

MA2001

MULTI-VARIABLE CALCULUS AND LINEAR ALGEBRA

Chapter 2

**Differential Calculus of Multivariable Functions
(Part B)**

Dr. Emily Chan

Chapter 2: Differential Calculus of Multivariable Functions (Part B)

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Taylor's Theorem for functions of one variable

Recall that if a function $f(x)$ of one variable is differentiable up to order N in the interval $a - l < x < a + l$ where $l > 0$, then for any $h \in \mathbb{R}$ such that $a - l < a + h < a + l$, we have

$$f(a + h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{N-1}}{(N-1)!} f^{(N-1)}(a) + R_N(h)$$

where the remainder term $R_N(h) = \frac{h^N}{N!} f^{(N)}(a + \theta h)$ for some $0 < \theta < 1$.

Alternatively, by putting $x = a + h$, we have the equivalent form:

$$f(x) = f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \cdots + \frac{f^{(N-1)}(a)}{(N-1)!} (x - a)^{N-1} + R_N(x),$$

where the remainder term $R_N(x) = \frac{f^{(N)}[a + \theta(x-a)]}{N!} (x - a)^N$ for some $0 < \theta < 1$.

If $a = 0$, we have **Maclaurin's Theorem**:

$$f(h) = f(0) + \frac{h}{1!} f'(0) + \frac{h^2}{2!} f''(0) + \cdots + \frac{h^{N-1}}{(N-1)!} f^{(N-1)}(0) + R_N(h)$$

where $R_N(h) = \frac{h^N}{N!} f^{(N)}(\theta h)$ for some $0 < \theta < 1$.

OR

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(N-1)}(0)}{(N-1)!} x^{N-1} + R_N(x),$$

where $R_N(x) = \frac{f^{(N)}(\theta x)}{N!} x^N$ for some $0 < \theta < 1$.

Example 29

Express the Maclaurin series of $\cos x$ (up to x^4) and $\frac{1}{1+y}$ (up to y^4). Hence, find the Maclaurin series of $\frac{1}{\cos x}$ (up to x^4).

Solution

Let $f(x) = \cos x \Rightarrow f'(x) = -\sin x, f''(x) = -\cos x, f^{(3)}(x) = \sin x, f^{(4)}(x) = \cos x$.

Then $f(0) = \cos 0 = 1, f'(0) = -\sin 0 = 0, f''(0) = -1, f^{(3)}(0) = 0, f^{(4)}(0) = 1$.

Maclaurin series of $\cos x$:

$$\cos x = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Let $g(y) = \frac{1}{1+y} \Rightarrow g'(y) = -\frac{1}{(1+y)^2}, g''(y) = \frac{2}{(1+y)^3}, g^{(3)}(y) = \frac{-6}{(1+y)^4}, g^{(4)}(y) = \frac{24}{(1+y)^5}$.

Then $g(0) = \frac{1}{1+0} = 1, g'(0) = -\frac{1}{(1+0)^2} = -1, g''(0) = 2, g^{(3)}(0) = -6, g^{(4)}(0) = 24$.

Maclaurin series of $\frac{1}{1+y}$:

$$\begin{aligned}\frac{1}{1+y} &= g(0) + \frac{g'(0)}{1!}y + \frac{g''(0)}{2!}y^2 + \frac{g^{(3)}(0)}{3!}y^3 + \frac{g^{(4)}(0)}{4!}y^4 + \dots \\ &= 1 - y + y^2 - y^3 + y^4 - \dots\end{aligned}$$

Hence, the Maclaurin series of $\frac{1}{\cos x}$ is

$$\begin{aligned}\frac{1}{\cos x} &= \frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots} = \frac{1}{1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)} \\ &= 1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)^2 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)^3 + \dots \\ &= 1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right) + \frac{x^4}{(2!)^2} + \dots \\ &= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots \text{ (up to } x^4\text{)}\end{aligned}$$

□

Taylor's Theorem for functions of two variables

Taylor's Theorem can be generalized to a function of two variables $f(x, y)$ with continuous partial derivatives of all orders up to N inside the rectangle with vertices $(a - p, b - q)$, $(a - p, b + q)$, $(a + p, b - q)$, $(a + p, b + q)$, where $p, q > 0$.

At the point $(a + h, b + k)$, where $a - p < a + h < a + p$, $b - q < b + k < b + q$, we have

$$\begin{aligned}
 f(a + h, b + k) &= f(a, b) + \frac{1}{1!} [hf_x(a, b) + kf_y(a, b)] \\
 &\quad + \frac{1}{2!} [h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b)] \\
 &\quad + \frac{1}{3!} [h^3 f_{xxx}(a, b) + 3h^2 k f_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)] \\
 &\quad + \cdots + R_N(h, k) \\
 &= \sum_{r=0}^{N-1} \frac{1}{r!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^r f(a, b) + R_N(h, k)
 \end{aligned}$$

where the remainder term $R_N(h, k)$ is given by

$$\begin{aligned}
R_N(h, k) &= \frac{1}{N!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^N f(a + \theta h, b + \theta k) \\
&= \frac{1}{N!} \left[\sum_{i=0}^N \binom{N}{i} h^i k^{N-i} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a + \theta h, b + \theta k) \right] \\
&= \frac{1}{N!} \left[\sum_{i=0}^N \frac{N!}{i! (N-i)!} h^i k^{N-i} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a + \theta h, b + \theta k) \right] \\
&= \sum_{i=0}^N \frac{h^i k^{N-i}}{i! (N-i)!} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a + \theta h, b + \theta k)
\end{aligned}$$

for some $0 < \theta < 1$.

If $R_N(h, k) \rightarrow 0$ as $N \rightarrow \infty$, we have the **Taylor Series** representation of $f(a + h, b + k)$.

Alternatively, by putting $x = a + h$ and $y = b + k$, we have the equivalent form:

$$\begin{aligned}
 f(x, y) &= f(a, b) + \frac{1}{1!} [(x - a)f_x(a, b) + (y - b)f_y(a, b)] \\
 &\quad + \frac{1}{2!} [(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] \\
 &\quad + \cdots + R_N(x, y) \\
 &= \sum_{r=0}^{N-1} \frac{1}{r!} \left((x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y} \right)^r f(a, b) + R_N(x, y)
 \end{aligned}$$

$$\begin{aligned}
 \text{where } R_N(x, y) &= \frac{1}{N!} \left[(x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y} \right]^N f[a + \theta(x - a), b + \theta(y - b)] \\
 &= \frac{1}{N!} \left[\sum_{i=0}^N \binom{N}{i} (x - a)^i (y - b)^{N-i} \frac{\partial^N f}{\partial x^i \partial y^{N-i}} (a + \theta(x - a), b + \theta(y - b)) \right] \\
 &= \sum_{i=0}^N \frac{(x - a)^i (y - b)^{N-i}}{i! (N - i)!} \frac{\partial^N f}{\partial x^i \partial y^{N-i}} (a + \theta(x - a), b + \theta(y - b))
 \end{aligned}$$

for some $0 < \theta < 1$.

If $R_N(x, y) \rightarrow 0$ as $N \rightarrow \infty$, we have the **Taylor Series** representation of $f(x, y)$.

The **equation of the tangent plane to the surface** $z = f(x, y)$ at the point $P(a, b, c)$ where $c = f(a, b)$ is

$$z = f(a, b) + [(x - a)f_x(a, b) + (y - b)f_y(a, b)].$$

Example 30

Find the tangent plane to the surface $z = x \cos y - ye^x$ at $(x, y, z) = (0, 0, 0)$.

Solution

Let $f(x, y) = x \cos y - ye^x$.

Then $f_x = \cos y - ye^x$ and $f_y = -x \sin y - e^x$.

Substitute $x = 0$ and $y = 0$ into f , f_x and f_y :

$$f(0, 0) = 0 - 0 = 0, \quad f_x(0, 0) = (\cos 0) - 0 = 1, \quad f_y(0, 0) = 0 - e^0 = -1$$

$\therefore$ The **tangent plane to the surface** at $(0, 0, 0)$ is

$$z = f(0, 0) + [(x - 0)f_x(0, 0) + (y - 0)f_y(0, 0)] = 0 + x \cdot 1 + y \cdot (-1) = x - y$$

i.e. $z = x - y$. □

Successive terms given by Taylor's Theorem give the **tangent plane approximation** to the surface $z = f(x, y)$ (by taking the **first two terms** of the series), a **quadratic surface approximation** (by taking the **first three terms** of the series), a **cubic surface approximation** (by taking the **first four terms** of the series) etc., all meeting $z = f(x, y)$ at the point (a, b) .

Example 31

Find the tangent plane and quadratic surface approximations given by Taylor's Theorem at the point $(1, 0)$ for the cone $f(x, y) = (x^2 + y^2)^{\frac{1}{2}}$. Hence estimate $f(0.9, 0.1)$.

Solution

$$f_x = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}$$

$$f_y = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}} \cdot 2y = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}}$$

$$\begin{aligned}
 f_{xx} &= \frac{\partial}{\partial x} \left[x(x^2 + y^2)^{-\frac{1}{2}} \right] = x \cdot \frac{\partial}{\partial x} \left[(x^2 + y^2)^{-\frac{1}{2}} \right] + (x^2 + y^2)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial x} (x) \\
 &= x \cdot \left[-\frac{1}{2} (x^2 + y^2)^{-\frac{3}{2}} \cdot 2x \right] + (x^2 + y^2)^{-\frac{1}{2}} = \frac{-x^2 + (x^2 + y^2)}{(x^2 + y^2)^{\frac{3}{2}}} = \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}}
 \end{aligned}$$

$$f_{xy} = \frac{\partial}{\partial y} \left[x(x^2 + y^2)^{-\frac{1}{2}} \right] = x \cdot \frac{\partial}{\partial y} \left[(x^2 + y^2)^{-\frac{1}{2}} \right] = x \cdot \left[-\frac{1}{2} (x^2 + y^2)^{-\frac{3}{2}} \cdot 2y \right] = \frac{-xy}{(x^2 + y^2)^{\frac{3}{2}}}$$

$$\begin{aligned}
 f_{yy} &= \frac{\partial}{\partial y} \left[y(x^2 + y^2)^{-\frac{1}{2}} \right] = y \cdot \frac{\partial}{\partial y} \left[(x^2 + y^2)^{-\frac{1}{2}} \right] + (x^2 + y^2)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial y} (y) \\
 &= y \cdot \left[-\frac{1}{2} (x^2 + y^2)^{-\frac{3}{2}} \cdot 2y \right] + (x^2 + y^2)^{-\frac{1}{2}} = \frac{-y^2 + (x^2 + y^2)}{(x^2 + y^2)^{\frac{3}{2}}} = \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}}
 \end{aligned}$$

which exist everywhere except at the origin $(0, 0)$.

Then evaluate f , f_x , f_y , f_{xx} , f_{xy} , f_{yy} at the point $(1, 0)$.

$$f(1, 0) = (1^2 + 0^2)^{\frac{1}{2}} = 1, \quad f_x(1, 0) = \frac{1}{(1^2 + 0^2)^{\frac{1}{2}}} = 1, \quad f_y(1, 0) = \frac{0}{(1^2 + 0^2)^{\frac{1}{2}}} = 0,$$

$$f_{xx}(1, 0) = \frac{0^2}{(1^2 + 0^2)^{\frac{3}{2}}} = 0, \quad f_{xy}(1, 0) = \frac{-(1)(0)}{(1^2 + 0^2)^{\frac{3}{2}}} = 0, \quad f_{yy}(1, 0) = \frac{1^2}{(1^2 + 0^2)^{\frac{3}{2}}} = 1.$$

The **tangent plane approximation** to the surface $z = f(x, y)$ at the point $(1, 0)$ is:

$$\begin{aligned}p_1(x, y) &= f(1, 0) + \frac{1}{1!} [(x - 1)f_x(1, 0) + (y - 0)f_y(1, 0)] \\&= 1 + [(x - 1) \times 1 + y \times 0] \\&= x\end{aligned}$$

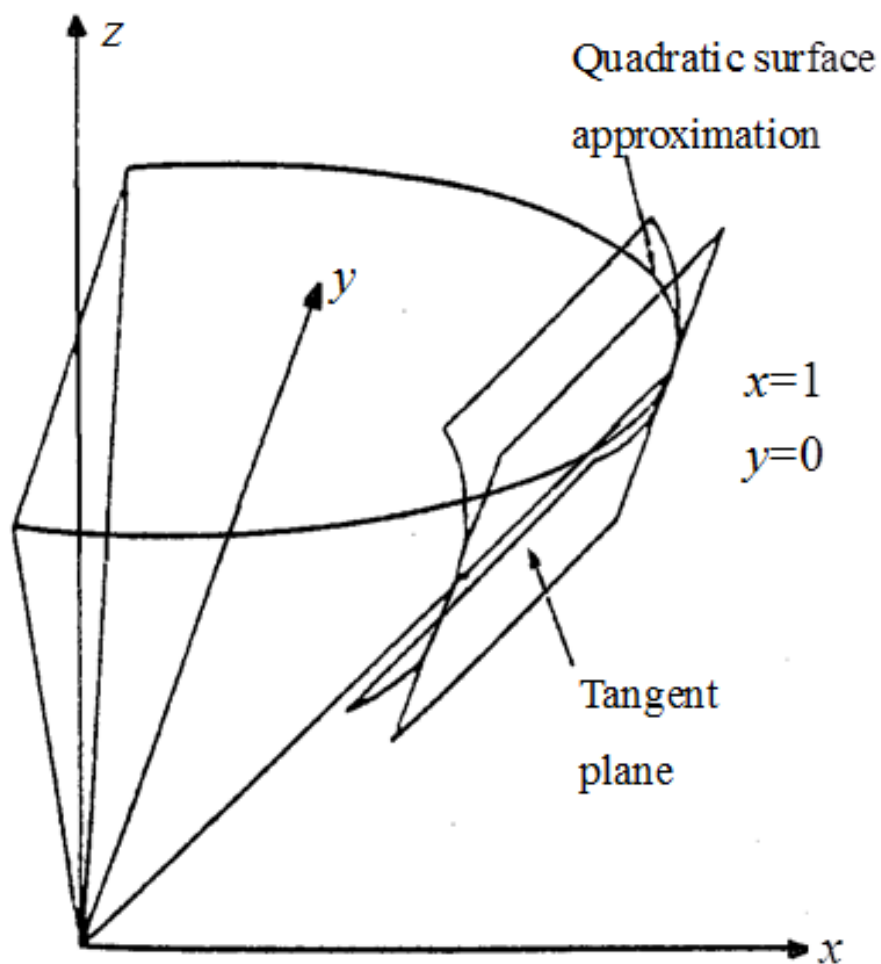
The **quadratic surface approximation** to the surface $z = f(x, y)$ at the point $(1, 0)$ is:

$$\begin{aligned}p_2(x, y) &= f(1, 0) + \frac{1}{1!} [(x - 1)f_x(1, 0) + (y - 0)f_y(1, 0)] \\&\quad + \frac{1}{2!} [(x - 1)^2 f_{xx}(1, 0) + 2(x - 1)(y - 0)f_{xy}(1, 0) + (y - 0)^2 f_{yy}(1, 0)] \\&= 1 + [(x - 1) \times 1 + y \times 0] + \frac{1}{2} [(x - 1)^2 \times 0 + 2(x - 1)y \times 0 + y^2 \times 1] \\&= x + \frac{1}{2} y^2\end{aligned}$$

Hence, using the **tangent plane**, $f(0.9, 0.1) \approx p_1(0.9, 0.1) = 0.9$

and using the **quadratic surface**, $f(0.9, 0.1) \approx p_2(0.9, 0.1) = 0.9 + \frac{1}{2} \times 0.1^2 = 0.905$.

Note: The true value is $f(0.9, 0.1) = 0.9055$ (to 4 significant figures).



Linear and quadratic approximations to the cone

$$z = (x^2 + y^2)^{\frac{1}{2}} \text{ near } (x, y) = (1, 0)$$



Example 32

$z^3 - 2xz + y = 0$ determines an implicit function $z = z(x, y)$ of x and y about the point $(1, 1)$. Use the Taylor's Theorem to expand $z = z(x, y)$ about the point $(1, 1, 1)$ up to the linear terms.

Solution

Partially differentiate both sides of $z^3 - 2xz + y = 0$ w.r.t. x :

$$3z^2 \frac{\partial z}{\partial x} - 2z - 2x \frac{\partial z}{\partial x} + 0 = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{2z}{3z^2 - 2x}$$

Thus,

$$\left. \frac{\partial z}{\partial x} \right|_{\substack{x=1 \\ y=1 \\ z=1}} = \frac{2(1)}{3(1)^2 - 2(1)} = 2$$

Partially differentiate both sides of $z^3 - 2xz + y = 0$ w.r.t. y :

$$3z^2 \frac{\partial z}{\partial y} - 2x \frac{\partial z}{\partial y} + 1 = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{1}{2x - 3z^2}$$

Thus,

$$\left. \frac{\partial z}{\partial y} \right|_{\substack{x=1 \\ y=1 \\ z=1}} = \frac{1}{2(1) - 3(1)^2} = -1$$

Hence,

$$\begin{aligned} z(x, y) &= z(1, 1) + \left[\left. \frac{\partial z}{\partial x} \right|_{\substack{x=1 \\ y=1 \\ z=1}} (x - 1) + \left. \frac{\partial z}{\partial y} \right|_{\substack{x=1 \\ y=1 \\ z=1}} (y - 1) \right] + \dots \\ &= 1 + 2(x - 1) - 1(y - 1) + \dots \end{aligned}$$

□

Example 33 (Taylor series of a function in 3 variables)

Let $f(x)$ be a smooth function, that is, $f(x)$ has continuous derivatives up to any order.

Then the Taylor series of $f(x)$ about the point a up to order 2 is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots$$

Hence, or otherwise, find the Taylor series of

$$g(x, y, z) = \cos(x + y + z) - \cos x \cos y \cos z$$

about the point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$ up to order 2.

Solution

$$\cos(x + y + z) = \cos x \cos(y + z) - \sin x \sin(y + z)$$

$$= \cos x (\cos y \cos z - \sin y \sin z) - \sin x (\sin y \cos z + \cos y \sin z)$$

$$= \cos x \cos y \cos z - \cos x \sin y \sin z - \sin x \sin y \cos z - \sin x \cos y \sin z$$

$$\therefore g(x, y, z) = \cos(x + y + z) - \cos x \cos y \cos z$$

$$= -\cos x \sin y \sin z - \sin x \sin y \cos z - \sin x \cos y \sin z \dots (*)$$

We know that the Maclaurin series of $\cos x$ and $\sin x$ are

$$\cos x = \cos 0 + \frac{(-\sin 0)}{1!}x + \frac{(-\cos 0)}{2!}x^2 + \dots = 1 - \frac{x^2}{2} + \dots \quad (\text{up to order 2})$$

and

$$\sin x = \sin 0 + \frac{\cos 0}{1!}x + \frac{(-\sin 0)}{2!}x^2 + \dots = x - \dots \quad (\text{up to order 2}).$$

The Taylor series of $\cos y$ and $\sin y$ at $y = \frac{\pi}{2}$ are

$$\cos y = \cos \frac{\pi}{2} + \frac{\left(-\sin \frac{\pi}{2}\right)}{1!}\left(y - \frac{\pi}{2}\right) + \frac{\left(-\cos \frac{\pi}{2}\right)}{2!}\left(y - \frac{\pi}{2}\right)^2 + \dots = -\left(y - \frac{\pi}{2}\right) + \dots$$

and

$$\sin y = \sin \frac{\pi}{2} + \frac{\left(\cos \frac{\pi}{2}\right)}{1!}\left(y - \frac{\pi}{2}\right) + \frac{\left(-\sin \frac{\pi}{2}\right)}{2!}\left(y - \frac{\pi}{2}\right)^2 + \dots = 1 - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \dots$$

Putting these into (*), we obtain the Taylor series of $g(x, y, z)$ about the point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$\begin{aligned}
 g(x, y, z) &= -\cos x \sin y \sin z - \sin x \sin y \cos z - \sin x \cos y \sin z \\
 &= -\left(1 - \frac{x^2}{2} + \dots\right) \left[1 - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \dots\right] \left[1 - \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2 + \dots\right] \\
 &\quad - (x - \dots) \left[1 - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \dots\right] \left[-\left(z - \frac{\pi}{2}\right) + \dots\right] \\
 &\quad - (x - \dots) \left[-\left(y - \frac{\pi}{2}\right) + \dots\right] \left[1 - \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2 + \dots\right] \\
 &= -\left[1 - \frac{x^2}{2} - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 - \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2\right] - \left[-x\left(z - \frac{\pi}{2}\right)\right] - \left[-x\left(y - \frac{\pi}{2}\right)\right] + \dots \\
 &= -1 + \frac{x^2}{2} + \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2 + x\left(z - \frac{\pi}{2}\right) + x\left(y - \frac{\pi}{2}\right) + \dots
 \end{aligned}$$

(up to order 2)

□

Remark:

We can also derive the Taylor series of $g(x, y, z)$ about $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$ by using the formula

$$\begin{aligned} g(x, y, z) = & g\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) \\ & + \frac{1}{1!} \left[(x - 0) g_x\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right) g_y\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) + \left(z - \frac{\pi}{2}\right) g_z\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) \right] \\ & + \frac{1}{2!} \left[(x - 0)^2 g_{xx}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 g_{yy}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) + \left(z - \frac{\pi}{2}\right)^2 g_{zz}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) \right. \\ & \quad + 2(x - 0) \left(y - \frac{\pi}{2}\right) g_{xy}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) + 2(x - 0) \left(z - \frac{\pi}{2}\right) g_{xz}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) \\ & \quad \left. + 2 \left(y - \frac{\pi}{2}\right) \left(z - \frac{\pi}{2}\right) g_{yz}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) \right] \\ & + \dots \end{aligned}$$

A bound for the error involved in approximating a function near a point (a, b) may be obtained by summing over the estimates of the moduli of the terms in $R_N(x, y)$.

Example 34

- (a) Use the Taylor's formula to find a linear approximation of $f(x, y) = \cos x \cos y$ near the origin.
- (b) Estimate the error in the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

Solution

- (a) By the Taylor's Theorem, we have

$$\begin{aligned} f(x, y) &= f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)] \\ &\quad + \frac{1}{2} [x^2 f_{xx}(\theta x, \theta y) + 2xy f_{xy}(\theta x, \theta y) + y^2 f_{yy}(\theta x, \theta y)], \end{aligned}$$

where $0 < \theta < 1$.

Then the linear approximation of $f(x, y) = \cos x \cos y$ near the origin is

$$p_1(x, y) = f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)].$$

$$f(x, y) = \cos x \cos y, \quad f(0, 0) = \cos x \cos y \Big|_{\substack{x=0 \\ y=0}} = 1$$

$$f_x(x, y) = -\sin x \cos y, \quad f_x(0, 0) = -\sin x \cos y \Big|_{\substack{x=0 \\ y=0}} = 0$$

$$f_y(x, y) = -\cos x \sin y, \quad f_y(0, 0) = -\cos x \sin y \Big|_{\substack{x=0 \\ y=0}} = 0$$

Thus, $p_1(x, y) = 1$.

If (x, y) is close to $(0, 0)$, $f(x, y) = \cos x \cos y \approx p_1(x, y) = 1$.

(b) The error in the approximation is

$$\begin{aligned} E(x, y) &= f(x, y) - f(0, 0) - [x f_x(0, 0) + y f_y(0, 0)] \\ &= \frac{1}{2} [x^2 f_{xx}(\theta x, \theta y) + 2xy f_{xy}(\theta x, \theta y) + y^2 f_{yy}(\theta x, \theta y)] \end{aligned}$$

for some $0 < \theta < 1$.

$$f_{xx}(x, y) = -\cos x \cos y, \quad f_{yy}(x, y) = -\cos x \cos y, \quad f_{xy}(x, y) = \sin x \sin y$$

$$\begin{aligned}
|E(x, y)| &= \frac{1}{2} |x^2 f_{xx}(\theta x, \theta y) + 2xy f_{xy}(\theta x, \theta y) + y^2 f_{yy}(\theta x, \theta y)| \\
&\leq \frac{1}{2} (|x^2 f_{xx}(\theta x, \theta y)| + |2xy f_{xy}(\theta x, \theta y)| + |y^2 f_{yy}(\theta x, \theta y)|) \\
&\quad \because |a + b| \leq |a| + |b| \text{ for all } a, b \in \mathbb{R} \\
&= \frac{1}{2} (|x|^2 |f_{xx}(\theta x, \theta y)| + 2|x||y| |f_{xy}(\theta x, \theta y)| + |y|^2 |f_{yy}(\theta x, \theta y)|) \\
&= \frac{1}{2} (|x|^2 |-\cos \theta x \cos \theta y| + 2|x||y| |\sin \theta x \sin \theta y| + |y|^2 |-\cos \theta x \cos \theta y|) \\
&= \frac{1}{2} (|x|^2 |\cos \theta x| |\cos \theta y| + 2|x||y| |\sin \theta x| |\sin \theta y| + |y|^2 |\cos \theta x| |\cos \theta y|) \\
&\stackrel{\substack{|\cos \theta x| \leq 1 \\ |\sin \theta x| \leq 1}}{\leq} \frac{1}{2} (|x|^2 + 2|x||y| + |y|^2) \\
&\stackrel{\substack{|\cos \theta x| \leq 1 \\ |\sin \theta x| \leq 1}}{\leq} \frac{1}{2} [(0.1)^2 + 2(0.1)(0.1) + (0.1)^2] \\
&= 0.02
\end{aligned}$$

Example 35

- (a) Use Taylor's formula to find a quadratic approximation of $f(x, y) = e^x \sin y$ at the origin.
- (b) Show that in the approximation, the error $|E(x, y)| < 1.474 \times 10^{-3}$ if $|x| \leq 0.1$, $|y| \leq 0.1$ and $e^{0.1} \approx 1.1052$.
- (c) Find an approximation of $e^{-0.09} \sin 0.06$ with an error less than 1.474×10^{-3} .

Solution

- (a) By Taylor's Theorem, we have

$$\begin{aligned} f(x, y) &= f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)] \\ &\quad + \frac{1}{2!} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] \\ &\quad + \frac{1}{3!} [x^3 f_{xxx}(\theta x, \theta y) + 3x^2 y f_{xxy}(\theta x, \theta y) + 3xy^2 f_{xyy}(\theta x, \theta y) + y^3 f_{yyy}(\theta x, \theta y)] \end{aligned}$$

where $0 < \theta < 1$.

Then the quadratic approximation of $f(x, y) = e^x \sin y$ at the origin is

$$p_2(x, y) = f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)] + \frac{1}{2} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)].$$

$$f(x, y) = e^x \sin y, \quad f(0, 0) = e^x \sin y \Big|_{\substack{x=0 \\ y=0}} = 0$$

$$f_x(x, y) = e^x \sin y, \quad f_x(0, 0) = e^x \sin y \Big|_{\substack{x=0 \\ y=0}} = 0$$

$$f_y(x, y) = e^x \cos y, \quad f_y(0, 0) = e^x \cos y \Big|_{\substack{x=0 \\ y=0}} = 1$$

$$f_{xx}(x, y) = e^x \sin y, \quad f_{xx}(0, 0) = e^x \sin y \Big|_{\substack{x=0 \\ y=0}} = 0$$

$$f_{xy}(x, y) = e^x \cos y, \quad f_{xy}(0, 0) = e^x \cos y \Big|_{\substack{x=0 \\ y=0}} = 1$$

$$f_{yy}(x, y) = -e^x \sin y, \quad f_{yy}(0, 0) = -e^x \sin y \Big|_{\substack{x=0 \\ y=0}} = 0$$

Thus, $p_2(x, y) = y + xy$.

If (x, y) is close to $(0, 0)$, $f(x, y) = e^x \sin y \approx p_2(x, y) = y + xy$.

(b) The error in the approximation is

$$E(x, y) = f(x, y) - p_2(x, y) \\ = \frac{1}{6} [x^3 f_{xxx}(\theta x, \theta y) + 3x^2 y f_{xxy}(\theta x, \theta y) + 3xy^2 f_{xyy}(\theta x, \theta y) + y^3 f_{yyy}(\theta x, \theta y)]$$

for some $0 < \theta < 1$.

$$f_{xxx}(x, y) = e^x \sin y, \quad f_{xxy}(x, y) = e^x \cos y, \\ f_{xyy}(x, y) = -e^x \sin y, \quad f_{yyy}(x, y) = -e^x \cos y$$

$$|E(x, y)| = |f(x, y) - p_2(x, y)| \\ = \frac{1}{6} |x^3 f_{xxx}(\theta x, \theta y) + 3x^2 y f_{xxy}(\theta x, \theta y) + 3xy^2 f_{xyy}(\theta x, \theta y) + y^3 f_{yyy}(\theta x, \theta y)| \\ \leq \frac{1}{6} [|x|^3 |f_{xxx}(\theta x, \theta y)| + 3|x|^2 |y| |f_{xxy}(\theta x, \theta y)| + 3|x| |y|^2 |f_{xyy}(\theta x, \theta y)| \\ + |y|^3 |f_{yyy}(\theta x, \theta y)|]$$

($\because |a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$ (triangle inequality))

$$\begin{aligned}
&= \frac{1}{6} \left[|x|^3 |e^{\theta x} \sin \theta y| + 3|x|^2 |y| |e^{\theta x} \cos \theta y| + 3|x| |y|^2 |-e^{\theta x} \sin \theta y| \right. \\
&\quad \left. + |y|^3 |-e^{\theta x} \cos \theta y| \right] \\
&= \frac{1}{6} \left[|x|^3 |e^{\theta x}| |\sin \theta y| + 3|x|^2 |y| |e^{\theta x}| |\cos \theta y| + 3|x| |y|^2 |e^{\theta x}| |\sin \theta y| \right. \\
&\quad \left. + |y|^3 |e^{\theta x}| |\cos \theta y| \right]
\end{aligned}$$

Since $0 < \theta < 1$ and $-0.1 \leq x \leq 0.1$, we have $-0.1 < \theta x < 0.1 \Rightarrow |\theta x| < 0.1$.

Then $|\sin \theta y| \leq 1$, $|\cos \theta y| \leq 1$, $|e^{\theta x}| = e^{\theta x} \leq e^{|\theta x|} < e^{0.1}$.

$$\begin{aligned}
\therefore |E(x, y)| &< \frac{1}{6} [(0.1)^3 e^{0.1} + 3(0.1)^3 e^{0.1} + 3(0.1)^3 e^{0.1} + (0.1)^3 e^{0.1}] \\
&= \frac{0.008 e^{0.1}}{6} \\
&\approx \frac{0.008 \times 1.1052}{6} \\
&\approx 1.474 \times 10^{-3}
\end{aligned}$$

(c) From (b), with an error less than 1.474×10^{-3} , the value

$$e^{-0.09} \sin 0.06 = f(-0.09, 0.06)$$

can be approximated by

$$\begin{aligned} p_2(-0.09, 0.06) &= (y + xy)|_{\substack{x=-0.09 \\ y=0.06}} \\ &= 0.06 + (-0.09)(0.06) \\ &= 0.0546 \end{aligned}$$

Estimation of Small Changes

For (x, y) near (a, b) , we may expect

$$\begin{aligned} f(x, y) &\approx f(a, b) + (x - a) f_x(a, b) + (y - b) f_y(a, b) \\ \Rightarrow f(x, y) - f(a, b) &\approx (x - a) f_x(a, b) + (y - b) f_y(a, b) \end{aligned}$$

Writing $f(x, y) - f(a, b) = \delta f$, $x - a = \delta x$, $y - b = \delta y$, we have

$$\delta f \approx \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y,$$

which gives the approximate change in f due to small changes δx and δy in x and y , respectively. (This result will be used in Section 8 (Directional derivative and gradient).)

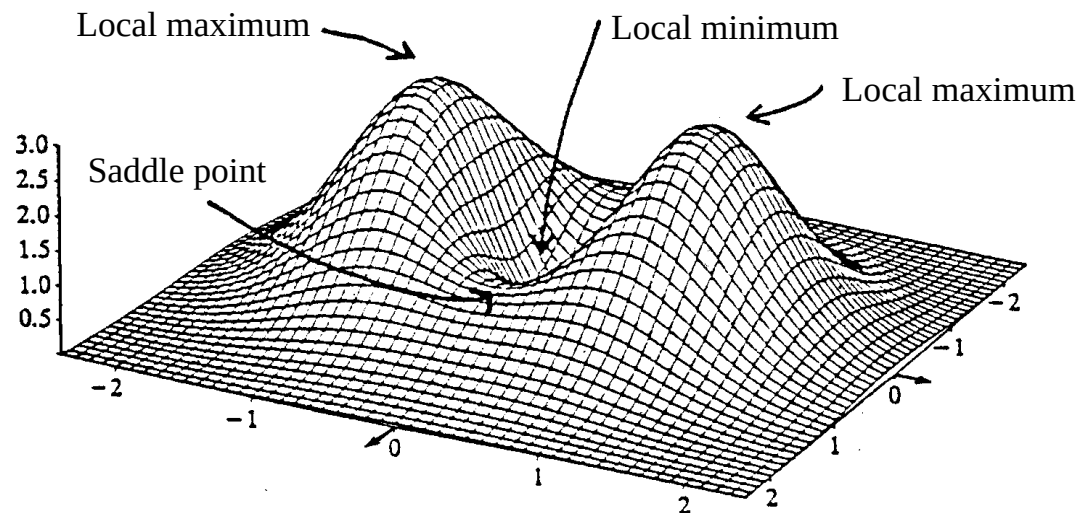
□

7 Maxima and Minima

A function $f(x, y)$ has a **local maximum** at a point (a, b) if for all points (x, y) close to (a, b) , we have $f(x, y) \leq f(a, b)$. The point (a, b) is called a local maximum point.

A function $f(x, y)$ has a **local minimum** at a point (a, b) if for all points (x, y) close to (a, b) , we have $f(x, y) \geq f(a, b)$. The point (a, b) is called a local minimum point.

In a region, the **global or absolute maximum (minimum)** of a function is either the greatest (least) of the local maxima (minima) or it occurs on the boundary of the region.



$$z = f(x, y) = (x^2 + 3y^2)e^{1-x^2-y^2}$$

For a local maximum or minimum at (a, b) , the curves lying in the two planes $x = a$ and $y = b$ must also have a maximum or minimum at (a, b) . Hence we must have

$$\boxed{\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0}.$$

If this condition holds, we say that (a, b) is a **stationary point**.

It is possible to have a stationary point which is neither a local maximum nor a local minimum. If along one path through (a, b) the point is a local minimum, while along a different path it is a local maximum, we say that the point is a **saddle point**. In other words, the stationary point (a, b) is a saddle point of f if in every open region containing (a, b) , there exist a point (x_0, y_0) with $f(x_0, y_0) > f(a, b)$ and a point (x_1, y_1) with $f(x_1, y_1) < f(a, b)$.

Theorem (Second Partial Derivative Test)

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that

$$f_x(a, b) = f_y(a, b) = 0.$$

That is, (a, b) is a **stationary point** of f .

- If $f_{xx}(a, b) > 0$ and $f_{xx}(a, b) f_{yy}(a, b) - f_{xy}^2(a, b) > 0$, then $f(x, y)$ has a **local minimum** at (a, b) .
- If $f_{xx}(a, b) < 0$ and $f_{xx}(a, b) f_{yy}(a, b) - f_{xy}^2(a, b) > 0$, then $f(x, y)$ has a **local maximum** at (a, b) .
- If $f_{xx}(a, b) f_{yy}(a, b) - f_{xy}^2(a, b) < 0$, then $f(x, y)$ has a **saddle point** at (a, b) .
- If $f_{xx}(a, b) f_{yy}(a, b) - f_{xy}^2(a, b) = 0$, **the test is inconclusive**. We must find some other way to determine the nature of $f(x, y)$ at (a, b) .

Illustrations

(i) Consider the function $f(x, y) = x^2 + 2y^2$.

To find the stationary point(s):

$$f_x(x, y) = 2x, \quad f_y(x, y) = 4y$$

$$\text{Set } \begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases} \Rightarrow \begin{cases} 2x = 0 \\ 4y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$\therefore (x, y) = (0, 0)$ is the only stationary point.

To determine the nature of the stationary point(s):

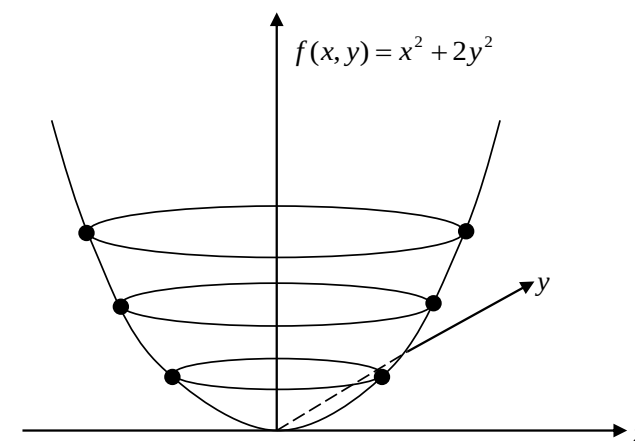
$$f_{xx}(x, y) = 2, \quad f_{yy}(x, y) = 4, \quad f_{xy}(x, y) = 0$$

Since $f_{xx}(0, 0) = 2 > 0$ and $f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = 8 > 0$,

f has a **local minimum** at $(0, 0)$, and the value of f at this point is $f(0, 0) = 0$.

Clearly, $f(x, y) \geq 0$ for all x and y , hence $(0, 0)$ is a **global minimum** point and the global minimum value of f is $f(0, 0) = 0$.

The graph of $f(x, y) = x^2 + 2y^2$ is shown on the right.



(ii) Consider the function $f(x, y) = 2x - x^2 - 2y^2$.

To find the stationary point(s):

$$f_x(x, y) = 2 - 2x, \quad f_y(x, y) = -4y$$

$$\text{Set } \begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases} \Rightarrow \begin{cases} 2 - 2x = 0 \\ -4y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \end{cases}$$

$\therefore (x, y) = (1, 0)$ is the only stationary point.

To determine the nature of the stationary point(s):

$$f_{xx}(x, y) = -2, \quad f_{yy}(x, y) = -4, \quad f_{xy}(x, y) = 0$$

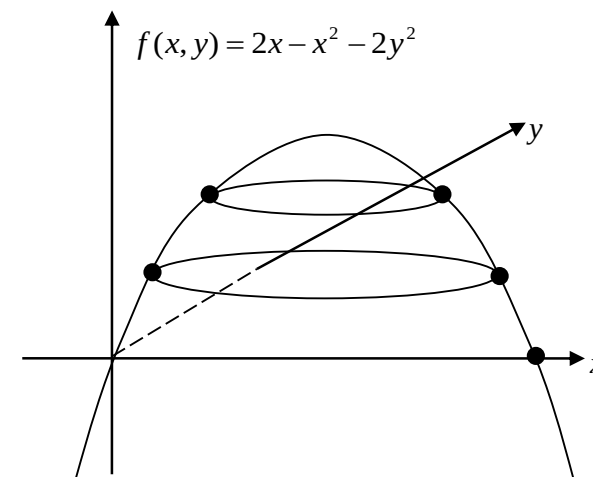
Since $f_{xx}(1, 0) = -2 < 0$ and $f_{xx}(1, 0) f_{yy}(1, 0) - f_{xy}^2(1, 0) = 8 > 0$,

f has a **local maximum** at $(1, 0)$, and the value of f at this point is $f(1, 0) = 1$.

In addition, for all $x, y \in \mathbb{R}$,

$$f(x, y) - f(1, 0) = 2x - x^2 - 2y^2 - 1 = -(x - 1)^2 - 2y^2 \leq 0.$$

Thus, $f(x, y) \leq f(1, 0)$ for all $x, y \in \mathbb{R}$ and therefore $(1, 0)$ is a **global maximum** point and the global maximum value of f is $f(1, 0) = 1$.



(iii) Consider the function $f(x, y) = x^2 - 2y^2$.

To find the stationary point(s):

$$f_x(x, y) = 2x, \quad f_y(x, y) = -4y$$

$$\text{Set } \begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases} \Rightarrow \begin{cases} 2x = 0 \\ -4y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$\therefore (x, y) = (0, 0)$ is the only stationary point.

To determine the nature of the stationary point(s):

$$f_{xx}(x, y) = 2, \quad f_{yy}(x, y) = -4, \quad f_{xy}(x, y) = 0$$

Since $f_{xx}(0, 0) f_{yy}(0, 0) - f_{xy}^2(0, 0) = -8 < 0$, then f has a **saddle point** at $(0, 0)$.

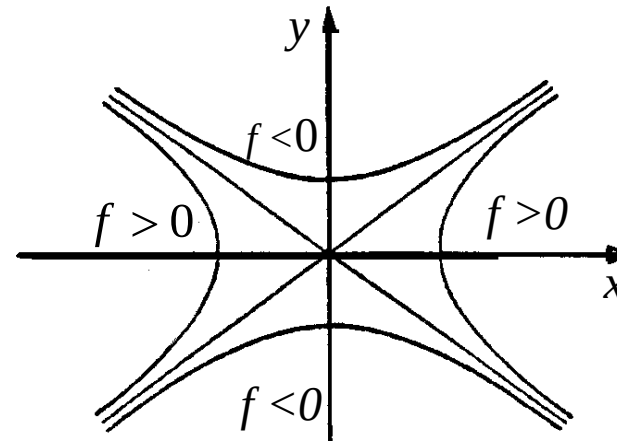
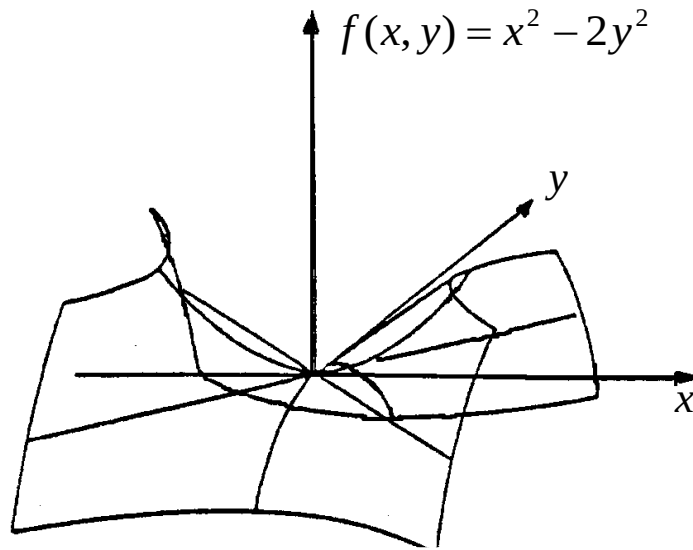
$(0, 0)$ being a saddle point can also be observed by noting that $f(x, y)$ can be either positive or negative depending on the relative magnitudes of $|x|$ and $|y|$.

If $y = 0$ and $x \neq 0$, then $f(x, 0) - f(0, 0) = x^2 > 0 \Rightarrow f(x, 0) > f(0, 0)$.

If $x = 0$ and $y \neq 0$, then $f(0, y) - f(0, 0) = -2y^2 < 0 \Rightarrow f(0, y) < f(0, 0)$.

Thus, $(0, 0)$ is a saddle point.

The graph of $f(x, y) = x^2 - 2y^2$ is shown below:



$$f > 0 \text{ if } |x| > \sqrt{2}|y|$$

$$f < 0 \text{ if } |x| < \sqrt{2}|y|$$

(iv) Consider the function $f(x, y) = x^4 + y^4$.

Then $f_x = 4x^3$ and $f_y = 4y^3$.

$$\text{Set } \begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases} \Rightarrow \begin{cases} 4x^3 = 0 \\ 4y^3 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$\therefore (x, y) = (0, 0)$ is the only stationary point.

$$f_{xx}(x, y) = 12x^2, \quad f_{yy}(x, y) = 12y^2, \quad f_{xy}(x, y) = 0$$

$f_{xx}(0, 0) f_{yy}(0, 0) - f_{xy}^2(0, 0) = 0$, the **test is inconclusive**.

However, $f(x, y) = x^4 + y^4 \geq 0 = f(0, 0)$ for all $x, y \in \mathbb{R}$, so clearly $(0, 0)$ is a local minimum point. In fact, $(0, 0)$ is also a global minimum point.

Example 36

Find and classify the stationary points of the function $f(x, y) = x^3 + 6xy + 3y^2$.

Solution

$$f_x = 3x^2 + 6y, \quad f_y = 6x + 6y$$

$$\text{Set } \begin{cases} f_x = 3x^2 + 6y = 0 \\ f_y = 6x + 6y = 0 \end{cases} \Rightarrow \begin{cases} x^2 + 2y = 0 \\ x = -y \end{cases} \Rightarrow \begin{cases} y^2 + 2y = 0 \\ x = -y \end{cases} \Rightarrow \begin{cases} y(y + 2) = 0 \\ x = -y \end{cases}$$

$$\Rightarrow \begin{cases} y = 0 \\ x = -y = 0 \end{cases} \quad \text{or} \quad \begin{cases} y = -2 \\ x = -y = 2 \end{cases}$$

$\therefore (0, 0)$ and $(2, -2)$ are the **stationary points**.

$$f_{xx} = 6x, \quad f_{yy} = 6, \quad f_{xy} = 6 = f_{yx}. \quad \text{Let } \Delta(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 36x - 36.$$

For $(0, 0)$, we have $\Delta(0, 0) = -36 < 0$.

$\therefore f$ has a **saddle point** at $(0, 0)$.

For $(2, -2)$, we have $f_{xx}(2, -2) = 12 > 0$ and $\Delta(2, -2) = 36 > 0$.

$\therefore f$ has a **local minimum** at $(2, -2)$.

Example 37

Determine the nature of the stationary points of $f(x, y) = x^3 + y^3 - 3(x + y)$.

Solution

$$f_x = 3x^2 - 3, \quad f_y = 3y^2 - 3$$

$$\text{Set } \begin{cases} f_x = 3x^2 - 3 = 0 \\ f_y = 3y^2 - 3 = 0 \end{cases} \Rightarrow \begin{cases} x^2 = 1 \\ y^2 = 1 \end{cases} \Rightarrow \begin{cases} x = \pm 1 \\ y = \pm 1 \end{cases}$$

Hence there are 4 stationary points:

$$(1, 1), (-1, -1), (1, -1), (-1, 1).$$

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = f_{yx} = 0$$

Stationary Point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - (f_{xy})^2$	Conclusion
(1, 1)	6 (> 0)	6	0	36 (> 0)	Local minimum
(-1, -1)	-6 (< 0)	-6	0	36 (> 0)	Local maximum
(1, -1)	6	-6	0	-36 (< 0)	Saddle point
(-1, 1)	-6	6	0	-36 (< 0)	Saddle point

Example 38

Consider a point $P(x_0, y_0)$ and a straight line $y = ax + b$. The value $[y_0 - (ax_0 + b)]^2$ is called the square of the vertical displacement of the data point $P(x_0, y_0)$ from the line $y = ax + b$.

Determine the straight line $y = ax + b$ such that the sum S of the squares of the vertical displacements of the data points $P_1(1, 3.1)$, $P_2(2, 6.1)$, $P_3(3, 9.2)$ from the line $y = ax + b$ is the smallest.

In other words, determine the values of a and b such that S is the smallest where

$$S = (3.1 - a - b)^2 + (6.1 - 2a - b)^2 + (9.2 - 3a - b)^2 \cdots (*)$$

Solution

$$S(a, b) = (3.1 - a - b)^2 + (6.1 - 2a - b)^2 + (9.2 - 3a - b)^2$$

Partially differentiate both sides of $(*)$ w.r.t. a :

$$\frac{\partial S}{\partial a} = -2(3.1 - a - b) - 4(6.1 - 2a - b) - 6(9.2 - 3a - b)$$

Partially differentiate both sides of (*) w.r.t. b :

$$\frac{\partial S}{\partial b} = -2(3.1 - a - b) - 2(6.1 - 2a - b) - 2(9.2 - 3a - b)$$

Set $\frac{\partial S}{\partial a} = 0$ and $\frac{\partial S}{\partial b} = 0$:

$$\begin{cases} \frac{\partial S}{\partial a} = -2(3.1 - a - b) - 4(6.1 - 2a - b) - 6(9.2 - 3a - b) = 0 \\ \frac{\partial S}{\partial b} = -2(3.1 - a - b) - 2(6.1 - 2a - b) - 2(9.2 - 3a - b) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 28a + 12b = 85.8 \\ 12a + 6b = 36.8 \end{cases}$$

$$\Rightarrow \begin{cases} a = \frac{85.8 - 2 \times 36.8}{4} = 3.05 \\ b = \frac{36.8 - 12 \times 3.05}{6} = \frac{1}{30} \end{cases}$$

$\therefore S$ has a stationary point when $a = 3.05$, $b = \frac{1}{30}$.

Next, we determine whether this stationary point is a local minimum.

$$S_{aa} = 28, \quad S_{bb} = 6, \quad S_{ab} = 12.$$

$$\text{Thus, } S_{aa}\left(3.05, \frac{1}{30}\right) = 28, \quad S_{bb}\left(3.05, \frac{1}{30}\right) = 6, \quad S_{ab}\left(3.05, \frac{1}{30}\right) = 12.$$

Since

$$\Delta\left(3.05, \frac{1}{30}\right) = S_{aa}\left(3.05, \frac{1}{30}\right) S_{bb}\left(3.05, \frac{1}{30}\right) - S_{ab}^2\left(3.05, \frac{1}{30}\right) = 28 \times 6 - 12^2 = 24 > 0$$

and

$$S_{aa}\left(3.05, \frac{1}{30}\right) = 28 > 0,$$

it follows that $S(a, b)$ has a **local minimum** value at $a = 3.05$, $b = \frac{1}{30}$.

Observe that $S(a, b) = (3.1 - a - b)^2 + (6.1 - 2a - b)^2 + (9.2 - 3a - b)^2$ is a degree 2 polynomial in a, b and $\left(3.05, \frac{1}{30}\right)$ is the only local minimum point, thus $S(a, b)$ actually has the global minimum value

$$S\left(3.05, \frac{1}{30}\right) = \left(3.1 - 3.05 - \frac{1}{30}\right)^2 + \left[6.1 - 2(3.05) - \frac{1}{30}\right]^2 + \left[9.2 - 3(3.05) - \frac{1}{30}\right]^2 = \frac{1}{600}$$

at $\left(3.05, \frac{1}{30}\right)$.

This fact can be shown as follows:

Due to the Taylor's expansion theorem and notice that $S(a, b)$ is a degree 2 polynomial in a and b , we have

$$\begin{aligned}
 & S(a, b) - S\left(3.05, \frac{1}{30}\right) \\
 &= \left[\underbrace{S_a\left(3.05, \frac{1}{30}\right)}_{=0} (a - 3.05) + \underbrace{S_b\left(3.05, \frac{1}{30}\right)}_{=0} \left(b - \frac{1}{30}\right) \right] \\
 &+ \frac{1}{2} \left[\underbrace{S_{aa}\left(3.05, \frac{1}{30}\right)}_{=28} (a - 3.05)^2 + 2 \underbrace{S_{ab}\left(3.05, \frac{1}{30}\right)}_{=12} (a - 3.05) \left(b - \frac{1}{30}\right) + \underbrace{S_{bb}\left(3.05, \frac{1}{30}\right)}_{=6} \left(b - \frac{1}{30}\right)^2 \right] \\
 &= \frac{1}{2} \left[28(a - 3.05)^2 + 2(12)(a - 3.05) \left(b - \frac{1}{30}\right) + 6 \left(b - \frac{1}{30}\right)^2 \right] \\
 &= \frac{1}{2} \left[\begin{pmatrix} a - 3.05 & b - \frac{1}{30} \end{pmatrix} \begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix} \begin{pmatrix} a - 3.05 \\ b - \frac{1}{30} \end{pmatrix} \right]
 \end{aligned}$$

Observe that $\begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix}$ is a 2×2 real symmetric matrix which has eigenvalues $17 \pm \sqrt{265} (> 0)$. Then there exists a 2×2 orthogonal matrix P and the diagonal matrix $D = \begin{pmatrix} 17 + \sqrt{265} & 0 \\ 0 & 17 - \sqrt{265} \end{pmatrix}$ such that

$$\begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix} = P \begin{pmatrix} 17 + \sqrt{265} & 0 \\ 0 & 17 - \sqrt{265} \end{pmatrix} P^{-1} \underset{P^{-1}=P^T}{=} P \begin{pmatrix} 17 + \sqrt{265} & 0 \\ 0 & 17 - \sqrt{265} \end{pmatrix} P^T.$$

$$\text{Let } \begin{pmatrix} a - 3.05 \\ b - \frac{1}{30} \end{pmatrix} = P \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P^{-1} \begin{pmatrix} a - 3.05 \\ b - \frac{1}{30} \end{pmatrix} = P^T \begin{pmatrix} a - 3.05 \\ b - \frac{1}{30} \end{pmatrix}.$$

Then

$$\begin{aligned} & \begin{pmatrix} a - 3.05 & b - \frac{1}{30} \end{pmatrix} \begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix} \begin{pmatrix} a - 3.05 \\ b - \frac{1}{30} \end{pmatrix} \\ &= \begin{pmatrix} a - 3.05 & b - \frac{1}{30} \end{pmatrix} P \begin{pmatrix} 17 + \sqrt{265} & 0 \\ 0 & 17 - \sqrt{265} \end{pmatrix} P^T \begin{pmatrix} a - 3.05 \\ b - \frac{1}{30} \end{pmatrix} \\ &= \left[P^T \begin{pmatrix} a - 3.05 \\ b - \frac{1}{30} \end{pmatrix} \right]^T \begin{pmatrix} 17 + \sqrt{265} & 0 \\ 0 & 17 - \sqrt{265} \end{pmatrix} \left[P^T \begin{pmatrix} a - 3.05 \\ b - \frac{1}{30} \end{pmatrix} \right] \\ &= (y_1 \ y_2) \begin{pmatrix} 17 + \sqrt{265} & 0 \\ 0 & 17 - \sqrt{265} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= (17 + \sqrt{265}) y_1^2 + (17 - \sqrt{265}) y_2^2 \\ &\geq 0 \end{aligned}$$

It follows that

$$S(a, b) - S\left(3.05, \frac{1}{30}\right) = \frac{1}{2} \left[\begin{pmatrix} a - 3.05 & b - \frac{1}{30} \end{pmatrix} \begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix} \begin{pmatrix} a - 3.05 \\ b - \frac{1}{30} \end{pmatrix} \right] \geq 0.$$

Therefore, $S(a, b)$ has the global minimum value $S\left(3.05, \frac{1}{30}\right) = \frac{1}{600}$ at $\left(3.05, \frac{1}{30}\right)$.

□

Example 39 (Global maximum/Global minimum)

Find the global (absolute) maximum and global minimum values of the function

$$f(x, y) = x^2 + y^2 - x - y + 1$$

in the disk defined by $x^2 + y^2 \leq 1$.

Solution

Since $f(x, y) = x^2 + y^2 - x - y + 1$ is a continuous function defined on a **bounded** closed disk $x^2 + y^2 \leq 1$, $f(x, y)$ can attain global maximum and global minimum at stationary points or at those points where $f(x, y)$ do not have partial derivatives or at the points on the boundary.

We first consider $f(x, y) = x^2 + y^2 - x - y + 1$ **on $x^2 + y^2 = 1$** , where $x^2 + y^2 = 1$ is the **boundary** of $x^2 + y^2 \leq 1$.

By expressing $x^2 + y^2 = 1$ in parametric form, we let

$$x = \cos t, \quad y = \sin t, \quad \text{where } 0 \leq t \leq 2\pi.$$

Consider $g(t) \equiv f(\cos t, \sin t) = \cos^2 t + \sin^2 t - \cos t - \sin t + 1 = 2 - \cos t - \sin t$,

where $0 \leq t \leq 2\pi$. Then $g'(t) = \sin t - \cos t$.

Set $g'(t) = 0 \Rightarrow \sin t - \cos t = 0 \Rightarrow \tan t = 1 \Rightarrow t = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

$$g''(t) = \cos t + \sin t$$

$$g''\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} > 0$$

$$g''\left(\frac{5\pi}{4}\right) = \cos\frac{5\pi}{4} + \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} < 0$$

$$g(0) = g(2\pi) = 2 - 1 = 1$$

$$g\left(\frac{\pi}{4}\right) = 2 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 2 - \sqrt{2}$$

$$g\left(\frac{5\pi}{4}\right) = 2 - \left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) = 2 + \sqrt{2}$$

$\therefore g(t)$ attains the greatest value $g\left(\frac{5\pi}{4}\right) = 2 + \sqrt{2}$ at $t = \frac{5\pi}{4}$ and the smallest value $g\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$ at $t = \frac{\pi}{4}$.

When $t = \frac{5\pi}{4}$, we have $x = \cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$, $y = \sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$.

When $t = \frac{\pi}{4}$, we have $x = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $y = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Hence, **on the boundary curve** $x^2 + y^2 = 1$, the function $f(x, y) = x^2 + y^2 - x - y + 1$

has a maximum value $f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 2 + \sqrt{2}$ at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$;

and a minimum value $f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 2 - \sqrt{2}$ at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Next, we find the maximum and minimum values of $f(x, y)$ **inside the disk** $x^2 + y^2 \leq 1$, i.e.

when $x^2 + y^2 < 1$. Then compare their f values with $f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 2 + \sqrt{2}$ and

$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 2 - \sqrt{2}$ to obtain the global maximum and global minimum.

Consider $f(x, y) = x^2 + y^2 - x - y + 1$ **inside** $x^2 + y^2 = 1$:

$$\begin{cases} \frac{\partial f}{\partial x} = 2x - 1 = 0 \\ \frac{\partial f}{\partial y} = 2y - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ y = \frac{1}{2} \end{cases} \Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \text{ is a stationary point.}$$

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$\Rightarrow f_{xx}\left(\frac{1}{2}, \frac{1}{2}\right) = 2, \quad f_{yy}\left(\frac{1}{2}, \frac{1}{2}\right) = 2, \quad f_{xy}\left(\frac{1}{2}, \frac{1}{2}\right) = 0.$$

Since

$$\Delta\left(\frac{1}{2}, \frac{1}{2}\right) = f_{xx}\left(\frac{1}{2}, \frac{1}{2}\right)f_{yy}\left(\frac{1}{2}, \frac{1}{2}\right) - f_{xy}^2\left(\frac{1}{2}, \frac{1}{2}\right) = 2 \times 2 - 0^2 = 4 > 0$$

and

$$f_{xx}\left(\frac{1}{2}, \frac{1}{2}\right) = 2 > 0,$$

it follows that $f(x, y) = x^2 + y^2 - x - y + 1$ has a local minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the minimum value is $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$.

Compare $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$ and $f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 2 - \sqrt{2}$.

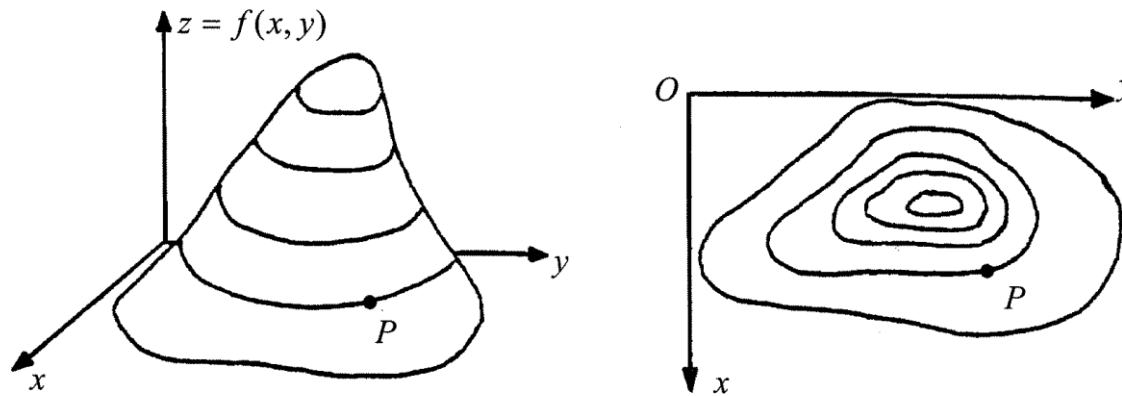
Since $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} < 2 - \sqrt{2} = f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, we conclude that $f(x, y) = x^2 + y^2 - x - y + 1$ has an **absolute minimum** at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the absolute minimum value is $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$.

Also, $f(x, y) = x^2 + y^2 - x - y + 1$ has an **absolute maximum** at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and the absolute maximum value is $f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 2 + \sqrt{2}$.

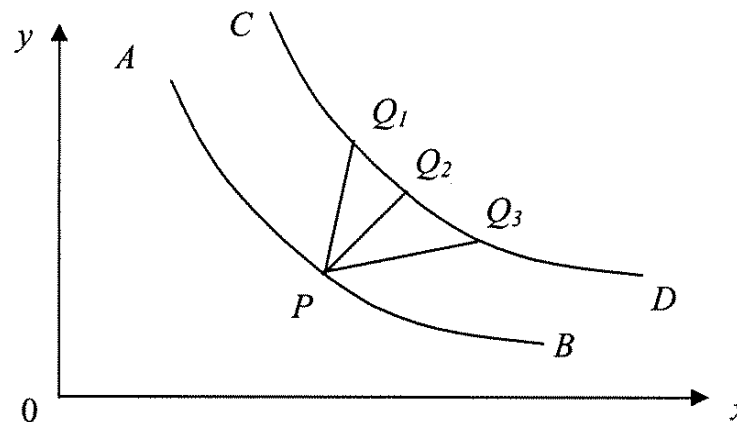


8 Directional Derivative and Gradient

For the scalar function $z = f(x, y)$, the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ measure the rates of change of $z = f(x, y)$ in the x and y directions, respectively. We would like to know the rate of change in any direction.



Consider a point P which lies on some level curve AB and a (variable) point Q on a nearby level curve CD .



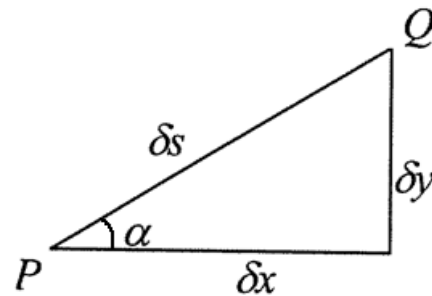
The rate of change of f in the direction PQ , also known as the **directional derivative** of $f(x, y)$ in the direction PQ , is given by

$$\frac{df}{ds} = \lim_{\delta s \rightarrow 0} \frac{\delta f}{\delta s} = \lim_{\delta s \rightarrow 0} \frac{f(P) - f(Q)}{PQ}.$$

Geometrically, it is clear that $\frac{df}{ds}$ will be a maximum in the direction normal to the level curve at P (since $f(P) - f(Q)$ is constant and PQ is least in that direction).

The **directional derivative** of $f(x, y)$ in the direction PQ , i.e. $\frac{df}{ds}$, is also denoted as $D_{\vec{v}}f$, where $\vec{v}$ is the unit vector in the direction PQ .

If the direction PQ is inclined an angle α to the x -axis, we have



$$\delta x = \delta s \cos \alpha, \quad \delta y = \delta s \sin \alpha$$

From Taylor's Theorem (page 29),

$$\delta f \approx \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y.$$

Then $\frac{\delta f}{\delta s} \approx \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \sin \alpha.$

As $\delta s \rightarrow 0,$

$$\frac{df}{ds} = D_{\vec{v}}f = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \sin \alpha \cdots (*),$$

where $\vec{v} = (\cos \alpha) \vec{i} + (\sin \alpha) \vec{j}$ is the **unit** vector in the direction PQ .

By considering $\frac{df}{ds}$ as a function of α , it is easy to show that $\frac{df}{ds}$ has a maximum value of $\left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right]^{\frac{1}{2}}$ when $\alpha = \tan^{-1} \left(\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}} \right).$

We define the **vector differential operator**, denoted by ∇ (read as “del”), as

$$\boxed{\nabla \equiv \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y}}.$$

The **gradient of f** , denoted by **$\text{grad } f$** , is defined as the vector

$$\text{grad } f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}.$$

In terms of ∇ , if we write $\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} \right) f$, then

$$\text{grad } f = \nabla f.$$

The gradient $\text{grad } f$ (or ∇f) is a vector whose magnitude is the maximum value of $\frac{df}{ds}$ (or $D_{\vec{v}}f$) and its direction is that in which $\frac{df}{ds}$ attains its maximum value.

Hence, ∇f gives us the magnitude and direction (normal to the level curve at P) of the greatest rate of change of f .

Also, from (*), if $\vec{v}$ is a unit vector in direction PQ , then $\vec{v} = (\cos \alpha) \vec{i} + (\sin \alpha) \vec{j}$ and the **directional derivative of f in the direction of $\vec{v}$** is

$$D_{\vec{v}}f = (\text{grad } f) \cdot \vec{v} = \nabla f \cdot \vec{v}$$

Summary of **directional derivative** and **gradient** for functions of two independent variables

Let $\vec{u} = u_1\vec{i} + u_2\vec{j}$ be a unit vector in $\mathbb{R}^2$.

The **directional derivative** of $f(x, y)$ at (x_0, y_0) in the direction of $\vec{u}$ is the number

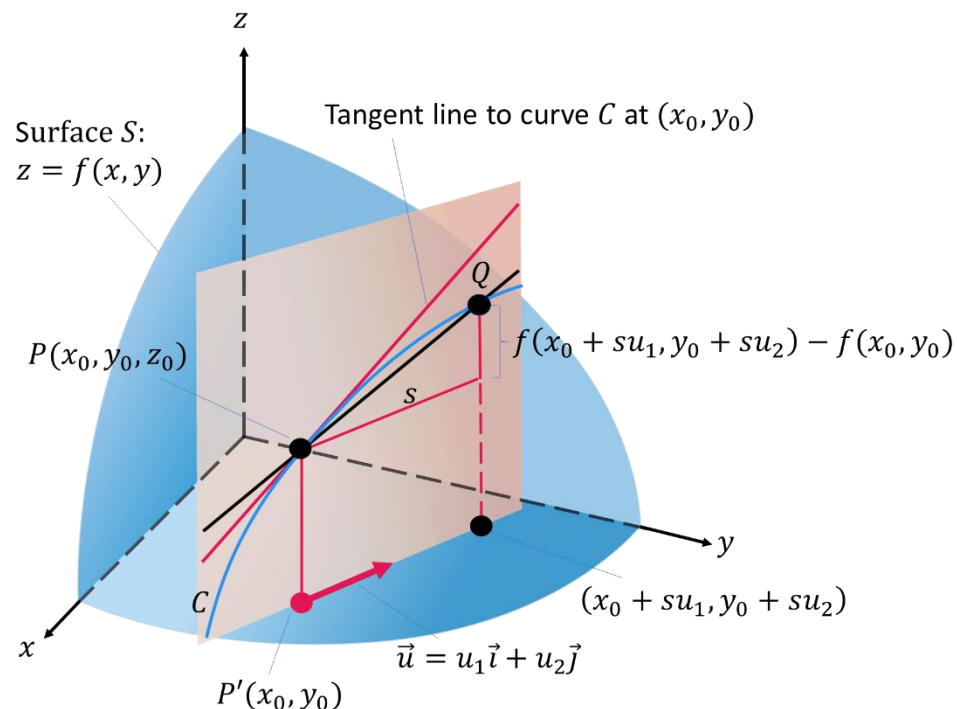
$$D_{\vec{u}}f(x_0, y_0) = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided the limit exists. This is the rate of change of $f(x, y)$ at (x_0, y_0) in the direction of $\vec{u}$.

To compute $D_{\vec{u}}f(x_0, y_0)$, we may also use the formula

$$D_{\vec{u}}f(x_0, y_0) = u_1 \frac{\partial f}{\partial x}(x_0, y_0) + u_2 \frac{\partial f}{\partial y}(x_0, y_0).$$

The slope of curve C at $P'(x_0, y_0)$ is the directional derivative $D_{\vec{u}}f(x_0, y_0)$.



The **gradient** of $f(x, y)$, denoted by $\text{grad } f$, is the **vector**

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$$

which gives the magnitude and direction of the greatest rate of change of f at a point.

In terms of the gradient vector ∇f , the **directional derivative** of $f(x, y)$ at (x_0, y_0) in the direction of the unit vector $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ can be computed as

$$D_{\vec{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

The above discussion can be generalized to functions in 3 or more independent variables.

For example, the **directional derivative** of $f(x, y, z)$ at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ in $\mathbb{R}^3$ is defined as

$$D_{\vec{u}}f(x_0, y_0, z_0) = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2, z_0 + su_3) - f(x_0, y_0, z_0)}{s}$$

provided the limit exists.

The **directional derivative** of $f(x, y, z)$ in the direction of $\vec{u}$, that is $D_{\vec{u}}f(x, y, z)$, can be computed by using the formula

$$D_{\vec{u}}f(x, y, z) = \nabla f \cdot \vec{u} = u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} + u_3 \frac{\partial f}{\partial z},$$

where

$$\nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

is the **gradient of f** . Here, $\nabla \equiv \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$ is the vector differential operator.

Now, we would like to show that the gradient $\nabla f(a_1, \dots, a_n) = \begin{pmatrix} f_{x_1}(a_1, \dots, a_n) \\ \vdots \\ f_{x_n}(a_1, \dots, a_n) \end{pmatrix} \neq \vec{0}$ of f at $(a_1, \dots, a_n)$ points in the direction in which f increases most rapidly.

Let $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ be a unit vector, i.e. $|\vec{v}| = 1$. Then by Cauchy-Schwarz inequality, we have

$$|D_{\vec{v}}f(a_1, \dots, a_n)| = |\nabla f(a_1, \dots, a_n) \cdot \vec{v}| \leq |\nabla f(a_1, \dots, a_n)| |\vec{v}| = |\nabla f(a_1, \dots, a_n)|.$$

In addition, let $\vec{w} = \frac{\nabla f(a_1, \dots, a_n)}{|\nabla f(a_1, \dots, a_n)|}$ be a unit vector in the direction of $\nabla f(a_1, \dots, a_n)$.

Then,

$$\begin{aligned} D_{\vec{w}}f(a_1, \dots, a_n) &= \nabla f(a_1, \dots, a_n) \cdot \vec{w} = \nabla f(a_1, \dots, a_n) \cdot \frac{\nabla f(a_1, \dots, a_n)}{|\nabla f(a_1, \dots, a_n)|} = \frac{|\nabla f(a_1, \dots, a_n)|^2}{|\nabla f(a_1, \dots, a_n)|} \\ &= |\nabla f(a_1, \dots, a_n)| \end{aligned}$$

It follows that the magnitude of $D_{\vec{v}}f(a_1, \dots, a_n)$ is the maximum when $\vec{v}$ points in the direction of $\nabla f(a_1, \dots, a_n)$.

Therefore, at $(a_1, \dots, a_n)$, **f increases most rapidly in the direction of $\nabla f(a_1, \dots, a_n)$** , and the **maximum rate of increase is $|\nabla f(a_1, \dots, a_n)|$** , where $\nabla f(a_1, \dots, a_n) = \begin{pmatrix} f_{x_1}(a_1, \dots, a_n) \\ \vdots \\ f_{x_n}(a_1, \dots, a_n) \end{pmatrix}$.

Example 40

Find the directional derivatives of the following functions in the given directions:

(a) $f(x, y) = x^2 + 2xy$ at $(1, 2)$ in the direction of $\vec{v} = \vec{i} + \vec{j}$;

(b) $f(x, y, z) = x^2z + y^3z^2 - xyz$ at $(0, -1, 2)$ in the direction of $\vec{v} = \vec{i} - 2\vec{j} + 2\vec{k}$.

Solution

(a) The **unit** vector $\vec{u}$ in the direction of $\vec{v} = \vec{i} + \vec{j}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}.$$

Method 1: Using the formula $D_{\vec{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

$$f(x, y) = x^2 + 2xy$$

$$f_x = 2x + 2y \Rightarrow f_x(1, 2) = 2(1) + 2(2) = 6$$

$$f_y = 2x \Rightarrow f_y(1, 2) = 2(1) = 2$$

$$\therefore D_{\vec{u}}f(1, 2) = \nabla f(1, 2) \cdot \vec{u} = (6\vec{i} + 2\vec{j}) \cdot \left(\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right) = (6)\left(\frac{1}{\sqrt{2}}\right) + (2)\left(\frac{1}{\sqrt{2}}\right) = \frac{8}{\sqrt{2}}$$

Method 2: Using the definition of directional derivative

$$D_{\vec{u}}f(x_0, y_0) = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s},$$

where $x_0 = 1$, $y_0 = 2$, $u_1 = \frac{1}{\sqrt{2}}$ and $u_2 = \frac{1}{\sqrt{2}}$, we have

$$\begin{aligned} D_{\vec{u}}f(1, 2) &= \lim_{s \rightarrow 0} \frac{f\left(1 + \frac{s}{\sqrt{2}}, 2 + \frac{s}{\sqrt{2}}\right) - f(1, 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left[\left(1 + \frac{s}{\sqrt{2}}\right)^2 + 2\left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right)\right] - [1^2 + 2(1)(2)]}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(5 + \frac{8s}{\sqrt{2}} + \frac{3s^2}{2}\right) - 5}{s} \\ &= \lim_{s \rightarrow 0} \left(\frac{8}{\sqrt{2}} + \frac{3s}{2}\right) \\ &= \frac{8}{\sqrt{2}} \end{aligned}$$

(b) The **unit** vector $\vec{u}$ in the direction of $\vec{v} = \vec{i} - 2\vec{j} + 2\vec{k}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{3}\vec{i} - \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}.$$

$$f(x, y, z) = x^2z + y^3z^2 - xyz$$

$$f_x = 2xz - yz \Rightarrow f_x(0, -1, 2) = 2$$

$$f_y = 3y^2z^2 - xz \Rightarrow f_y(0, -1, 2) = 12$$

$$f_z = x^2 + 2y^3z - xy \Rightarrow f_z(0, -1, 2) = -4$$

$$\therefore D_{\vec{u}}f(0, -1, 2) = \nabla f(0, -1, 2) \cdot \vec{u}$$

$$= (2\vec{i} + 12\vec{j} - 4\vec{k}) \cdot \left(\frac{1}{3}\vec{i} - \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right)$$

$$= \frac{2}{3} - \frac{24}{3} - \frac{8}{3} = -10$$

Example 41

For each of the following functions, find the direction in which the function increases most rapidly at the given point P , and find the maximum rate of increase.

(a) $f(x, y) = x^2 - 6x - y^2$ at $P(-1, 5)$;

(b) $f(x, y, z) = ze^{x^2y}$ at $P(2, 0, 3)$.

Solution

(a) $f(x, y) = x^2 - 6x - y^2$ increases most rapidly at $P(-1, 5)$ in the direction

$$\nabla f(-1, 5) = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \Big|_{(-1, 5)} = \begin{pmatrix} 2x - 6 \\ -2y \end{pmatrix} \Big|_{(-1, 5)} = \begin{pmatrix} -8 \\ -10 \end{pmatrix}$$

and the maximum rate of increase is

$$|\nabla f(-1, 5)| = \sqrt{(-8)^2 + (-10)^2} = \sqrt{164} = 2\sqrt{41}$$

(b) $f(x, y, z) = ze^{x^2y}$ increases most rapidly at $P(2, 0, 3)$ in the direction

$$\nabla f(2, 0, 3) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \bigg|_{(2,0,3)} = \begin{pmatrix} 2xyz e^{x^2y} \\ x^2 z e^{x^2y} \\ e^{x^2y} \end{pmatrix} \bigg|_{(2,0,3)} = \begin{pmatrix} 0 \\ 12 \\ 1 \end{pmatrix}$$

and the maximum rate of increase is

$$|\nabla f(2, 0, 3)| = \sqrt{0^2 + 12^2 + 1^2} = \sqrt{145}$$

Example 42

Find the rate of change of $f(x, y, z) = 10 + xy + xz + yz$ at the point $(1, 1, 1)$ in the direction of $\vec{v} = \vec{i} + \vec{j}$.

Solution

The **unit** vector $\vec{u}$ in the direction of $\vec{v} = \vec{i} + \vec{j}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}.$$

$$\nabla f(1, 1, 1) = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \bigg|_{(1,1,1)} = \begin{pmatrix} y + z \\ x + z \\ x + y \end{pmatrix} \bigg|_{(1,1,1)} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

The rate of change of f at $(1, 1, 1)$ in the direction of $\vec{v} = \vec{i} + \vec{j}$ is the directional derivative

$$D_{\vec{u}}f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \vec{u} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

Example 43

Suppose that the temperature (in degree Celsius) at the point (x, y, z) is

$$w = f(x, y, z) = \frac{1}{180} (7400 - 4x - 9y - 0.03xy) - 2z.$$

Suppose a hawk hovering at the point $P(200, 200, 5)$ above the airport suddenly dives at a speed of 3 km/min in the direction specified by the vector $(3, 4, -12)^T$. What instantaneous rate of change of temperature does the bird experience?

Solution

The unit vector in the direction of $3\vec{i} + 4\vec{j} - 12\vec{k}$ is

$$\vec{u} = \frac{3\vec{i} + 4\vec{j} - 12\vec{k}}{\sqrt{3^2 + 4^2 + (-12)^2}} = \frac{3}{13}\vec{i} + \frac{4}{13}\vec{j} - \frac{12}{13}\vec{k}.$$

The temperature gradient vector

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \frac{-4 - 0.03y}{180}\vec{i} + \frac{-9 - 0.03x}{180}\vec{j} - 2\vec{k}$$

has the value

$$\begin{aligned}\nabla f(200, 200, 5) &= \frac{-4 - 0.03 \times 200}{180} \vec{i} + \frac{-9 - 0.03 \times 200}{180} \vec{j} - 2\vec{k} \\ &= -\frac{10}{180} \vec{i} - \frac{15}{180} \vec{j} - 2\vec{k}\end{aligned}$$

at the initial position $P(200, 200, 5)$ of the hawk.

Therefore, the hawk's initial rate of change of temperature with respect to distance is

$$\begin{aligned}D_{\vec{u}}f(200, 200, 5) &= \nabla f(200, 200, 5) \cdot \vec{u} \\ &= \left(-\frac{10}{180} \vec{i} - \frac{15}{180} \vec{j} - 2\vec{k}\right) \cdot \left(\frac{3}{13} \vec{i} + \frac{4}{13} \vec{j} - \frac{12}{13} \vec{k}\right) \\ &= 1.808 \text{ } ^\circ\text{C}/\text{km}\end{aligned}$$

Its speed is $\frac{ds}{dt} = 3 \text{ km}/\text{min}$, so the instantaneous rate of change of temperature experienced by the hawk is

$$\frac{dw}{dt} = \frac{dw}{ds} \frac{ds}{dt} = D_{\vec{u}}f \frac{ds}{dt} = 1.808 \times 3 = 5.424 \text{ } ^\circ\text{C}/\text{min}.$$

Thus, the hawk initially gets warmer by almost 5.5 degrees per minute as it dives toward the ground. □