

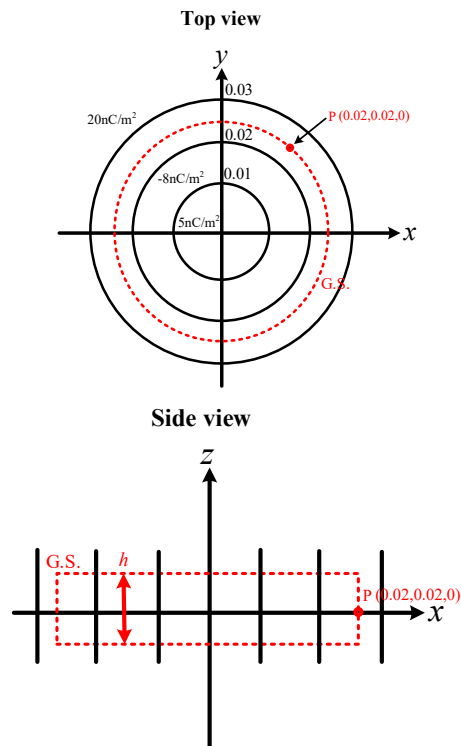
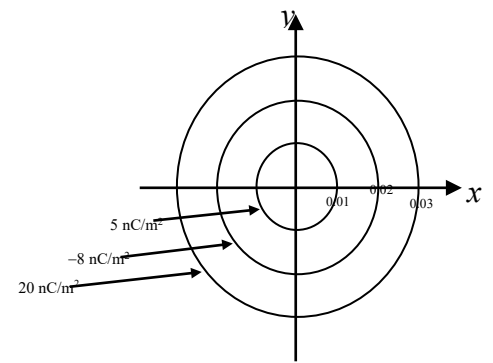
EE2104 Introduction to Electromagnetics
Assignment 3 Solution

1. In the figure below, suppose that we have three concentric infinitely long cylinders. They are centered around the origin and extends along the z direction. The innermost cylinder has a radius of $a = 0.01$ m and it carries a surface charge density of $A = 5$ nC/m². The middle one has a radius of $b = 0.02$ m and it carries a surface charge density of $B = -8$ nC/m². The outermost cylinder has a radius of $c = 0.03$ m and it carries a surface charge density of $C = 20$ nC/m².

Using Gauss's law, find $\mathbf{D}$ at point $(x, y, z) = (0.02$ m, 0.02 m, $0)$.

Solution:

Based on the structure in the figure, we know that the charges are distributed symmetrically to the origin. So, $\mathbf{D}$ points along the radial direction and its magnitude only depends on the distance between the point and the origin. We can have $\mathbf{D}(\rho, \phi, z) = \mathbf{D}(\rho) = D(\rho) \hat{\mathbf{a}}_\rho$.



We can construct a Gaussian Surface as shown in the figure above.

From Gauss's law, we have $\oint_S \mathbf{D}(\rho, \varphi, z) \cdot d\mathbf{S} = \oint_S \mathbf{D}(\rho) \cdot d\mathbf{S} = Q$,

where Q is the total charge enclosed by the Gaussian surface.

$$\oint_S \mathbf{D}(\rho) \cdot d\mathbf{S} = \int_{top} \mathbf{D}(\rho) \cdot d\mathbf{S} + \int_{bottom} \mathbf{D}(\rho) \cdot d\mathbf{S} + \int_{side} \mathbf{D}(\rho) \cdot d\mathbf{S}$$

$$\text{We have } \int_{top} \mathbf{D}(\rho) \cdot d\mathbf{S} = \int_{top} \mathbf{D}(\rho) \cdot d\mathbf{S} = 0$$

On the side surface, $\mathbf{D}$ is always parallel with $d\mathbf{S}$ and $|\mathbf{D}(\rho)| = D(\rho)$ is constant.

$$\text{Then we have } \oint_S \mathbf{D}(\rho) \cdot d\mathbf{S} = D(\rho) \int_{side} dS = D(\rho) \times 2\pi P_d \times h = 0.04\sqrt{2}\pi h D(\rho),$$

where h is the thickness of the Gaussian surface.

$$\text{The charge } Q = Q_1 + Q_2 = \rho_1 S_1 + \rho_2 S_2$$

$$= 5 \times 10^{-9} \times 2\pi \times 0.01h - 8 \times 10^{-9} \times 2\pi \times 0.02h = -0.22 \times 10^{-9} \pi h (\text{C})$$

$$\text{Then we can get } 0.04\sqrt{2}\pi h D(\rho) = -0.22\pi h \times 10^{-9}$$

$$D(\rho) = -3.89 \times 10^{-9} \text{ C/m}^2 = -3.89 \text{ nC/m}^2$$

The direction of $\mathbf{D}(\rho)$ is parallel to the vector pointing from origin to P.

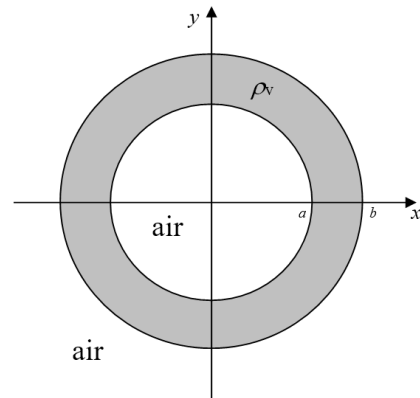
$$\text{So } \mathbf{D} \text{ at point } (0.02 \text{ m}, 0.02 \text{ m}, 0) \text{ is } (-2.75\hat{\mathbf{a}}_x - 2.75\hat{\mathbf{a}}_y) \text{ nC/m}^2$$

2. Consider a hollow spherical structure with inner radius a and outer radius b . The figure below shows the cross-section of the structure. In the spherical coordinate system, the charge density of the structure for $a < \rho < b$ is

$$\rho_v = \rho_{v0} \frac{\rho}{b}$$

where ρ_{v0} (C/m³) is a constant.

Find the electric field $\mathbf{E}$. Then, sketch its magnitude as a function of the radial coordinate ρ .



Solution:

To match the geometry of the structure, let us choose an imaginary sphere of radius ρ , centered at the origin as the gaussian surface S .

Thus, the differential surface on S is $d\mathbf{S} = dS \hat{\mathbf{a}}_\rho$, where it points radially away from the origin.

By symmetry, the $\mathbf{D}$ -field points along $\hat{\mathbf{a}}_\rho$ and $|\mathbf{D}| = D$ is independent of (θ, ϕ) .

So, we have $\mathbf{D}(\rho, \theta, \phi) = D(\rho) \hat{\mathbf{a}}_\rho$.

From Gauss's law, we have $\oint_S \mathbf{D}(\rho, \theta, \phi) \cdot d\mathbf{S} = Q_{enc}$, where Q_{enc} is the enclosed charge within S .

Thus,

$$\oint_S D(\rho) \hat{\mathbf{a}}_\rho \cdot (dS \hat{\mathbf{a}}_\rho) = Q_{enc}$$

$$\oint_S D(\rho) (\hat{\mathbf{a}}_\rho \cdot \hat{\mathbf{a}}_\rho) dS = Q_{enc}$$

$$\oint_S D(\rho) dS = Q_{enc}$$

$$D(\rho) \oint_S dS = Q_{enc}$$

$$D(\rho) (4\pi\rho^2) = Q_{enc}$$

$$D(\rho) = \frac{Q_{enc}}{4\pi\rho^2}$$

Therefore,

(1) for $0 < \rho < a$, $Q_{enc} = 0$, $D(\rho) = \frac{Q_{enc}}{4\pi\rho^2} = 0$, $E = 0$.

(2) for $a < \rho < b$,

$$\begin{aligned} Q_{enc} &= \int_a^\rho \int_0^\pi \int_0^{2\pi} \rho_v \rho^2 \sin\theta \, d\rho \, d\theta \, d\varphi = \int_a^\rho \int_0^\pi \int_0^{2\pi} \rho_{v0} \frac{\rho}{b} \rho^2 \sin\theta \, d\rho \, d\theta \, d\varphi \\ &= \frac{\rho_{v0}}{b} \int_a^\rho \int_0^\pi \int_0^{2\pi} \rho^3 \sin\theta \, d\rho \, d\theta \, d\varphi = \frac{\rho_{v0}}{b} \int_a^\rho \rho^3 d\rho \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi \\ &= \frac{\rho_{v0}}{b} \frac{\rho^4 - a^4}{4} \cdot 2 \cdot 2\pi = \frac{\rho_{v0}\pi(\rho^4 - a^4)}{b} \end{aligned}$$

So, we have

$$\begin{aligned} D(\rho) &= \frac{Q_{enc}}{4\pi\rho^2} = \frac{\rho_{v0}(\rho^4 - a^4)}{4\rho^2 b} \\ E &= \frac{\rho_{v0}(\rho^4 - a^4)}{4\rho^2 b \epsilon_0} \end{aligned}$$

(3) for $\rho > b$,

$$\begin{aligned} Q_{enc} &= \int_a^b \int_0^\pi \int_0^{2\pi} \rho_v \rho^2 \sin\theta \, d\rho \, d\theta \, d\varphi = \int_a^b \int_0^\pi \int_0^{2\pi} \rho_{v0} \frac{\rho}{b} \rho^2 \sin\theta \, d\rho \, d\theta \, d\varphi \\ &= \frac{\rho_{v0}}{b} \int_a^b \int_0^\pi \int_0^{2\pi} \rho^3 \sin\theta \, d\rho \, d\theta \, d\varphi = \frac{\rho_{v0}}{b} \int_a^b \rho^3 d\rho \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi \\ &= \frac{\rho_{v0}}{b} \frac{b^4 - a^4}{4} \cdot 2 \cdot 2\pi = \frac{\rho_{v0}\pi(b^4 - a^4)}{b} \end{aligned}$$

So, we have

$$\begin{aligned} D(\rho) &= \frac{Q_{enc}}{4\pi\rho^2} = \frac{\rho_{v0}(b^4 - a^4)}{4\rho^2 b} \\ E &= \frac{\rho_{v0}(b^4 - a^4)}{4\rho^2 b \epsilon_0} \end{aligned}$$

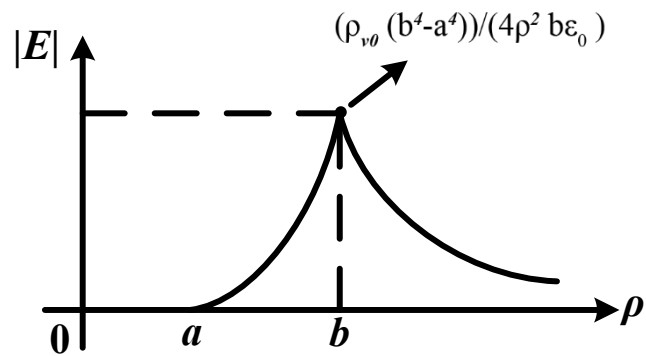
In the end, we have

$$\mathbf{E}(\rho, \theta, \varphi) = 0, \quad \text{for } 0 < \rho < a$$

$$\mathbf{E}(\rho, \theta, \varphi) = \frac{\rho_{v0}(\rho^4 - a^4)}{4\rho^2 b \epsilon_0} \hat{\mathbf{a}}_\rho, \quad \text{for } a < \rho < b$$

$$\mathbf{E}(\rho, \theta, \varphi) = \frac{\rho_{v0}(b^4 - a^4)}{4\rho^2 b \epsilon_0} \hat{\mathbf{a}}_\rho, \quad \text{for } \rho > b$$

So, the magnitude of the electric field $\mathbf{E}$ as a function of ρ can be plotted as:



3. In a dielectric material, the electric flux density is 1.5 times the polarization. Find the dielectric constant of the material.

Solution:

Assuming that the dielectric constant (also called relative permittivity) of the material is ϵ_r , we have the relations as follows.

i: The electric flux density $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$

ii: The polarization $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (\epsilon_r - 1) \mathbf{E}$

Moreover, $\mathbf{D} = 1.5\mathbf{P}$ is given.

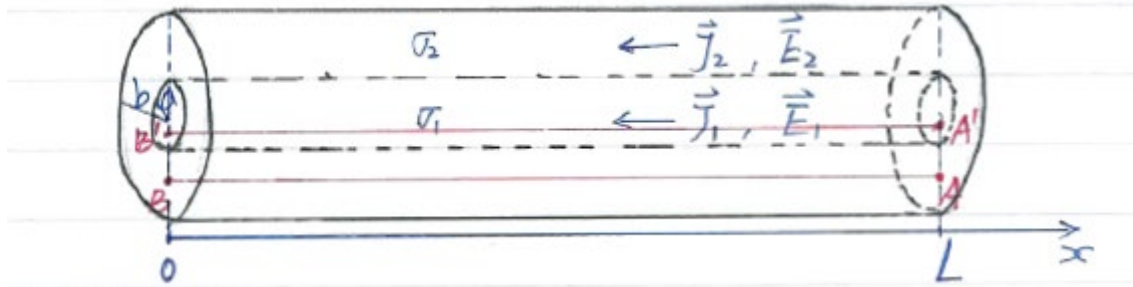
Thus, $\epsilon_0 \epsilon_r \mathbf{E} = 1.5 \epsilon_0 (\epsilon_r - 1) \mathbf{E}$

$$\epsilon_r = 1.5(\epsilon_r - 1)$$

$$\epsilon_r = 3$$

4. Consider a solid wire consisting of two conductive materials. The core has radius a and conductivity σ_1 . The jacket has conductivity σ_2 . Its inner radius is a and outer radius is b . Show that the ratio of the current densities in the two materials is independent of a and b .

Solution:



Consider the conductive wire ended at $x=0$ and $x=L$

When the end at $x=0$ is grounded and a voltage V is applied at $x=L$, current flows from $x=L$ to $x=0$.

Assuming the current densities are J_1 and J_2 in the core and jacket,

We have $J_1 = \sigma_1 E_1$ and $J_2 = \sigma_2 E_2$

Considering $V_{AB} = - \int_B^A \mathbf{E} \cdot d\mathbf{L}$ is path independent,

We can get $V_{AB} = V_{A'B'}$ and the electric fields should satisfy $E_1 = E_2$

Thus, $\frac{J_1}{\sigma_1} = \frac{J_2}{\sigma_2}$

$$\frac{J_1}{J_2} = \frac{\sigma_1}{\sigma_2}$$

So, the ratio of the current densities in the two materials is independent of a and b .