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# **EE3121**

# **Differential Equations for Engineers**

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## **Tutorial 4-1**



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# Tutorial

## Problem 1:

Compute the first four iterations  $y_0(t)$ ,  $y_1(t)$ ,  $y_2(t)$  and  $y_3(t)$  in successive approximation to the solution of

$$\frac{dy}{dt} - y = 4t, \quad y(0) = -2$$

# Tutorial

**Solution:**

$$y_{n+1}(t) = y(0) + \int_{t_0}^t f(\tau, y_n(\tau)) d\tau,$$

$$y_0(t) = y(0),$$

$$\frac{dy_1(t)}{dt} = 4t + y_0(t) = 4t - 2.$$

**Integrate:**

$$y_1(t) = y(0) + \int_0^t (4\tau - 2) d\tau = -2 + 4 \cdot \frac{1}{2} t^2 - 2t = 2(t^2 - t - 1)$$

$$\frac{dy_2(t)}{dt} = f(t, y_1(t)) = y_1(t) + 4t = 2(t^2 + t - 1)$$

# Tutorial

$$\begin{aligned}y_2(t) &= y(0) + \int_0^t (\tau, y_1(\tau)) d\tau = -2 + \int_0^t 2(\tau^2 + \tau - 1) d\tau \\&= \frac{2}{3}t^3 + t^2 - 2t - 2\end{aligned}$$

$$\frac{dy_3(t)}{dt} = f(t, y_2(t)) = y_2(t) + 4t = \frac{2}{3}t^3 + t^2 + 2t - 2$$

$$\begin{aligned}y_3(t) &= y(0) + \int_0^t (\tau, y_2(\tau)) d\tau = -2 + \int_0^t \left( \frac{2}{3}\tau^3 + \tau^2 + 2\tau - 2 \right) d\tau \\&= \frac{1}{6}t^4 + \frac{1}{3}t^3 + t^2 - 2t - 2\end{aligned}$$

# Tutorial

## Problem 2:

Determine the equilibrium points for

$$\frac{dy}{dt} = 3y^2 - 33y + 84$$

Draw the vector field associated with the equation. Determine which equilibrium points are stable and which are not stable.

# Tutorial

**Solution:** Equilibrium Points:

$$\frac{dy}{dt} = 0$$

$$\Rightarrow 3y^2 - 33y + 84 = 0$$

$$\Rightarrow 3(y - 4)(y - 7) = 0$$

Equilibrium Points:

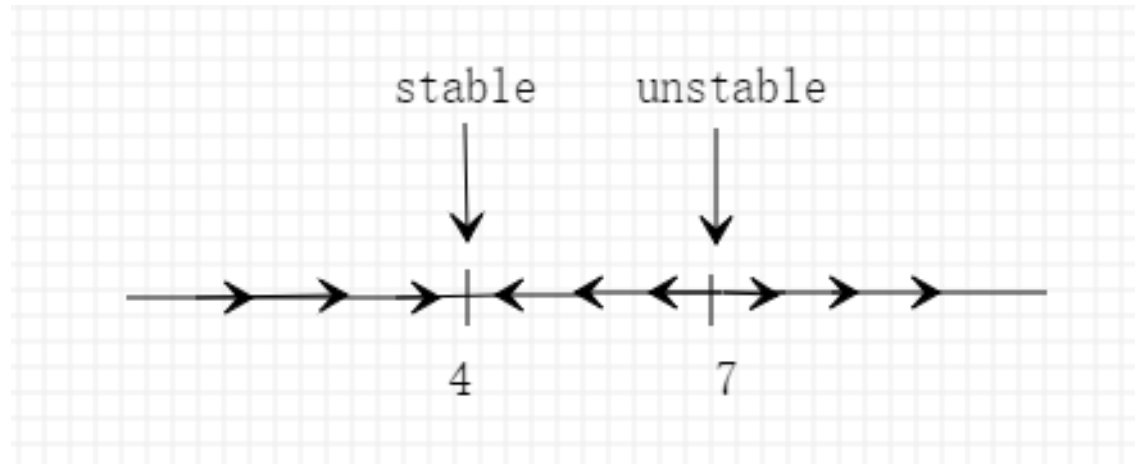
$$y_1 = 4, \quad y_2 = 7$$

# Tutorial

$$y > 7 : \frac{dy}{dt} = 3(y - 4)(y - 7) > 0$$

$$4 < y < 7 : \frac{dy}{dt} < 0$$

$$y < 4 : \frac{dy}{dt} > 0$$



# Tutorial

## Problem 3:

Solve the differential equation

$$\frac{dy}{dt} = \frac{1}{\sqrt{1-y}}, \quad y(0) = 0. \quad (1)$$

Determine the maximum time a solution exists. Compare the maximum time with that estimated by using Lipschitz condition.

# Tutorial

**Solution:** Solve the D.E.

$$\int_{y(0)}^{y(t)} \sqrt{1-y} dy = \int_0^t dt$$

$$\int_{y(0)}^{y(t)} d\left(-\frac{2}{3}(1-y)^{\frac{3}{2}}\right) = \int_0^t dt$$

$$\Rightarrow -\frac{2}{3}(1-y(t))^{\frac{3}{2}} + \frac{2}{3} = t$$

$$\Rightarrow \frac{2}{3} - t = \frac{2}{3}(1-y(t))^{\frac{3}{2}}$$

# Tutorial

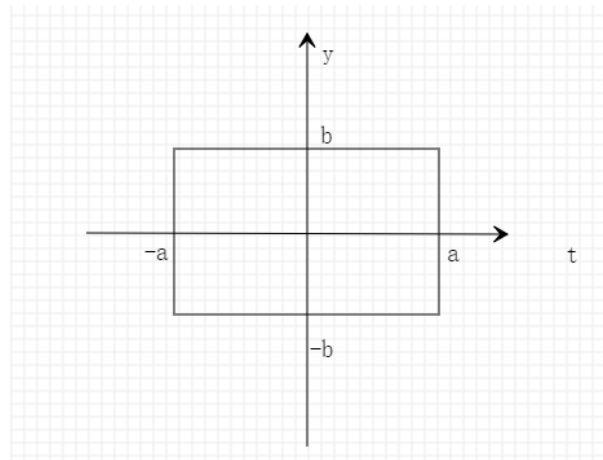
Real range:

$$t < \frac{2}{3}$$

$$1 - \frac{3}{2}t = (1 - y(t))^{\frac{3}{2}}$$

$$\Rightarrow y(t) = 1 - \left(1 - \frac{3}{2}t\right)^{\frac{2}{3}}$$

Estimate  $t_E$  without solving the equation



# Tutorial

$$|t - t_0| \leq \min \left\{ a, \frac{b}{M} \right\}$$
$$|f(t, y)| \leq M$$

Recall the equation

$$\frac{dy}{dt} = \frac{1}{\sqrt{1-y}}, \quad y(0) = 0$$

$$|f(t, y_1) - f(t, y_2)| = \left| \frac{1}{\sqrt{1-y_1}} - \frac{1}{\sqrt{1-y_2}} \right| = \left| \frac{\sqrt{1-y_2} - \sqrt{1-y_1}}{\sqrt{1-y_1} \cdot \sqrt{1-y_2}} \right|$$

Use Taylor series to show that the *error*  $\leq K$ .

# Tutorial

$$y \leq b < 1$$
$$|f(t, y)| = \frac{1}{\sqrt{1-y}}$$

$$y \uparrow \rightarrow \frac{1}{\sqrt{1-y}} \uparrow$$

$$|f(t, y)| = \frac{1}{\sqrt{1-y}} \leq \frac{1}{\sqrt{1-b}} = M$$

Select  $a > \frac{b}{M}$

$$t_E = \min \left\{ a, \frac{b}{M} \right\} = \frac{b}{M} = b\sqrt{1-b}$$

# Tutorial

$$g(b) = b\sqrt{1-b}$$

$$g'(b) = \sqrt{1-b} - \frac{b}{2} \cdot \frac{1}{\sqrt{1-b}} = \frac{1}{\sqrt{1-b}} \left(1 - b - \frac{b}{2}\right) = \frac{1}{\sqrt{1-b}} \left(1 - \frac{3}{2}b\right) = 0$$

$$b^* = \frac{2}{3}$$

Estimate:

$$\max t_E = b^* \sqrt{1-b^*} = \frac{2}{3} \sqrt{1 - \frac{2}{3}} = \left(\frac{1}{\sqrt{3}}\right) \frac{2}{3}$$

**END**