

Homework 4 Solutions

Question 1: (*20 points*) A consumer's preferences are represented by the utility function

$$u(x_1, x_2) = 2\ln x_1 + \ln x_2$$

If the budget constraint is $p_1x_1 + p_2x_2 = M$, determine the demand functions for x_1 and x_2 using the Lagrangean method.

The Lagrangean for the problem is

$$(2 \text{ points}) \quad \mathcal{L} = 2\ln x_1 + \ln x_2 - \lambda(p_1x_1 + p_2x_2 - M)$$

The resulting conditions are

$$(2 \text{ points}) \quad \frac{\partial \mathcal{L}}{\partial x_1} = \frac{2}{x_1} - \lambda p_1 = 0$$

$$(2 \text{ points}) \quad \frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{x_2} - \lambda p_2 = 0$$

$$(2 \text{ points}) \quad p_1x_1 + p_2x_2 - M = 0$$

From the first two equations, we have

$$\frac{2}{p_1x_1} = \frac{1}{p_2x_2}$$

$$(2 \text{ points}) \quad \implies x_1 = \left(\frac{2p_2}{p_1}\right) x_2$$

Substituting this into the third equation, we get

$$p_1 \left(\frac{2p_2}{p_1}\right) x_2 + p_2x_2 = M$$

$$(5 \text{ points}) \quad x_2 = \frac{M}{3p_2}$$

And,

$$(5 \text{ points}) \quad x_1 = \frac{2M}{3p_1}$$

Question 2: (20 points) A firm has production function

$$q(k, l) = 50k^{2/3}l^{1/3},$$

and unit capital and labor costs of 6 and 4, respectively. What is the maximum output achievable in a week if the firm spends no more than 1000 each week? Use the Lagrangean method to solve the problem.

(2 points) The problem is to maximize the output $q(k, l) = 50k^{2/3}l^{1/3}$ subject to the constraint $6k + 4l = 1000$.

The Lagrangean for the problem is

$$(2 \text{ points}) \quad \mathcal{L} = 50k^{2/3}l^{1/3} - \lambda(6k + 4l - 1000)$$

The conditions are

$$(2 \text{ points}) \quad \frac{\partial \mathcal{L}}{\partial k} = \frac{100}{3}k^{-1/3}l^{1/3} - 6\lambda = 0$$

$$(2 \text{ points}) \quad \frac{\partial \mathcal{L}}{\partial l} = \frac{50}{3}k^{2/3}l^{-2/3} - 4\lambda = 0$$

$$(2 \text{ points}) \quad 6k + 4l - 1000 = 0$$

From the first two equations, we have

$$\frac{100}{18}k^{-1/3}l^{1/3} = \frac{50}{12}k^{2/3}l^{-2/3}$$

$$\implies l = 3k/4$$

The third equation then gives

$$6k + 4\left(\frac{3k}{4}\right) = 1000 \implies (3 \text{ points}) \quad k = \frac{1000}{9}$$

And,

$$(3 \text{ points}) \quad l = \frac{1000}{12}$$

Maximal output is

$$(4 \text{ points}) \quad 50 \left(\frac{1000}{9}\right)^{2/3} \left(\frac{1000}{12}\right)^{1/3} \approx 5047.6$$

Question 3: (20 points) Use the method of elementary row operations to solve the following system of equations

$$x_1 + 2x_2 + 2x_3 = 7$$

$$4x_1 + 5x_2 + x_3 = 11$$

$$7x_1 - 4x_2 - x_3 = 1$$

The system of equations can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 4 & 5 & 1 \\ 7 & -4 & -1 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 7 \\ 11 \\ 1 \end{pmatrix}$$

The augmented matrix is given by

$$(2 \text{ points}) \quad (A|\mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 2 & 2 & 7 \\ 4 & 5 & 1 & 11 \\ 7 & -4 & -1 & 1 \end{array} \right)$$

Using elementary row operations, we have to reduce the above matrix to echelon form:

$$\xrightarrow{R_3 - 7R_1 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 7 \\ 4 & 5 & 1 & 11 \\ 0 & -18 & -15 & -48 \end{array} \right)$$

$$\xrightarrow{R_2 - 4R_1 \rightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 7 \\ 0 & -3 & -7 & -17 \\ 0 & -18 & -15 & -48 \end{array} \right)$$

$$\xrightarrow{6R_2 - R_3 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 7 \\ 0 & -3 & -7 & -17 \\ 0 & 0 & -27 & -54 \end{array} \right)$$

$$(10 \text{ points}) \quad \xrightarrow{(-1/27)R_3 \rightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 7 \\ 0 & -3 & -7 & -17 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

(2 points) Then, the above system of equations can be written as

$$\begin{aligned}x_1 + 2x_2 + 2x_3 &= 7 \\-3x_2 - 7x_3 &= -17 \\x_3 &= 2\end{aligned}$$

Using back substitution, we get

$$\begin{aligned}-3x_2 &= -17 + 14 = -3 \implies x_2 = 1 \\x_1 + (2)(1) + (2)(2) &= 7 \implies x_1 = 1\end{aligned}$$

(6 points) Therefore, $x_1 = 1, x_2 = 1, x_3 = 2$.

There are many ways of reducing the augmented matrix to echelon form.

Question 4: (20 points) Use the method of elementary row operations to find the inverse of

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 5 & 5 \\ 2 & 11 & 13 \end{pmatrix}$$

We form the matrix

$$(2 \text{ points}) \quad (A|I) = \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 1 & 5 & 5 & 0 & 1 & 0 \\ 2 & 11 & 13 & 0 & 0 & 1 \end{array} \right)$$

Performing elementary row operations, we have

$$\begin{aligned}(A|I) &= \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 1 & 5 & 5 & 0 & 1 & 0 \\ 2 & 11 & 13 & 0 & 0 & 1 \end{array} \right) \\&\xrightarrow{R_3 - 2R_1 \rightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 1 & 5 & 5 & 0 & 1 & 0 \\ 0 & 3 & 7 & -2 & 0 & 1 \end{array} \right) \\&\xrightarrow{R_2 - R_1 \rightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 7 & -2 & 0 & 1 \end{array} \right)\end{aligned}$$

$$\begin{aligned}
 & \xrightarrow{R_3 - 3R_2 \rightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array} \right) \\
 & \xrightarrow{R_2 - 2R_3 \rightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 7 & -2 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array} \right) \\
 & \xrightarrow{R_1 - 3R_3 \rightarrow R_1} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -2 & 9 & -3 \\ 0 & 1 & 0 & -3 & 7 & -2 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array} \right) \\
 (14 \text{ points}) \quad & \xrightarrow{R_1 - 4R_2 \rightarrow R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -19 & 5 \\ 0 & 1 & 0 & -3 & 7 & -2 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array} \right)
 \end{aligned}$$

Therefore, the inverse of the matrix is

$$(4 \text{ points}) \quad \begin{pmatrix} 10 & -19 & 5 \\ -3 & 7 & -2 \\ 1 & -3 & 1 \end{pmatrix}$$

There are many ways of reducing the augmented matrix $(A|I)$ to $(I|A^{-1})$.

Question 5: (20 points) Calculate the determinant of the matrix

$$\begin{pmatrix} 5 & 7 & -2 \\ 10 & 15 & -2 \\ 5 & 8 & -1 \end{pmatrix}$$

The determinant of the matrix is given by

$$(2 \text{ points}) \quad \begin{vmatrix} 5 & 7 & -2 \\ 10 & 15 & -2 \\ 5 & 8 & -1 \end{vmatrix}$$

$$(8 \text{ points}) \quad = 5 \begin{vmatrix} 15 & -2 \\ 8 & -1 \end{vmatrix} - 7 \begin{vmatrix} 10 & -2 \\ 5 & -1 \end{vmatrix} + (-2) \begin{vmatrix} 10 & 15 \\ 5 & 8 \end{vmatrix}$$

$$(6 \text{ points}) \quad = 5 \left((15)(-1) - (8)(-2) \right) - 7 \left((10)(-1) - (5)(-2) \right) - 2 \left((10)(8) - (5)(15) \right)$$

$$= 5(-15 + 16) - 7(-10 + 10) - 2(80 - 75)$$

$$(4 \text{ points}) \quad = 5 - (2)(5) = -5$$

You may use any alternative method to get to the same answer.