

Solutions

Name: _____

Student ID: _____

Answer ALL questions. (Full marks: 40)

Total Marks: _____

Q1: _____ Q2: _____ Q3: _____

Question 1 (10 marks)It is given that $w = f(x, y, z)$, where $x = r \cos t$, $y = r \sin t$ and $z = t^3$.(i) Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial t}$. (5 marks)(ii) If $\frac{\partial f}{\partial x} = 2$, $\frac{\partial f}{\partial y} = -1$, and $\frac{\partial f}{\partial z} = 3$ at the point $(x, y, z) = (-2, 0, \pi^3)$, find the values of $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial t}$ at the corresponding point $(r, t) = (2, \pi)$. (5 marks)

$$\text{Solution. (i)} \quad \frac{\partial w}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= \frac{\partial f}{\partial x} \cdot \cos t + \frac{\partial f}{\partial y} \cdot \sin t$$

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$= \frac{\partial f}{\partial x} \cdot (-r \sin t) + \frac{\partial f}{\partial y} \cdot (r \cos t) + \frac{\partial f}{\partial z} \cdot 3t^2$$

$$(ii) \quad \frac{\partial w}{\partial r}(2, \pi) = 2 \cdot \cos \pi + (-1) \cdot \sin \pi = -2$$

$$\frac{\partial w}{\partial t}(2, \pi) = 2 \cdot (-2 \sin \pi) + (-1) \cdot (2 \cdot \cos \pi) + (3) \cdot 3 \cdot \pi^2 = 2 + 9\pi^2$$

Question 2 (10 marks)

Suppose near the point $(x, y, u, v) = (1, 1, 1, 1)$, we can solve $\begin{cases} xu + yvu^2 = 2 \\ xu^4 + y^2v^3 = 2 \end{cases}$ uniquely for u

and v as functions of x and y . Compute $\frac{\partial u}{\partial x}(1, 1)$.

Solution. $\begin{cases} xu + yvu^2 = 2 \\ xu^4 + y^2v^3 = 2 \end{cases} \Rightarrow \begin{cases} u + x u_x + y(v_x u^2 + v \cdot 2u \cdot u_x) = 0 \\ u^4 + 4xu^3 \cdot u_x + y^2 \cdot 3v^2 \cdot v_x = 0 \end{cases}$

At $(x, y, u, v) = (1, 1, 1, 1)$, we have

$$\begin{cases} 1 + u_x + v_x + 2u_x = 0 \\ 1 + 4u_x + 3v_x = 0 \end{cases} \Leftrightarrow \begin{cases} 3u_x + v_x = -1 \\ 4u_x + 3v_x = -1 \end{cases}$$

$$\Leftrightarrow \begin{cases} 9u_x + 3v_x = -3 \\ 4u_x + 3v_x = -1 \end{cases}$$

$$\Rightarrow 5u_x = -2$$

$$\Rightarrow u_x(1, 1) = -\frac{2}{5}.$$

Question 3 (20 marks)

Diagonalize the matrix $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 5 \end{pmatrix}$ by the following steps.

- (i) Find the eigenvalues of A . (4 marks)
- (ii) Find three linearly independent eigenvectors of A . (7 marks)
- (iii) Construct a matrix P from the vectors in (ii) and calculate the inverse P^{-1} using Gauss elimination. (3 marks)
- (iv) Calculate $B = P^{-1}A$. (2 marks)
- (v) Calculate $D = BP$. Is D a diagonal matrix? (2 marks)
- (vi) Deduce the eigenvalues and eigenvectors of $C = A^2 + 2A + I$, where I is the identity matrix. [Note: Do not attempt to compute matrix C]. (2 marks)

Solution. (i) $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & 5-\lambda \end{vmatrix} = (4-\lambda) \begin{vmatrix} 3-\lambda & 0 \\ 0 & 5-\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 \\ 3-\lambda & 0 \end{vmatrix}$

$$= (4-\lambda)(3-\lambda)(5-\lambda) + 2(- (3-\lambda))$$

$$= (3-\lambda)[(4-\lambda)(5-\lambda) - 2] = (3-\lambda)[20 - 9\lambda + \lambda^2 - 2]$$

$$= (3-\lambda)(\lambda^2 - 9\lambda + 18) = (3-\lambda)(\lambda-3)(\lambda-6)$$

eigenvalues $\lambda = 3$ (multiplicity 2) and $\lambda = 6$

(ii)(a) For $\lambda = 3$,

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix} = s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s, t \neq 0$$

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(b) For $\lambda = 6$

$$\left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \Rightarrow \begin{cases} -2x_1 + x_3 = 0 \\ -3x_2 = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{t}{2} \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}, \quad t \neq 0$$

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Note that the determinant of $[\vec{v}_1, \vec{v}_2, \vec{v}_3] = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ is

$$\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -2 - 1 = -3 \neq 0.$$

hence $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly indep. eigenvectors.

(iii) One can choose $p = [\vec{v}_1, \vec{v}_2, \vec{v}_3] = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Then

$$(p | I) = \left(\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array} \right) \xrightarrow{r_1+r_3} \left(\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{\substack{r_1 \cdot (-1) \\ r_3/3}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 0 & 1/3 \end{array} \right) \xrightarrow{r_1+r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & 0 & 4/3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 0 & 1/3 \end{array} \right)$$

$$\Rightarrow p^{-1} = \frac{1}{3} \begin{pmatrix} -2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

$$(iv) \quad B = p^{-1}A = \frac{1}{3} \begin{pmatrix} -2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 5 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -6 & 0 & 3 \\ 0 & 9 & 0 \\ 6 & 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$(v) \quad D = BP = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

D is a diagonal matrix

(vi) The eigenvectors of C are the same as those of A .

The eigenvalues of C are

$$3^2 + 2 \cdot 3 + 1 = (3+1)^2 = 4^2 = 16$$

$$\text{and } 6^2 + 2 \cdot 6 + 1 = (6+1)^2 = 7^2 = 49.$$