0}{m} \right) \end{aligned}$$

### 3.1.1.1 Maximum Speed Going Up

At rocket burnout, that is, the moment the rocket fuel is exhausted,

$$V_b = c \ln \left( \frac{m_0}{m_b} \right) \quad (5)$$

$V_b$  is the maximum speed the sounding rocket attains on its way up. Equation (5) is so-called *rocket* equation.

### 3.1.1.2 Maximum Altitude

- According to Newton's second law  $F = ma$ , after burnout and on the way up, the sounding rocket's trajectory is governed by the nonlinear differential equation

$$m_b \frac{dV}{dt} = -m_b g - \left( \frac{1}{2} \rho S C_D \right) V^2. \quad (6)$$

- Acceleration is  $\frac{dV}{dt}$ , the derivative of velocity with respect to time. The force are weight and air resistance(drag).
- Drag is proportional to the air density  $\rho$ , the rocket's cross-sectional area  $S$ , its drag coefficient  $C_D$ .

### 3.1.1.2 Maximum Altitude

Rewriting equation (6) as

$$\frac{dV}{dt} = -g - kV^2, \quad (7)$$

where  $k = \frac{\rho S C_D}{2m_b}$ .

Then, separating variables:

$$\frac{dV}{g + kV^2} = -dt. \quad (8)$$

We define  $v = V\sqrt{k/g}$ , equation (8) becomes

$$\frac{dV}{1 + V^2} = -\sqrt{gk}dt. \quad (9)$$

### 3.1.1.2 Maximum Altitude

Integrating the RHS of equation (9) from 0 to  $t$ , and the LHS correspondingly from  $v_b = V_b\sqrt{k/g}$  to  $v(t) = V(t)\sqrt{k/g}$ :

$$\arctan(v(t)) - \arctan(v_b) = -\sqrt{gk}t. \quad (10)$$

Solving equation (10)

$$v(t) = \tan(\arctan(v_b) - \sqrt{gk}t)$$
$$V(t) = \int_0^t \sqrt{g/k} \tan\left(\arctan\left(\sqrt{k/g}V_b\right) - \sqrt{gk}t\right) \quad (11)$$

Equation (11) shows that the maximum speed on the way up is  $V_b$ .

### 3.1.1.2 Maximum Altitude

The altitude  $h(t)$  is given by

$$h(t) = \int_0^t V(\tau) d\tau = \int_0^t \sqrt{g/k} \tan \left( \arctan \left( \sqrt{k/g} V_b \right) - \sqrt{gk} t \right) d\tau. \quad (12)$$

Let  $t' = \sqrt{gk} \tau$  and

$$\arctan \left( \sqrt{k/g} V_b \right) = t_a' \quad (13)$$

equation (12) becomes

$$\begin{aligned} h(t) &= \frac{1}{k} \int_0^{\sqrt{gk} t} \tan(t_a' - t') dt' = \frac{1}{k} \int_0^{\sqrt{gk} t} \frac{\sin(t_a' - t')}{\cos(t_a' - t')} dt' \\ &= \frac{1}{k} \ln \left( \frac{\cos(t_a' - \sqrt{gk} t)}{\cos(t_a')} \right) \end{aligned} \quad (14)$$

### 3.1.1.2 Maximum Altitude

From equation (11) and (13), the maximum altitude(apogee) is reached when  $V(t) = 0$ , i.e.  $\sqrt{gkt} = t_a'$ . From equation (14), the maximum altitude is given by

$$h_a = \frac{1}{k} \ln \left( \frac{1}{\cos(t_a')} \right), \quad (15)$$

from equation (13),  $\tan(t_a') = \sqrt{kV_b^2/g}$ ,

$$\begin{aligned} h_a &= \frac{1}{k} \ln \left( \frac{1}{\cos(t_a')} \right) = \frac{1}{k} \ln (\sec(t_a')) = \frac{1}{k} \ln \left( \sqrt{1 + kV_b^2/g} \right) \\ &= \frac{1}{2k} \ln (1 + kV_b^2/g). \end{aligned} \quad (16)$$

### 3.1.1.2 Maximum Altitude

In conclusion, the sounding rocket's maximum altitude is given by

$$h_a = \frac{1}{2k} \ln (1 + kV_b^2 / g) , \quad (17)$$

where  $k = \frac{\rho SC_D}{2m_b}$ . When  $k$  is small,

$$h_a = \frac{1}{2k} \ln (1 + kV_b^2 / g) \cong \frac{V_b^2}{2g}, \quad (18)$$

which is classic drag-free result.

### 3.1.1.2 Maximum Speed Coming Down

When the sounding rocket is on the way down, its equation of motion is

$$m_b \frac{dV}{dt} = -m_b g + \left( \frac{1}{2} \rho S C_D \right) V^2 \quad (19)$$

Separating equation (19) into

$$\frac{dv}{1 - v^2} = -\sqrt{gk} dt \quad (20)$$

Via a partial fraction expansion

$$\frac{dv}{1 - v^2} = \left( \frac{1}{1 - v^2} \right) dv = \left( \frac{c_1}{1 - v} + \frac{c_2}{1 + v} \right) dv = -\sqrt{gk} dt \quad (21)$$

## 3.1.1.2 Maximum Speed Coming Down

To find  $c_1$  and  $c_2$ , we have

$$\frac{c_1}{1-v} + \frac{c_2}{1+v} = \frac{(c_1 + c_2) + (c_1 - c_2)v}{1-v^2} = \frac{1}{1-v^2},$$

then,

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 0, \end{aligned}$$

therefore,

$$c_1 = c_2 = \frac{1}{2}.$$

Returning to equation (21), we have

$$\left( \frac{1/2}{1-v} + \frac{1/2}{1+v} \right) dv = -\sqrt{gk} dt = -dt'. \quad (22)$$

## 3.1.1.2 Maximum Speed Coming Down

Integrating the RHS of equation (22) from  $t' = t'_a$  and the LHS correspondingly from  $v = 0$ ,

$$-(1/2) \ln(1 - v) + (1/2) \ln(1 + v) = -(t' - t'_a)$$

$$\ln \left( \frac{1+v}{1-v} \right) = -2(t' - t'_a)$$

$$\frac{1+v}{1-v} = e^{-2(t' - t'_a)}$$

$$1 + v = (1 - v)e^{-2(t' - t'_a)}$$

$$-v(1 + e^{-2(t' - t'_a)}) = 1 - e^{-2(t' - t'_a)}$$

$$v = -\frac{1 - e^{-2(t' - t'_a)}}{1 + e^{-2(t' - t'_a)}}$$

## 3.1.1.2 Maximum Speed Coming Down

Multiplying numerator and denominator by  $e^{t'-t'_a}$  yields a more favored form

$$v = -\frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}, \quad (23)$$

where  $\theta = t' - t'_a = \sqrt{gk}(t - t_a)$ .

- The RHS of equation (23) is the negative of a function called the *hyperbolic tangent* of  $\theta$ , written as  $\tanh(\theta)$ .
- In dimensional form, the sounding rocket's velocity on the way down is given by

$$v = -\sqrt{g/k} \left( \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}} \right) = -\sqrt{g/k} \tanh(\theta). \quad (24)$$

## 3.1.1.2 Maximum Speed Coming Down

- The maximum speed is

$$V_m = \frac{-V_b}{\sqrt{1 + kV_b^2/g}}. \quad (25)$$

To determine the maximum speed is a challenge in CP3.2.

- For vanishing  $k$ (very low drag),

$$V_m = -V_b; \quad (26)$$

- When there is no atmosphere( $kV_b^2/g \gg 1$ ),

$$V_m = -\sqrt{g/k}, \quad (27)$$

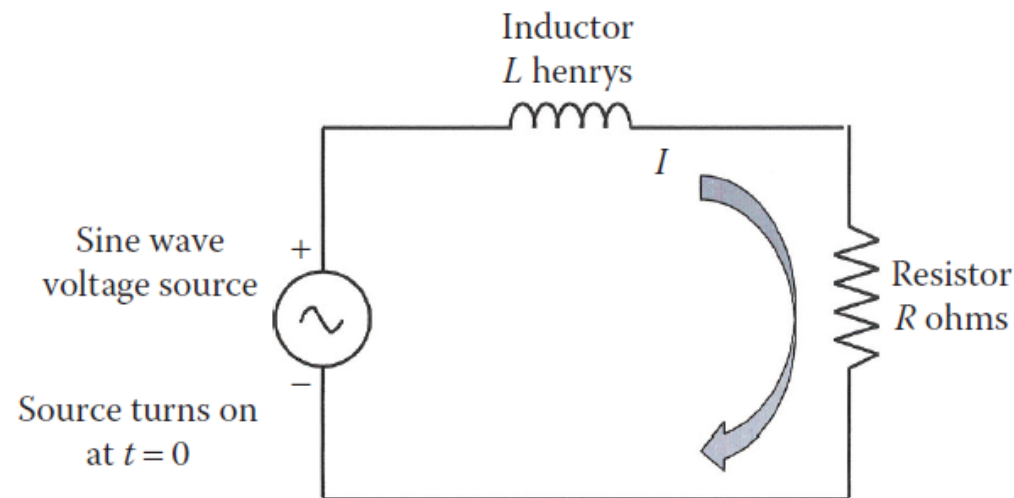
$\sqrt{g/k}$  is called the *terminal velocity*.

## 3.2 Possibility of Transforming Nonlinear Equations into Linear Ones

- As it happens, the nonlinear equations for both upward and downward rocket flight can be transformed into linear ones by changing the independent variable from  $t$  to  $h$  and the dependent variable from  $V$  to  $V^2/2$ .
- This shortcut method cannot determine the way velocity varies with time, but it can yield both the maximum altitude and the maximum speed on the way down.
  - WP3.1 outlines this approach;
  - WP3.2 and 3.3 are variations on this approach for two different and important applications;
  - Problem 3.13 presents another transformation with some generality.

## 3.3 Successive Approximations for Almost Linear Systems

Consider the electrical circuit shown below:



Applying Kirchhoff's voltage law to this circuit results in

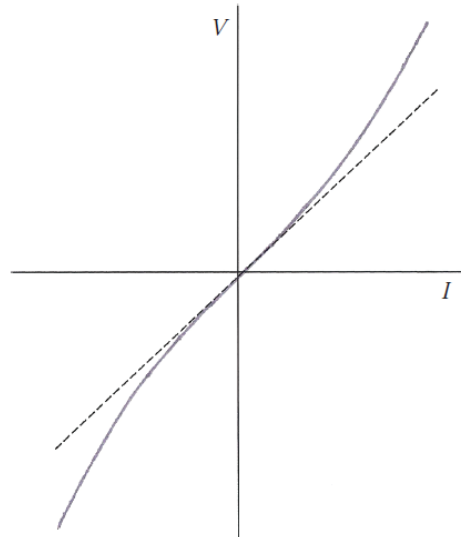
$$L \frac{dI}{dt} + IR = V_0 \sin \omega t. \quad (28)$$

## 3.3 Successive Approximations for Almost Linear Systems

Dividing through by  $L$ ,

$$\frac{dI}{dt} + \lambda I = \lambda I_0 \sin \omega t, \quad (29)$$

where  $\lambda = \frac{R}{L}$ ,  $I_0 = \frac{V_0}{R}$ . Suppose the resistor  $R$  is nonlinear,



## 3.3 Successive Approximations for Almost Linear Systems

The circuit equation is modified to

$$\frac{dI}{dt} + \lambda I + \lambda \varepsilon I^3 = \lambda I_0 \sin \omega t \quad (30)$$

- If the nonlinear term  $\varepsilon I^3$  is not too large compared to  $I_0$  (equation (30) is *almost linear*). We can employ a technique called *successive approximations*.
- Solution  $I_1$ : Ignoring the nonlinearity and solving equation (29);
- Solution  $I_2$ : Approximating the nonlinear term as the *known function of time*  $\lambda \varepsilon I_1^3$ .

## 3.3 Successive Approximations for Almost Linear Systems

We obtain  $I_2$  through

$$\frac{dI_2}{dt} + \lambda I_2 = \lambda I_0 \sin \omega t - \lambda \varepsilon I_1^3, \quad (31)$$

$$I_2(t) = \int_0^t e^{-\lambda(t-\tau)} (\lambda I_0 \sin \omega \tau - \lambda \varepsilon I_1^3(\tau)) d\tau. \quad (32)$$

Now

$$\int_0^t e^{-\lambda(t-\tau)} \lambda I_0 \sin \omega \tau d\tau = I_1(t), \quad (33)$$

then

$$I_2(t) = I_1(t) - \lambda \varepsilon \int_0^t e^{-\lambda(t-\tau)} I_1^3(\tau) d\tau. \quad (34)$$

## 3.3 Successive Approximations for Almost Linear Systems

Proceeding in this way, we calculate

$$I_{n+1}(t) = I_1(t) - \lambda \varepsilon \int_0^t e^{-\lambda(t-\tau)} I_n^3(\tau) d\tau. \quad (35)$$

Performing the integral in equation (34) leads to

$$I_2(t) = G_1 I_0(\sin \omega t - \eta_2(t)), \quad (36)$$

where

$$\eta_2(t) = \frac{\alpha}{4} (3G_1 \sin(\omega t - 2\theta_1) - G_3 \sin(3\omega t - 3\theta_1 - \theta_3)), \quad (37)$$

the effect of nonlinearity is captured in  $\eta_2(t)$ .

## 3.3 Successive Approximations for Almost Linear Systems

And

$$G_n = \frac{\lambda}{\sqrt{\lambda^2 + (n\omega)^2}}, \quad (38)$$

$$\theta_n = \arctan\left(\frac{n\omega}{\lambda}\right), \quad (39)$$

$$\alpha = \varepsilon(G_1 I_0)^2. \quad (40)$$

If we continue the sequence we would find

$$I_n(t) = I_1 - G_1 I_0 \eta_n(t), \quad (41)$$

where

$$\eta_n(t) = \sum_{m=1}^{N_n} k_{mn} \sin(m\omega t - \phi_{mn}). \quad (42)$$

## 3.3 Successive Approximations for Almost Linear Systems

- Equation (41) and (42) are specific examples of a general fact.
- Nonlinear systems introduce multiples of that frequency, called *harmonics*. The phenomenon is called *distortion*.

For higher order systems

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y = g(t), \quad (43)$$

with zero initial conditions the solution is

$$y(t) = \int_0^t K(t - \tau) g(\tau) d\tau,$$

## 3.3 Successive Approximations for Almost Linear Systems

where  $K(t - \tau)$  is the kernel of the system.

Suppose a system is described by the ODE

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y + f(t, y) = g(t), \quad (44)$$

where  $f(t, y)$  is nonlinear in the variable  $y$ .

## 3.3 Successive Approximations for Almost Linear Systems

If  $y_1(t)$  is a solution of equation (43) for  $t_0 < t < T$  and  $f(t, y_1(t))$  is sufficiently small for the same time interval, then a solution to equation (44) in the successive approximations

$$y_{n+1}(t) = \int_{t_0}^t K(t - \tau)(g(\tau) - f(\tau, y_n(\tau)))d\tau, \quad (45)$$

$$y_{n+1}(t) = y_1(t) - \int_{t_0}^t K(t - \tau)f(\tau, y_n(\tau))d\tau. \quad (46)$$

**END**