

EF2452: Mathematics for Economics and Finance

Lecture 10

(Last Lecture)

Department of Economics and Finance
City University of Hong Kong
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Main Topics

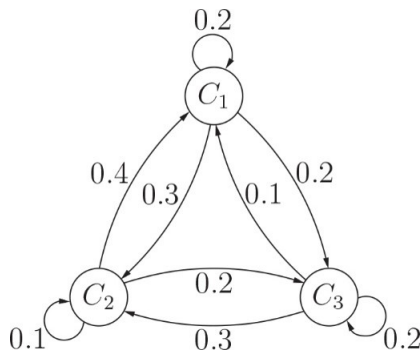
Applications of matrix algebra:

- Input-output model; technology matrix
- Arbitrage portfolios and state prices

- We assume an economy with n interdependent production processes, manufacturing commodities $C_1, C_2, \dots, C_n$.
- The production process for any one of the commodities requires an input, which uses part of the output of some of the others.
- In addition, there is an external demand for each commodity.
- The problem is to determine the production schedule that enables each process to meet all the demands for its product.

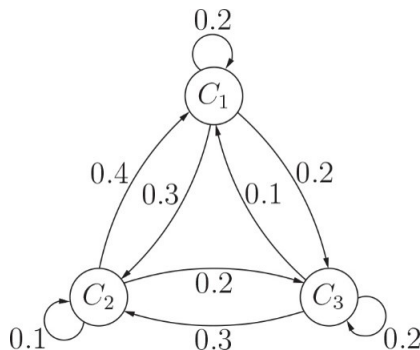
An Economy with Three Industries: Example

- The production processes for three goods, C_1, C_2, C_3 , are interlinked.
- To produce one dollar's worth of C_1 requires the input of \$0.2 worth of C_1 , \$0.4 worth of C_2 , and \$0.1 worth of C_3 .



An Economy with Three Industries: Example

- To produce one dollar's worth of C_2 requires the input of \$0.3 worth of C_1 , \$0.1 worth of C_2 , and \$0.3 worth of C_3 .
- To produce one dollar's worth of C_3 requires the input of \$0.2 worth of C_1 , \$0.2 worth of C_2 , and \$0.2 worth of C_3 .



An Economy with Three Industries: Example

- Suppose that, in a given time period, there is an external demand for d_1 dollars' worth of C_1 , d_2 of C_2 , and d_3 of C_3 .
- Our goal is to know the production levels x_1, x_2, x_3 of C_1, C_2, C_3 required to satisfy all demands in the given period.
- Consider first the total demand for C_1 . This is d_1 , the external demand, plus the quantity required to produce C_1, C_2 and C_3 .
- Each unit of C_1 requires 0.2 units of C_1 , each unit of C_2 requires 0.3 of C_1 and each unit of C_3 requires 0.2 of C_1 . Since the quantities of C_1, C_2 and C_3 being produced are x_1, x_2 and x_3 , the total demand for C_1 is therefore

$$x_1 = d_1 + 0.2x_1 + 0.3x_2 + 0.2x_3$$

An Economy with Three Industries: Example

- Similarly, we have

$$x_2 = d_2 + 0.4x_1 + 0.1x_2 + 0.2x_3$$

$$x_3 = d_3 + 0.1x_1 + 0.3x_2 + 0.2x_3$$

The Technology Matrix

- We measure the amounts of each commodity in dollars. In other words, the unit used is dollars.
- a_{ij} = amount of C_i needed to produce one unit of C_j
- Considering the above quantities as entries of a matrix $A = (a_{ij})$ gives us the technology matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

The System of Equations

- d_i = amount of C_i required to satisfy the external demand
- x_i = amount of C_i required to satisfy all needs
- Our problem is: given the technology matrix A and the external demands d_i , can we find the production schedules x_i ?
- We have a system of n equations (one equation for each $i = 1, 2, \dots, n$)

$$x_i = d_i + a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$$

The System of Equations

- The system of equations is

$$\mathbf{x} = \mathbf{d} + A\mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{pmatrix}$$

The System of Equations

- Rearranging, we get

$$\mathbf{x} - A\mathbf{x} = \mathbf{d}$$

$$(I - A)\mathbf{x} = \mathbf{d}$$

where I is the identity matrix.

- Thus, provided the matrix $(I - A)$ is invertible, there is a unique solution

$$\mathbf{x} = (I - A)^{-1}\mathbf{d}$$

- The matrix $I - A$ is known as the Leontief matrix, after the economist Wassily Leontief (who won the Nobel Prize in Economics in 1973 for his work in the input-output model).

The System of Equations

- For the three industries example discussed before, we have the following technology matrix

$$A = \begin{pmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{pmatrix}$$

- The input-output equations in matrix form are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} + \begin{pmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{d} + A\mathbf{x}$$

The System of Equations

- We have

$$I - A = \begin{pmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{pmatrix}$$

$$(I - A)^{-1} = \begin{pmatrix} 1.72 & 0.78 & 0.63 \\ 0.89 & 1.61 & 0.63 \\ 0.55 & 0.70 & 1.56 \end{pmatrix}$$

(Verify the calculation of the inverse yourself!)

- For any demand $\mathbf{d}$, we have the solution for $\mathbf{x}$:

$$\mathbf{x} = (I - A)^{-1} \mathbf{d} = \begin{pmatrix} 1.72 & 0.78 & 0.63 \\ 0.89 & 1.61 & 0.63 \\ 0.55 & 0.70 & 1.56 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

The System of Equations

- Then,

$$\mathbf{x} = \begin{pmatrix} 1.72d_1 + 0.78d_2 + 0.63d_3 \\ 0.89d_1 + 1.61d_2 + 0.63d_3 \\ 0.55d_1 + 0.70d_2 + 1.56d_3 \end{pmatrix}$$

The Technology Matrix: A Restriction

- Given a technology matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

we often place the following restriction

$$\sum_{i=1}^n a_{ij} < 1$$

for each $j = 1, 2, \dots, n$. In other words, for each column, the sum of entries along that column must be less than 1.

Why $\sum_{i=1}^n a_{ij} < 1$?

- Consider the production of commodity C_j . Each dollar's worth produced requires a_{1j} dollars' worth of C_1 , a_{2j} dollars' worth of C_2 , and so on. Thus the total cost of producing a dollar's worth of C_j is (summing along a column of the technology matrix):

$$a_{1j} + a_{2j} + \cdots + a_{nj}$$

- It is reasonable to assume that this total is less than one dollar, because otherwise the production of C_j would make a loss. Therefore, we need

$$\sum_{i=1}^n a_{ij} < 1$$

for each $j = 1, 2, \dots, n$.

Application to Investment Portfolios

- The basic idea is that an investor can choose between a number of different assets; these may be thought of as shares in companies, holdings of foreign exchange, old-master paintings, land (real estate), etc.
- After a fixed time period, say one year, the assets will have a new value, and this will depend on what has happened in the meantime. We model this uncertainty by assuming that there are a number of states which may occur.

Application to Investment Portfolios

- Suppose, for example, the citizens of a country can invest in either of the three assets: land, bonds (which yield a fixed return) and stocks (which yield an uncertain return).
- An election is due, after which one of the two political parties, party A or party B, will be in power.
- There are three assets:

Asset 1 = land

Asset 2 = bonds

Asset 3 = stocks

- There are two states:
State 1 = party A in power State 2 = party B in power.

Application to Investment Portfolios

- Clearly, the value of an investment will depend upon which of the states occur (due to the policies of the party in power).
- For example, government by party A may result in an increase in the value of land and a decrease in the value of stocks, while government by party B may have the opposite effect.
- We can describe this by writing down the value in a year's time of the amount of each asset which costs one dollar now, under the assumption that a given state occurs. In this example, we could get a table like the following:

	State 1	State 2
Land	1.25	0.95
Bonds	1.05	1.05
Stocks	0.90	1.15

Application to Investment Portfolios

- We have a 3×2 matrix $R = (r_{ij})$ which determines the return of each asset in the various states.
- In general, if there are m assets and n states, then the returns matrix R will be an $m \times n$ matrix.
- Suppose an investor decides, before the election, to invest \$5000 in land, \$1000 in bonds, and \$4000 in stocks. Then the row vector

$$Y = (5000 \quad 1000 \quad 4000)$$

is her portfolio. its value in one year's time depends upon which party wins the election.

Application to Investment Portfolios

- If party A wins, the value of the portfolio will be

$$(5000 \times 1.25) + (1000 \times 1.05) + (4000 \times 0.90) = 10,900$$

- If party B wins, the value of the portfolio will be

$$(5000 \times 0.95) + (1000 \times 1.05) + (4000 \times 1.15) = 10,400$$

- The portfolio is a 1×3 matrix Y , the table of returns R is a 3×2 matrix, and their product

$$YR = (5000 \quad 1000 \quad 4000) \begin{pmatrix} 1.25 & 0.95 \\ 1.05 & 1.05 \\ 0.90 & 1.15 \end{pmatrix} = (10,900 \quad 10,400)$$

- When R is given, YR tells us the possible values of the portfolio Y in the various states.

Investment: Zero-cost portfolios

- Within this example, an investor might look for a portfolio which costs nothing. For example, this could be a portfolio such as

$$Y = (5000 \quad -10000 \quad 5000),$$

which means that the investor borrows \$10,000 from the bank and uses it to buy land and stocks.

- At the end of the year, she owes the bank more (10000×1.05 dollars), but the changes in value of land and stocks may offset this. In fact, we calculate that

$$YR = (5000 \quad -10000 \quad 5000) \begin{pmatrix} 1.25 & 0.95 \\ 1.05 & 1.05 \\ 0.90 & 1.15 \end{pmatrix} = (250 \quad 0)$$

Investment: Arbitrage portfolios

- Thus if party A wins, the investor gains \$250 without having to 'invest' anything in real terms, while if party B wins, she is no worse off.
- A portfolio like the one just described, which **costs nothing, cannot lose** and **in at least one state yields a profit**, is called an **arbitrage portfolio**.

Arbitrage Portfolios and State Prices

- If there are m assets and n states, the $m \times n$ returns matrix R is defined so that the element r_{ij} is the return from investing in one unit of asset i when state j occurs.
- A portfolio $Y = (y_1 \ y_2 \ y_3 \dots \ y_m)$ is a row vector (or $1 \times m$ matrix), in which y_i is the number of units of asset i held by the investor. We often use “dollar” units, implying how much money in dollars is invested in asset i to facilitate computations.
- The return on a portfolio Y in the state j is $(YR)_j$, the j th component of YR . In particular, Y is an arbitrage portfolio if

$$y_1 + y_2 + \dots + y_m = 0,$$

and YR has no negative entries and at least one positive entry.

Arbitrage Portfolios and State Prices

- We now introduce another idea, which turns out to be closely linked to the existence of arbitrage portfolios.
- The positive numbers $p_1, p_2, \dots, p_n$ are said to be state prices for R if the column vector

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ \vdots \\ \vdots \\ p_n \end{pmatrix} \quad \text{satisfies} \quad R\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

- If the above system is inconsistent, then state prices for R do not exist.

Arbitrage Portfolios and State Prices

- It is helpful to think of p_j as a ‘weight’ assigned to state j . The weights are chosen so that the weighted return on one unit of asset i , that is,

$$r_{i1}p_1 + r_{i2}p_2 + \cdots + r_{in}p_n = (R\mathbf{p})_i,$$

is equal to 1 for each asset i .

- Note that what is important is that the above weighted return for each (dollar) unit of asset i must be equalized across assets; we are merely normalizing the equalized weighted return to 1.

Arbitrage Portfolios and State Prices: Example

- Suppose that there are three assets and three states, and the returns matrix is

$$R = \begin{pmatrix} 1.00 & 1.00 & 1.20 \\ 1.05 & 1.05 & 1.05 \\ 0.95 & 1.10 & 0.95 \end{pmatrix}$$

- A vector of state prices is a solution of the system whose augmented matrix is

$$\left(\begin{array}{ccc|c} 1.00 & 1.00 & 1.20 & 1 \\ 1.05 & 1.05 & 1.05 & 1 \\ 0.95 & 1.10 & 0.95 & 1 \end{array} \right)$$

- Use elementary row operations to solve the system and get the following state prices for R (verify this yourself!)

$$p_1 = 5/63 \quad p_2 = 40/63 \quad p_3 = 5/21$$

Arbitrage Portfolios and State Prices: Example

- We can use this result to show that there is no arbitrage portfolio for the given R .
- Suppose Y is a portfolio for which $(YR)_j$, the return on Y is state j , is non-negative for $j = 1, 2, 3$, and positive for at least one j .
- Then, since the state prices p_j are positive, the expression

$$YR\mathbf{p} = (YR)_1p_1 + (YR)_2p_2 + (YR)_3p_3 > 0$$

Arbitrage Portfolios and State Prices: Example

- But, we have

$$R\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- So,

$$YR\mathbf{p} = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = y_1 + y_2 + y_3$$

- Then, $y_1 + y_2 + y_3 > 0$ (since $YR\mathbf{p} > 0$ from last slide).
- Therefore, **there is no arbitrage portfolio when state prices exist.**
- The converse is also true: **if state prices do not exist, then there must be an arbitrage portfolio:** see next few slides.

No State Prices: Existence of Arbitrage Portfolios

- Suppose that an investor invests her money in three different assets and that three possible states can occur.
- The returns matrix is given by

$$R = \begin{pmatrix} 0.95 & 0.9 & 1.0 \\ 1.1 & 1.1 & 1.1 \\ 1.2 & 1.15 & 1.25 \end{pmatrix}$$

- $\mathbf{p}$ is a state-price vector for R if

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad R\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where p_1, p_2, p_3 are positive.

No State Prices: Existence of Arbitrage Portfolios

- Then the augmented matrix for the linear system which determines $\mathbf{p}$ is

$$\left(\begin{array}{ccc|c} 0.95 & 0.9 & 1.0 & 1 \\ 1.1 & 1.1 & 1.1 & 1 \\ 1.2 & 1.15 & 1.25 & 1 \end{array} \right)$$

- Using elementary row operations, you can show that the above system becomes (verify this yourself!)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 10/11 \\ 0 & 1 & -1 & -30/11 \\ 0 & 0 & 0 & -5/22 \end{array} \right)$$

- The form of the bottom row implies that there are no solutions to the system, and hence there is no state-price vector.

No State Prices: Existence of Arbitrage Portfolios

- You can create arbitrage portfolios. For example, the following are such portfolios

$$Y = (1000 \quad -5000 \quad 4000) \quad \text{and} \quad Z = (0 \quad -5000 \quad 5000)$$

- Note that each of the above portfolios costs nothing: the sum of the three entries add to zero.
- To show that they are indeed arbitrage portfolios, it must be that the portfolio cannot lose and in at least one of the states has a positive return. The product YR is

$$YR = (1000 \quad -5000 \quad 4000) \begin{pmatrix} 0.95 & 0.9 & 1.0 \\ 1.1 & 1.1 & 1.1 \\ 1.2 & 1.15 & 1.25 \end{pmatrix}$$

$$\implies YR = (250 \quad 0 \quad 500)$$

No State Prices: Existence of Arbitrage Portfolios

- Y is indeed an arbitrage portfolio: in none of the states does the value of the portfolio diminish (no negative entries in YR) and in two of them its value increases (strictly positive entries in two states). Similarly, ZR is given by

$$ZR = (0 \quad -5000 \quad 5000) \begin{pmatrix} 0.95 & 0.9 & 1.0 \\ 1.1 & 1.1 & 1.1 \\ 1.2 & 1.15 & 1.25 \end{pmatrix}$$

$$\implies ZR = (500 \quad 250 \quad 750)$$

- Z is also an arbitrage portfolio. In fact, since each entry of the column ZR is greater than the corresponding entry in YR , an investor with portfolio Z makes a greater return than one with portfolio Y , regardless of which of the three states occurs. In this sense, Z is 'better' than Y .
- **If there are no state prices, then arbitrage portfolios exist.**

References

Following sections/chapters of the textbook:

15.3 How to Make Money with Matrices

17.4 Arbitrage Portfolios and State Prices

19 The Input-Output Model