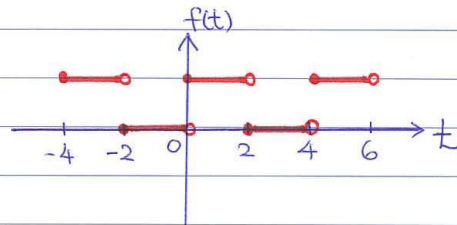


Solution to Problem Set 5:

Fourier Series

1.

$$f(t) = \begin{cases} 0 & \text{if } -2 \leq t < 0 \\ 1 & \text{if } 0 \leq t < 2 \end{cases}$$



(a) period $2T = 4 \Rightarrow T = 2$

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right) \right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{2}\right) + b_n \sin\left(\frac{n\pi t}{2}\right) \right]$$

where $a_0 = \frac{1}{T} \int_{-T}^T f(t) dt$

$$= \frac{1}{2} \left(\int_{-2}^0 0 dt + \int_0^2 1 dt \right)$$

$$= \frac{1}{2} [t]_0^2$$

$$= \frac{1}{2} (2 - 0)$$

$$= 1$$

$$a_n = \frac{1}{T} \int_{-T}^T f(t) \cos\left(\frac{n\pi t}{T}\right) dt$$

$$= \frac{1}{2} \left(\int_{-2}^0 0 \cdot \cos\left(\frac{n\pi t}{2}\right) dt + \int_0^2 1 \cdot \cos\left(\frac{n\pi t}{2}\right) dt \right)$$

$$= \frac{1}{2} \left[\frac{\sin\left(\frac{n\pi t}{2}\right)}{\frac{n\pi}{2}} \right]_0^2$$

$$= \frac{1}{2} \cdot \frac{2}{n\pi} [\sin(n\pi) - \sin 0]$$

$$= 0 \quad \because \sin(n\pi) = 0 \text{ for all } n=1, 2, 3, \dots$$

$$b_n = \frac{1}{T} \int_{-T}^T f(t) \sin\left(\frac{n\pi t}{T}\right) dt$$

$$= \frac{1}{2} \left(\int_{-2}^0 0 \cdot \sin\left(\frac{n\pi t}{2}\right) dt + \int_0^2 1 \cdot \sin\left(\frac{n\pi t}{2}\right) dt \right)$$

$$= \frac{1}{2} \left[\frac{-\cos\left(\frac{n\pi t}{2}\right)}{\frac{n\pi}{2}} \right]_0^2$$

$$= -\frac{1}{2} \cdot \frac{2}{n\pi} [\cos(n\pi) - \cos 0]$$

$$= \frac{1}{n\pi} [1 - (-1)^n] \quad \because \cos 0 = 1$$

$$\cos(n\pi) = (-1)^n \text{ for } n=1, 2, 3, \dots$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ The Fourier series for $f(t)$ is

$$f(t) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin\left[\frac{(2n-1)\pi t}{2}\right]$$

(b) Take $t=1$ so that

$$\sin\left[\frac{(2n-1)\pi t}{2}\right] = \sin\left(n\pi - \frac{\pi}{2}\right) = -\cos(n\pi) = (-1)^{n+1}$$

$f(t)$ is continuous at $t=1$.

$\therefore$ At $t=1$, the Fourier series converges to $f(1) = 1$.

Putting $t=1$ into the Fourier series :

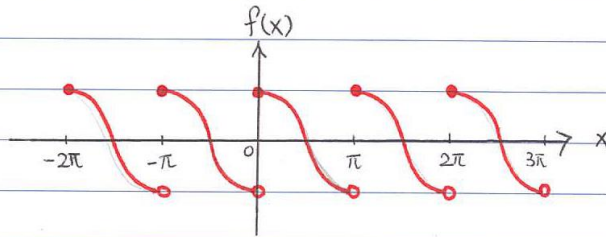
$$f(1) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \underbrace{\sin\left[\frac{(2n-1)\pi}{2}\right]}_{(-1)^{n+1}}$$

$$\Rightarrow 1 = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{2(-1)^n}{(2n-1)\pi}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = \frac{\pi}{2} \left(\frac{1}{2} - 1\right) = -\frac{\pi}{4}$$

2.

$$f(x) = \begin{cases} -\cos x, & \text{if } -\pi \leq x < 0 \\ \cos x, & \text{if } 0 \leq x < \pi \end{cases}$$



period $2T = 2\pi \Rightarrow T = \pi$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

Since $f(x)$ is an odd function (almost everywhere except at points of discontinuities $0, \pm\pi, \pm2\pi, \dots$),

$$a_0 = 0$$

$$a_n = 0 \text{ for } n=1, 2, 3, \dots$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos x \sin(nx) dx$$

$$= \frac{2}{\pi} \cdot \frac{1}{2} \int_0^{\pi} \{ \sin[(1+n)x] - \sin[(1-n)x] \} dx$$

$$\left(\because \cos A \sin B = \frac{1}{2} [\sin(A+B) - \sin(A-B)] \right)$$

$$= \frac{1}{\pi} \left[\frac{-\cos[(1+n)x]}{1+n} - \frac{-\cos[(1-n)x]}{1-n} \right]_0^{\pi}, \text{ for } \underline{n \neq 1}$$

$$= \frac{1}{\pi} \left\{ \left[\frac{-\cos[(1+n)\pi]}{1+n} + \frac{\cos[(1-n)\pi]}{1-n} \right] - \left[\frac{-\cos 0}{1+n} + \frac{\cos 0}{1-n} \right] \right\}$$

$$= \frac{1}{\pi} \left\{ \left[\frac{-\cos(\pi+n\pi)}{1+n} + \frac{\cos(\pi-n\pi)}{1-n} \right] - \left[\frac{-1}{1+n} + \frac{1}{1-n} \right] \right\}$$

$$= \frac{1}{\pi} \left[\frac{\cos(n\pi)}{1+n} + \frac{-\cos(n\pi)}{1-n} + \frac{1}{1+n} - \frac{1}{1-n} \right]$$

$$\left(\begin{aligned} \because \cos(\pi+\theta) &= -\cos \theta \\ \cos(\pi-\theta) &= -\cos \theta \end{aligned} \right)$$

$$= \frac{1}{\pi} \cdot \frac{[1+\cos(n\pi)](1-n) - [1+\cos(n\pi)](1+n)}{(1+n)(1-n)}$$

$$= \frac{-2n}{(1-n^2)\pi} [1 + (-1)^n] \quad , \text{ for } \underline{n \neq 1}$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd \& } n \neq 1 \\ \frac{4n}{(n^2-1)\pi} & \text{if } n \text{ is even} \end{cases}$$

$$b_1 = \frac{2}{\pi} \int_0^\pi \cos x \sin x \, dx$$

$$= \frac{2}{\pi} \int_0^\pi \frac{1}{2} \sin 2x \, dx \quad \because \sin 2x = 2 \sin x \cos x$$

$$= \frac{1}{\pi} \left[-\frac{\cos 2x}{2} \right]_0^\pi$$

$$= \frac{-1}{2\pi} [\cos(2\pi) - \cos 0]$$

$$= -\frac{1}{2\pi} (1-1)$$

$$= 0$$

$\therefore$ The Fourier series for $f(x)$ is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4(2n)}{[(2n)^2-1]\pi} \sin(2nx)$$

$$= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \sin(2nx)$$

3.(a) The Fourier series corresponding to $f(t)$ as $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$, where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (t^2 + 2t + 1) \cos ntdt \quad \begin{matrix} 2t \text{ is odd} \\ t^2+1 \text{ is even} \end{matrix} = \frac{2}{\pi} \int_0^{\pi} (t^2 + 1) \cos ntdt$$

$$= \frac{2}{\pi} \left[\left((t^2 + 1) \left(\frac{\sin nt}{n} \right) \right) \Big|_0^{\pi} - \int_0^{\pi} \left(\frac{\sin nt}{n} \right) 2t dt \right] = \frac{-4}{n\pi} \left[t \left(\frac{-\cos nt}{n} \right) \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos ntdt \right] = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (t^2 + 2t + 1) \sin ntdt = \frac{2}{\pi} \int_0^{\pi} 2t \sin ntdt = \frac{4}{\pi} \left[t \left(\frac{-\cos nt}{n} \right) \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos ntdt \right] = \frac{4(-1)^{n+1}}{n}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (t^2 + 2t + 1) dx = \frac{2}{\pi} \int_0^{\pi} (t^2 + 1) dt = \frac{2}{\pi} \left(\frac{t^3}{3} + t \right) \Big|_0^{\pi} = \frac{2(\pi^2 + 3)}{3}$$

Hence, the Fourier series for $f(t)$ is: $\frac{\pi^2 + 3}{3} + \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{n^2} \cos nt + \frac{4(-1)^{n+1}}{n} \sin nt \right)$

(b) The Fourier series corresponding to $f(t)$ is $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$ where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(t) dt = \frac{1}{\pi} \int_{\pi}^{2\pi} \sin t dt = \frac{1}{\pi} (-\cos t) \Big|_{\pi}^{2\pi} = \frac{-2}{\pi},$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt dt = \frac{1}{\pi} \int_{\pi}^{2\pi} \sin t \cos nt dt = \frac{1}{2\pi} \int_{\pi}^{2\pi} [\sin(n+1)t - \sin(n-1)t] dt$$

$$= \frac{1}{2\pi} \left[-\frac{\cos(n+1)t}{n+1} + \frac{\cos(n-1)t}{n-1} \right]_{\pi}^{2\pi}, n \neq 1$$

$$= \frac{1}{2\pi} \left[-\frac{1}{n+1} + \frac{1}{n-1} + \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} \right] = \frac{1}{\pi} \left[\frac{1}{n^2-1} - \frac{(-1)^{n+1}}{n^2-1} \right] = \begin{cases} 0 & \text{if } n \text{ is odd, } n \neq 1 \\ \frac{2}{(n^2-1)\pi} & \text{if } n \text{ is even} \end{cases}$$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos t dt = \frac{1}{\pi} \int_{\pi}^{2\pi} \sin t \cos t dt = \frac{1}{\pi} \left(\frac{\sin^2 t}{2} \right) \Big|_{\pi}^{2\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt dt = \frac{1}{\pi} \int_{\pi}^{2\pi} \sin t \sin nt dt = \frac{1}{2\pi} \int_{\pi}^{2\pi} [\cos(n-1)t - \cos(n+1)t] dt$$

$$= \frac{1}{2\pi} \left[\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1} \right]_{\pi}^{2\pi} = 0, n \neq 1$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin t dt = \frac{1}{\pi} \int_{\pi}^{2\pi} \sin^2 t dt = \frac{1}{2\pi} \int_{\pi}^{2\pi} (1 - \cos 2t) dt = \frac{1}{2\pi} \left(t - \frac{1}{2} \sin 2t \right) \Big|_{\pi}^{2\pi} = \frac{1}{2}.$$

Hence, the required series is $-\frac{1}{\pi} + \frac{1}{2} \sin t + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nt}{4n^2 - 1}$.

4.(a) At $t = \pi$, the infinite series converges to $\frac{1}{2}[(1-\pi)^2 + (1+\pi)^2]$, and hence

$$\frac{\pi^2 + 3}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} = 1 + \pi^2 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(b) Putting $t = 0$, the series converges to the function at $t = 0$, and hence

$$\frac{\pi^2 + 3}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

(c) Using the results in 4(a) and 4(b), we have $\sum_{n=1}^{\infty} [1 + (-1)^{n+1}] \frac{1}{n^2} = \frac{\pi^2}{6} + \frac{\pi^2}{12} \Rightarrow \sum_{\substack{n=1 \\ n \neq \text{even}}}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{4}$.

$$\text{So } \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

5. The Fourier cosine series for $f(x)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{2}{\pi} \left(\pi x - \frac{x^2}{2} \right) \Big|_0^{\pi} = \frac{2}{\pi} \left(\frac{\pi^2}{2} \right) = \pi,$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos nx dx = \frac{2}{\pi} \left[\left(\pi - x \frac{\sin nx}{n} \right) \Big|_0^{\pi} + \int_0^{\pi} \frac{\sin nx}{n} dx \right]$$

$$= \frac{2}{n\pi} \left(-\frac{\cos nx}{n} \right) \pi = -\frac{2}{n^2\pi} (\cos n\pi - \cos 0) = \begin{cases} 0 & , \text{if } n \text{ is even} \\ \frac{4}{n^2\pi} & , \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \text{ The Fourier series of } f(x) \text{ is } f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi} \cos(2n-1)x$$

The Fourier sine series for $f(x)$ is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where

$$b_n = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \sin nx dx = \frac{2}{\pi} \left[(\pi - x) \left(-\frac{\cos nx}{n} \right) \pi + \int_0^{\pi} \frac{-\cos nx}{n} dx \right]$$

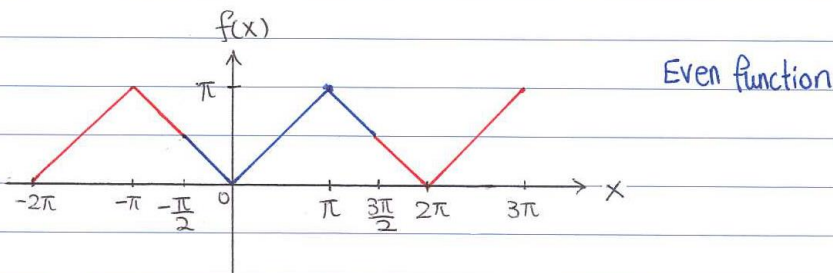
$$= \frac{2}{\pi} \left[\left(0 - \pi \left(\frac{-1}{n} \right) - \frac{1}{n} \left(\frac{\sin nx}{n} \right) \pi \right) \right] = \frac{2}{\pi} \left[\frac{\pi}{n} - \frac{1}{n^2} (\sin n\pi - \sin 0) \right] = \frac{2}{n}$$

$$\therefore \text{ The Fourier sine series of } f(x) \text{ is } f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin nt$$

6.

$$f(x) = \begin{cases} -x & , \quad -\frac{\pi}{2} \leq x < 0 \\ x & , \quad 0 \leq x < \pi \\ 2\pi - x & , \quad \pi \leq x \leq \frac{3\pi}{2} \end{cases} \quad , \text{ period} = 2\pi$$

(a)



$$(b) \text{ Period } 2T = 2\pi \Rightarrow T = \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

Since $f(x)$ is an even function, $b_n = 0$.

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left(\frac{\pi^2}{2} - 0 \right) = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x d\left(\frac{\sin(nx)}{n}\right)$$

$$= \frac{2}{\pi} \left\{ \left[x \cdot \frac{\sin(nx)}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n} dx \right\}$$

$$= \frac{2}{\pi} \left\{ (0-0) - \left[\frac{-\cos(nx)}{n^2} \right]_0^{\pi} \right\}$$

$$= \frac{2}{\pi} \cdot \frac{1}{n^2} [\cos(n\pi) - \cos 0]$$

$$= \frac{2}{n^2\pi} [(-1)^n - 1]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ The Fourier series for $f(x)$ is

$$f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi} \cos[(2n-1)x]$$

7.(a) Writing the Fourier cosine series of $\sin x$ on $[0, \pi]$, we have $\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x dx = \frac{2}{\pi} [-\cos x]_0^{\pi} = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx dx = \frac{2}{\pi} \int_0^{\pi} \left[\frac{\sin(n+1)x}{2} + \frac{\sin(1-n)x}{2} \right] dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} \sin(n+1)x dx - \int_0^{\pi} \sin(n-1)x dx \right] = \frac{1}{\pi} \left[\frac{-\cos(n+1)x}{n+1} \Big|_0^{\pi} + \frac{\cos(n-1)x}{n-1} \Big|_0^{\pi} \right] \quad (n \neq 1)$$

$$= \frac{1}{\pi} \left[(-1)^n \frac{1}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1} \right] = \frac{1}{\pi} \begin{cases} 0 & n \text{ odd } (n \neq 1) \\ \frac{2}{n+1} - \frac{2}{n-1} & n \text{ even} \end{cases}$$

$$= \begin{cases} 0 & n \text{ odd} \\ -\frac{4}{\pi(n^2-1)} & n \text{ even} \end{cases} \quad (n = 2, 3, 4, \dots)$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi} \sin x \cos x dx = \frac{2}{\pi} \int_0^{\pi} \sin x d(\sin x) = \frac{2}{\pi} \left[\frac{\sin^2 x}{2} \right]_0^{\pi} = 0$$

$$\therefore a_n = \begin{cases} 0, & n \text{ odd} \\ -\frac{4}{\pi(n^2-1)}, & n \text{ even} \end{cases}, \text{ for } n = 1, 2, 3, \dots$$

$$\Rightarrow a_{2n} = \frac{-4}{\pi(4n^2-1)}, \text{ for } n = 1, 2, 3, \dots$$

$$\text{So } \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n} \cos 2nx = \frac{4}{2\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2-1)} \cos 2nx = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos 2nx$$

(b) Put $x = \frac{\pi}{2}$, then

$$\begin{aligned} 1 &= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos n\pi \Rightarrow 1 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} \Rightarrow \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{2}{\pi} - 1 = \frac{2-\pi}{\pi} \\ &\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{2-\pi}{4} \end{aligned}$$

-End-