

Solution to Problem Set 4:

Systems of Linear differential Equations

1. $x_1 = x \Rightarrow x_1' = x' = x_2$,
 $x_2 = x' \Rightarrow x_2' = x'' = x_3$,
 $x_3 = x'' \Rightarrow x_3' = x''' = -x'' + x' + x = -x_3 + x_2 + x_1$.

The differential equation becomes $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

The equation is in the form $\frac{d\vec{u}}{dt} = A\vec{u}$, where $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

Trying $\vec{u} = e^{\lambda t} \vec{v}$, where $\vec{v} \neq \vec{0}$.

Solve $|A - \lambda I_3| = 0$.

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda[-\lambda(-1-\lambda) - 1] - (0 - 1) = 0$$

$$\Rightarrow -\lambda^3 - \lambda^2 + \lambda + 1 = 0$$

$$\Rightarrow -\lambda^2(\lambda + 1) + (\lambda + 1) = 0$$

$$\Rightarrow (1 - \lambda^2)(\lambda + 1) = 0$$

$$\Rightarrow (1 - \lambda)(1 + \lambda)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 1, -1, -1$$

For $\lambda = 1$, consider $\begin{pmatrix} -1 & 1 & 0 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 1 & 1 & -2 & | & 0 \end{pmatrix} \xrightarrow{r_3 - r_1} \begin{pmatrix} 1 & -1 & 0 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 0 & 2 & -2 & | & 0 \end{pmatrix} \xrightarrow{r_3 + 2r_2} \begin{pmatrix} 1 & -1 & 0 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$

So $v_2 = v_3, v_1 = v_2 = v_3$. Let $v_3 = 1$, then $v_1 = v_2 = v_3 = 1$. It follows that $\vec{u}_1 = e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a solution

to $\frac{d\vec{u}}{dt} = A\vec{u}$.

For $\lambda = -1$, consider $\begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 1 & 1 & 0 & | & 0 \end{pmatrix} \xrightarrow{r_3 - r_1} \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$

So $v_2 = -v_3, v_1 = -v_2 = v_3$. Let $v_3 = 1$, then $v_1 = v_3 = 1, v_2 = -1$. It follows that $e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is a

solution to $\frac{d\vec{u}}{dt} = A\vec{u}$. We need to find another linearly independent solution.

Find a vector $\vec{w}$ such that $(A - (-1)I_3)\vec{w} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 1 & 0 & | & 1 \\ 0 & 1 & 1 & | & -1 \\ 1 & 1 & 0 & | & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 1 & 0 & | & 1 \\ 0 & 1 & 1 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\therefore \begin{cases} w_1 + w_2 = 1 \\ w_2 + w_3 = -1 \end{cases}$$

Take $w_1 = 0$, then $w_2 = 1$ and $w_3 = -2$. $\therefore \vec{w} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

$\therefore \vec{u}_2 = e^{-t} \left[\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right] = e^{-t} \begin{pmatrix} t \\ 1-t \\ -2+t \end{pmatrix}$ is another linearly independent solution.

Hence the general solution is $\vec{u} = c_1 e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} t \\ 1-t \\ -2+t \end{pmatrix}$.

As a result, the general solution to $x''' + x'' - x' - x = 0$ is $x = c_1 e^t + c_2 e^{-t} + c_3 t e^{-t}$

2.
$$\begin{cases} x_1' = x_1 \\ x_2' = x_2 - x_3 \\ x_3' = x_2 + x_3 \end{cases} \Leftrightarrow \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Equation is in the form $\frac{d\vec{u}}{dt} = A\vec{u}$, where $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$.

Trying $\vec{u} = e^{\lambda t} \vec{v}$, where $\vec{v} \neq \vec{0}$.

Solve $|A - \lambda I_3| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(1-\lambda)^2 + 1] = 0$$

$$\Rightarrow \lambda = 1 \text{ or } (\lambda - 1)^2 + 1 = 0$$

$$\Rightarrow \lambda = 1, 1 \pm i$$

For $\lambda = 1$, consider $\begin{pmatrix} 0 & 0 & 0 & | & 0 \\ 0 & 0 & -1 & | & 0 \\ 0 & 1 & 0 & | & 0 \end{pmatrix}$

So $v_2 = v_3 = 0$. Let $v_1 = 1$. It follows that $e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is a solution to $\frac{d\vec{u}}{dt} = A\vec{u}$.

For $\lambda = 1 + i$, consider $\begin{pmatrix} -i & 0 & 0 & | & 0 \\ 0 & -i & -1 & | & 0 \\ 0 & 1 & -i & | & 0 \end{pmatrix} \sim \begin{pmatrix} i & 0 & 0 & | & 0 \\ 0 & i & 1 & | & 0 \\ 0 & 1 & -i & | & 0 \end{pmatrix} \xrightarrow{r_3 + ir_2} \begin{pmatrix} i & 0 & 0 & | & 0 \\ 0 & i & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$

So $v_2 = -\frac{1}{i}v_3 = iv_3, v_1 = 0$. Let $v_3 = 1$, then $v_2 = i$. It follows that $e^{(1+i)t} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$ is a complex solution

to $\frac{d\vec{u}}{dt} = A\vec{u}$.

$$\text{Now, } e^{(1+i)t} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} = e^t (\cos t + i \sin t) \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] = e^t \begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} + ie^t \begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix}$$

We know that if $\overline{z(t)} = \overline{x(t)} + \overline{y(t)}i$ is a complex-valued solution to $\frac{d\vec{u}}{dt} = A\vec{u}$, then both

$\overline{x(t)}, \overline{y(t)}$ are real-valued solution to $\frac{d\vec{u}}{dt} = A\vec{u}$, that is, $e^t \begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix}, e^t \begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix}$ are real-valued

solution to $\frac{d\vec{u}}{dt} = A\vec{u}$.

So the solution of the initial-valued problem is $\vec{u}(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix}$.

$$\text{Setting } t=0, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{u}(0) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -\sin 0 \\ \cos 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ \cos 0 \\ \sin 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_3 \\ c_2 \end{pmatrix} \Rightarrow c_1 = c_2 = c_3 = 1$$

$$\vec{u}(t) = e^t \begin{pmatrix} 1 \\ \cos t - \sin t \\ \cos t + \sin t \end{pmatrix}$$

$$\begin{aligned} 3. \quad \text{Consider } \det(A - \lambda I_5) &= \begin{vmatrix} 1-\lambda & 1 & 1 & 1 & 1 \\ 2 & 2-\lambda & 2 & 2 & 2 \\ 3 & 3 & 3-\lambda & 3 & 3 \\ 4 & 4 & 4 & 4-\lambda & 4 \\ 5 & 5 & 5 & 5 & 5-\lambda \end{vmatrix} \\ &\stackrel{R_1+R_2+R_3+R_4+R_5}{=} \begin{vmatrix} 15-\lambda & 15-\lambda & 15-\lambda & 15-\lambda & 15-\lambda \\ 2 & 2-\lambda & 2 & 2 & 2 \\ 3 & 3 & 3-\lambda & 3 & 3 \\ 4 & 4 & 4 & 4-\lambda & 4 \\ 5 & 5 & 5 & 5 & 5-\lambda \end{vmatrix} \stackrel{C_2-C_1}{=} \begin{vmatrix} 15-\lambda & 0 & 0 & 0 & 0 \\ 2 & -\lambda & 0 & 0 & 0 \\ 3 & 0 & -\lambda & 0 & 0 \\ 4 & 0 & 0 & -\lambda & 0 \\ 5 & 0 & 0 & 0 & -\lambda \end{vmatrix} \\ &\stackrel{C_3-C_1}{=} \begin{vmatrix} 15-\lambda & 0 & 0 & 0 & 0 \\ 2 & -\lambda & 0 & 0 & 0 \\ 3 & 0 & -\lambda & 0 & 0 \\ 4 & 0 & 0 & -\lambda & 0 \\ 5 & 0 & 0 & 0 & -\lambda \end{vmatrix} \stackrel{C_4-C_1}{=} \begin{vmatrix} 15-\lambda & 0 & 0 & 0 & 0 \\ 2 & -\lambda & 0 & 0 & 0 \\ 3 & 0 & -\lambda & 0 & 0 \\ 4 & 0 & 0 & -\lambda & 0 \\ 5 & 0 & 0 & 0 & -\lambda \end{vmatrix} \\ &\stackrel{C_5-C_1}{=} \begin{vmatrix} 15-\lambda & 0 & 0 & 0 & 0 \\ 2 & -\lambda & 0 & 0 & 0 \\ 3 & 0 & -\lambda & 0 & 0 \\ 4 & 0 & 0 & -\lambda & 0 \\ 5 & 0 & 0 & 0 & -\lambda \end{vmatrix} \\ &= (15-\lambda)(-\lambda)^4 \\ \therefore \det(A - \lambda I_5) &= 0 \Rightarrow \lambda = 0, 0, 0, 0, 15 \\ \text{For } \lambda = 0, \text{ solving } (A - \lambda I_5)\vec{v} &= \vec{0}. \end{aligned}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 3 & 3 & 0 \\ 4 & 4 & 4 & 4 & 4 & 0 \\ 5 & 5 & 5 & 5 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\therefore x_1 + x_2 + x_3 + x_4 + x_5 = 0$$

Let $x_1 = p$, $x_2 = q$, $x_3 = r$, $x_4 = s$. Then $x_5 = -p - q - r - s$.

$$\therefore \vec{v} = p \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + r \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Hence, $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ are four independent eigenvectors of A corresponding to eigenvalue 0.

Thus, $e^{0t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$, $e^{0t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$, $e^{0t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$, $e^{0t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ are independent solutions of $\frac{d\vec{u}}{dt} = A\vec{u}$.

Moreover, observe that $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ is an eigenvector of A corresponding to eigenvalue 15 since

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = (1+2+3+4+5) \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = 15 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}. \text{ So } e^{15t} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \text{ is an independent solution of}$$

$$\frac{d\vec{u}}{dt} = A\vec{u}.$$

Hence, the general solution to the system $\frac{d\vec{u}}{dt} = A\vec{u}$ is

$$\vec{u} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + c_5 e^{15t} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}.$$

$$4. \quad (a) \quad \begin{cases} \frac{dx}{dt} - 3x - 6y = 0 \\ \frac{dx}{dt} + \frac{dy}{dt} - 3y = 0 \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} - 3x - 6y = 0 \\ (3x + 6y) + \frac{dy}{dt} - 3y = 0 \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} - 3x - 6y = 0 \\ \frac{dy}{dt} + 3x + 3y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{dx}{dt} = 3x + 6y \\ \frac{dy}{dt} = -3x - 3y \end{cases} \Rightarrow \frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{pmatrix} 3 & 6 \\ -3 & -3 \end{pmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$(b) \quad e^{3it} \begin{pmatrix} 1+i \\ -1 \end{pmatrix} \text{ is a solution of } \frac{d\vec{u}}{dt} = A\vec{u}.$$

$$e^{3it} \begin{pmatrix} 1+i \\ -1 \end{pmatrix} = (\cos 3t + i \sin 3t) \begin{pmatrix} 1+i \\ -1 \end{pmatrix} = \begin{bmatrix} \cos 3t - \sin 3t + i(\sin 3t + \cos 3t) \\ -\cos 3t - i \sin 3t \end{bmatrix}$$

$$= \begin{pmatrix} \cos 3t - \sin 3t \\ -\cos 3t \end{pmatrix} + i \begin{pmatrix} \sin 3t + \cos 3t \\ -\sin 3t \end{pmatrix}.$$

Observe that the real part $\begin{pmatrix} \cos 3t - \sin 3t \\ -\cos 3t \end{pmatrix}$ and the imaginary part $\begin{pmatrix} \sin 3t + \cos 3t \\ -\sin 3t \end{pmatrix}$ of $e^{3it} \begin{pmatrix} 1+i \\ -1 \end{pmatrix}$ are also independent solutions of $\frac{d\vec{u}}{dt} = A\vec{u}$.

Thus, the general solution of $\frac{d\vec{u}}{dt} = A\vec{u}$ is $\vec{u}(t) = c_1 \begin{pmatrix} \cos 3t - \sin 3t \\ -\cos 3t \end{pmatrix} + c_2 \begin{pmatrix} \sin 3t + \cos 3t \\ -\sin 3t \end{pmatrix}$.

It is given that $\vec{u}(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$\vec{u}(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} \cos 3t - \sin 3t \\ -\cos 3t \end{pmatrix} \Big|_{t=0} + c_2 \begin{pmatrix} \sin 3t + \cos 3t \\ -\sin 3t \end{pmatrix} \Big|_{t=0}$$

$$= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow \dots \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\text{So } \vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = - \begin{pmatrix} \cos 3t - \sin 3t \\ -\cos 3t \end{pmatrix} + \begin{pmatrix} \sin 3t + \cos 3t \\ -\sin 3t \end{pmatrix} \Rightarrow \vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{pmatrix} 2 \sin 3t \\ \cos 3t - \sin 3t \end{pmatrix}$$

$$5. \quad \det \begin{pmatrix} 3-\lambda & 3 \\ 1 & 5-\lambda \end{pmatrix} = (3-\lambda)(5-\lambda) - 3 = \lambda^2 - 8\lambda + 12 = (\lambda-6)(\lambda-2)$$

A has eigenvalues 2 and 6.

For $\lambda = 2$, solving $(A - \lambda I_2)\vec{v} = \vec{0}$, i.e. $\left(\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array} \right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array} \right)$

$\therefore v_1 + 3v_2 = 0$. Let $v_2 = s$, then $v_1 = -3s$.

The eigenvectors of A corresponding to $\lambda = 2$ are $\vec{v} = s \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $s \neq 0$.

$\therefore \vec{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = 2$.

$\therefore \vec{u}_1 = e^{2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is a solution to $\frac{d\vec{u}}{dt} = A\vec{u}$.

For $\lambda = 6$, solving $(A - \lambda I_2)\vec{v} = \vec{0}$, i.e. $\left(\begin{array}{cc|c} -3 & 3 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{3R_2+R_1} \left(\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\therefore -3v_1 + 3v_2 = 0$. Let $v_2 = s$, then $v_1 = s$.

The eigenvectors of A corresponding to $\lambda = 6$ are $\vec{v} = s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $s \neq 0$.

$\therefore \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = 6$.

$\therefore \vec{u}_2 = e^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution to $\frac{d\vec{u}}{dt} = A\vec{u}$.

So the complementary solution to $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}$ is:

$$\vec{u}_c(t) = c_1 e^{2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Omega(t)C,$$

where $\Omega(t) = \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix}$, $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. The matrix $\Omega(t) = \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix}$ is called a

fundamental matrix of the homogeneous system $\frac{d\vec{u}}{dt} = A\vec{u}$.

$$\text{Let } \vec{u}_p = h_1(t)e^{2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + h_2(t)e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\text{Then } h_1'(t)e^{2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + h_2'(t)e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix} \dots \dots (*),$$

$$\text{i.e. } \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} h_1'(t) \\ h_2'(t) \end{pmatrix} = \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}.$$

$$\begin{aligned} \Rightarrow \begin{pmatrix} h_1'(t) \\ h_2'(t) \end{pmatrix} &= \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix} = \frac{1}{\begin{vmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{vmatrix}} \begin{pmatrix} e^{6t} & -e^{6t} \\ -e^{2t} & -3e^{2t} \end{pmatrix} \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix} \\ &= \frac{1}{-3e^{8t} - e^{8t}} \begin{pmatrix} 8e^{6t} - 4e^{9t} \\ -8e^{2t} - 12e^{5t} \end{pmatrix} = \begin{pmatrix} -2e^{-2t} + e^t \\ 2e^{-6t} + 3e^{-3t} \end{pmatrix} \end{aligned}$$

Alternative way to find $h_1'(t)$, $h_2'(t)$:

$$\text{From } (*), \text{ we have } \begin{cases} -3e^{2t}h_1'(t) + e^{6t}h_2'(t) = 8 \dots \dots (1) \\ e^{2t}h_1'(t) + e^{6t}h_2'(t) = 4e^{3t} \dots \dots (2) \end{cases}$$

$$3 \times (2) + (1) \text{ gives } 4e^{6t}h_2'(t) = 12e^{3t} + 8 \Rightarrow h_2'(t) = 3e^{-3t} + 2e^{-6t}.$$

$$(2) - (1) \text{ gives } 4e^{2t}h_1'(t) = 4e^{3t} - 8 \Rightarrow h_1'(t) = e^t - 2e^{-2t}.$$

$$\Rightarrow \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} \int (-2e^{-2t} + e^t) dt \\ \int (2e^{-6t} + 3e^{-3t}) dt \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}e^{-6t} - e^{-3t} \\ e^{-2t} + e^t \end{pmatrix}$$

$$\therefore \vec{u}_p = (e^{-2t} + e^t)e^{2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \left(-\frac{1}{3}e^{-6t} - e^{-3t}\right)e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{10}{3} - 4e^{3t} \\ \frac{2}{3} \end{pmatrix}.$$

Alternatively,

$$\vec{u}_p = \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} e^{-2t} + e^t \\ -\frac{1}{3}e^{-6t} - e^{-3t} \end{pmatrix} = \dots = \begin{pmatrix} -\frac{10}{3} - 4e^{3t} \\ \frac{2}{3} \end{pmatrix}$$

Hence, the general solution of the original system is

$$\vec{u} = C_1 e^{2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{10}{3} - 4e^{3t} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} -3C_1 e^{2t} + C_2 e^{6t} - \frac{10}{3} - 4e^{3t} \\ C_1 e^{2t} + C_2 e^{6t} + \frac{2}{3} \end{pmatrix}$$

6. Try $\vec{u} = e^{\lambda t} \vec{v}$, where $\vec{v} \neq \vec{0}$.

Solve $|A - \lambda I_2| = 0$, i.e.

$$\begin{vmatrix} 1-\lambda & -10 \\ -1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) - 10 = 0 \Rightarrow \lambda^2 - 5\lambda - 6 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda + 1) = 0 \Rightarrow \lambda = -1, 6 \text{ are the eigenvalues of } A.$$

For $\lambda = -1$, solving $(A - \lambda I_2)\vec{v} = \vec{0}$, i.e. $\left(\begin{array}{cc|c} 2 & -10 & 0 \\ -1 & 5 & 0 \end{array}\right) \xrightarrow{2R_2+R_1} \left(\begin{array}{cc|c} 2 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\therefore 2v_1 - 10v_2 = 0$. Let $v_2 = s$, then $v_1 = 5s$.

The eigenvectors of A corresponding to $\lambda = -1$ are $\vec{v} = s \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, $s \neq 0$.

$\therefore \vec{v} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = -1$.

$\therefore \vec{u}_1 = e^{-t} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is a solution to $\frac{d\vec{u}}{dt} = A\vec{u}$.

For $\lambda = 6$, solving $(A - \lambda I_2)\vec{v} = \vec{0}$, i.e. $\left(\begin{array}{cc|c} -5 & -10 & 0 \\ -1 & -2 & 0 \end{array}\right) \xrightarrow{5R_2-R_1} \left(\begin{array}{cc|c} -5 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\therefore -5v_1 - 10v_2 = 0$. Let $v_2 = s$, then $v_1 = -2s$.

The eigenvectors of A corresponding to $\lambda = 6$ are $\vec{v} = s \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $s \neq 0$.

$\therefore \vec{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = 6$.

$\therefore \vec{u}_2 = e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is a solution to $\frac{d\vec{u}}{dt} = A\vec{u}$.

Hence, the complementary solution to $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}$ is

$$\vec{u}_c = C_1 e^{-t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

For the particular solution to $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}$, we let

$$\vec{u}_p = h_1(t) e^{-t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + h_2(t) e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Then $h_1'(t) e^{-t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + h_2'(t) e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} e^t \\ \sin t \end{pmatrix}$.

$$\text{i.e. } \begin{pmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{pmatrix} \begin{pmatrix} h_1'(t) \\ h_2'(t) \end{pmatrix} = \begin{pmatrix} e^t \\ \sin t \end{pmatrix}.$$

$$\begin{aligned} \Rightarrow \begin{pmatrix} h_1'(t) \\ h_2'(t) \end{pmatrix} &= \begin{pmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{pmatrix}^{-1} \begin{pmatrix} e^t \\ \sin t \end{pmatrix} = \frac{1}{\begin{vmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{vmatrix}} \begin{pmatrix} e^{6t} & 2e^{6t} \\ -e^{-t} & 5e^{-t} \end{pmatrix} \begin{pmatrix} e^t \\ \sin t \end{pmatrix} \\ &= \frac{1}{5e^{5t} + 2e^{5t}} \begin{pmatrix} e^{7t} + 2e^{6t} \sin t \\ -1 + 5e^{-t} \sin t \end{pmatrix} = \begin{pmatrix} \frac{1}{7}e^{2t} + \frac{2}{7}e^t \sin t \\ -\frac{1}{7}e^{-5t} + \frac{5}{7}e^{-6t} \sin t \end{pmatrix} \end{aligned}$$

$$\therefore h_1(t) = \int \left(\frac{1}{7}e^{2t} + \frac{2}{7}e^t \sin t \right) dt = \dots = \frac{1}{14}e^{2t} + \frac{1}{7}e^t(\sin t - \cos t)$$

$$h_2(t) = \int \left(-\frac{1}{7}e^{-5t} + \frac{5}{7}e^{-6t} \sin t \right) dt = \dots = \frac{1}{35}e^{-5t} + \frac{5}{259}e^{-6t}(-6 \sin t - \cos t)$$

Thus,

$$\begin{aligned} \vec{u}_p &= \begin{pmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{pmatrix} \begin{pmatrix} \frac{1}{14}e^{2t} + \frac{1}{7}e^t(\sin t - \cos t) \\ \frac{1}{35}e^{-5t} + \frac{5}{259}e^{-6t}(-6 \sin t - \cos t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{10}e^t + \frac{35}{37}\sin t - \frac{25}{37}\cos t \\ \frac{1}{10}e^t + \frac{1}{37}\sin t - \frac{6}{37}\cos t \end{pmatrix} \end{aligned}$$

The general solution of $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}$ is

$$\begin{aligned} \vec{u} &= C_1 e^{-t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{3}{10}e^t + \frac{35}{37}\sin t - \frac{25}{37}\cos t \\ \frac{1}{10}e^t + \frac{1}{37}\sin t - \frac{6}{37}\cos t \end{pmatrix} \\ &= \begin{pmatrix} 5C_1 e^{-t} - 2C_2 e^{6t} + \frac{3}{10}e^t + \frac{35}{37}\sin t - \frac{25}{37}\cos t \\ C_1 e^{-t} + C_2 e^{6t} + \frac{1}{10}e^t + \frac{1}{37}\sin t - \frac{6}{37}\cos t \end{pmatrix} \end{aligned}$$

$$7. \quad \det(A - \lambda I_2) = \begin{vmatrix} 5-\lambda & -4 \\ 4 & -3-\lambda \end{vmatrix} = (5-\lambda)(-3-\lambda) + 16 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

Hence the eigenvalue is $\lambda = 1$ with multiplicity 2.

For $\lambda = 1$, we have

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 4 & -4 & | & 0 \\ 4 & -4 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \Rightarrow x_1 = x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}, k \neq 0 \Rightarrow_{k=1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

One solution of $\frac{d\vec{u}}{dt} = A\vec{u}$ is $\vec{u}_1 = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

To find another linearly independent solution of $\frac{d\vec{u}}{dt} = A\vec{u}$, we solve

$$(A - I_2)\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ i.e. } \left(\begin{array}{cc|c} 4 & -4 & 1 \\ 4 & -4 & 1 \end{array} \right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 4 & -4 & 1 \\ 0 & 0 & 0 \end{array} \right).$$

$$\therefore 4w_1 - 4w_2 = 1. \text{ Choose } w_2 = 0, \text{ then } w_1 = \frac{1}{4}. \therefore \vec{w} = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}.$$

Thus, $\vec{u}_2 = e^t \left[\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$ is also a solution to $\frac{d\vec{u}}{dt} = A\vec{u}$.

Hence, the complementary solution to $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}$ is

$$\vec{u}_c = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^t \left[\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right].$$

$$\text{Let } \vec{u}_p = h_1(t) e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + h_2(t) e^t \left[\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right].$$

$$\text{Then } h_1'(t) e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + h_2'(t) e^t \left[\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix},$$

$$\text{i.e. } \begin{pmatrix} e^t & \frac{1}{4}e^t + te^t \\ e^t & te^t \end{pmatrix} \begin{pmatrix} h_1'(t) \\ h_2'(t) \end{pmatrix} = \begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix}.$$

$$\begin{aligned} \Rightarrow \begin{pmatrix} h_1'(t) \\ h_2'(t) \end{pmatrix} &= \begin{pmatrix} e^t & \frac{1}{4}e^t + te^t \\ e^t & te^t \end{pmatrix}^{-1} \begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix} = \frac{1}{\begin{vmatrix} e^t & \frac{1}{4}e^t + te^t \\ e^t & te^t \end{vmatrix}} \begin{pmatrix} te^t & -\frac{1}{4}e^t - te^t \\ -e^t & e^t \end{pmatrix} \begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix} \\ &= \frac{1}{te^{2t} - \left(\frac{1}{4}e^{2t} + te^{2t}\right)} \begin{pmatrix} 2te^{2t} - \frac{1}{2}e^{2t} - 2te^{2t} \\ -2e^{2t} + 2e^{2t} \end{pmatrix} = \frac{1}{-\frac{1}{4}e^{2t}} \begin{pmatrix} -\frac{1}{2}e^{2t} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \end{aligned}$$

$$\Rightarrow \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} 2t \\ 0 \end{pmatrix} \quad (\text{The constants of integration are omitted.})$$

$$\text{Thus, } \vec{u}_p = \begin{pmatrix} e^t & \frac{1}{4}e^t + te^t \\ e^t & te^t \end{pmatrix} \begin{pmatrix} 2t \\ 0 \end{pmatrix} = \begin{pmatrix} 2te^t \\ 2te^t \end{pmatrix} = 2te^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution of $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}$ is

$$\vec{u} = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} \frac{1}{4} + t \\ t \end{pmatrix} + 2te^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\vec{u}_0 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \vec{u}(0) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}, \text{ i.e. } \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \frac{1}{\begin{vmatrix} 1 & \frac{1}{4} \\ 1 & 0 \end{vmatrix}} \begin{pmatrix} 0 & -\frac{1}{4} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \frac{1}{0 - \frac{1}{4}} \begin{pmatrix} -\frac{3}{4} \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -16 \end{pmatrix}$$

The solution of the given system $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}$ is

$$\vec{u} = 3e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 16e^t \begin{pmatrix} \frac{1}{4} + t \\ t \end{pmatrix} + 2te^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -e^t - 14te^t \\ 3e^t - 14te^t \end{pmatrix}.$$

8. (a) Let $\frac{dx}{dt} = z$, then $\frac{d^2x}{dt^2} = \frac{dx}{dt} + y \Leftrightarrow \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{dx}{dt} + y \xRightarrow{\frac{dx}{dt}=z} \frac{dz}{dt} = z + y$ and

$$\frac{dy}{dt} = \frac{dx}{dt} + y \Rightarrow \frac{dy}{dt} = z + y.$$

So

$$\begin{cases} \frac{d^2x}{dt^2} = \frac{dx}{dt} + y \\ \frac{dy}{dt} = \frac{dx}{dt} + y \end{cases} \Leftrightarrow \begin{cases} \frac{dx}{dt} = z \\ \frac{dy}{dt} = \frac{dx}{dt} + y \\ \frac{d^2x}{dt^2} = \frac{dx}{dt} + y \end{cases} \Leftrightarrow \begin{cases} \frac{dx}{dt} = z \\ \frac{dy}{dt} = z + y \\ \frac{dz}{dt} = z + y \end{cases} \Leftrightarrow \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$\text{We have } \frac{d[\vec{u}(t)]}{dt} = A\vec{u}(t) \text{ where } A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

- (b) Trying $\vec{u} = e^{\lambda t} \vec{v}$, where $\vec{v} \neq \vec{0}$.

$$\begin{aligned} \text{Solve } |A - \lambda I_3| = 0, \text{ i.e. } \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{vmatrix} = 0 &\Rightarrow (-\lambda) \begin{vmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = 0 \\ &\Rightarrow (-\lambda)[(1 - \lambda)^2 - 1] = 0 \\ &\Rightarrow \lambda = 0 \text{ or } \lambda^2 - 2\lambda = 0 \\ &\Rightarrow \lambda = 0 \text{ or } \lambda(\lambda - 2) = 0 \\ &\Rightarrow \lambda = 0, 0, 2 \end{aligned}$$

$$\text{For } \lambda = 2, \text{ solve } (A - \lambda I_3)\vec{v} = \vec{0}, \text{ i.e. } \begin{pmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \sim_{r_3+r_2} \begin{pmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\text{So } v_2 = v_3, v_1 = \frac{v_3}{2}.$$

$$\text{Let } v_3 = 1, \text{ then } v_1 = \frac{1}{2}, v_2 = v_3 = 1.$$

It follows that $\vec{u}_1(t) = e^{2t} \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$ is a solution to $\frac{d\vec{u}}{dt} = A\vec{u}$.

For $\lambda = 0$ (multiplicity 2), solve $(A - \lambda I_3)\vec{v} = \vec{0}$, i.e.

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right) \xrightarrow{r_3 - r_1} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

So $v_3 = 0$, $v_2 = 0$. Let $v_1 = 1$. It follows that $\vec{u}_2(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is a solution to $\frac{d\vec{u}}{dt} = A\vec{u}$.

We need to find another linearly independent solution for $\lambda = 0$.

$$\text{Solve } (A - \lambda I_3)\vec{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ i.e. } \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\therefore \begin{cases} w_3 = 1 \\ w_2 + w_3 = 0 \end{cases} \Rightarrow \begin{cases} w_3 = 1 \\ w_2 = -1 \end{cases}$$

$$\text{Choose } w_1 = 0. \text{ Then } \vec{w} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

$$\therefore \vec{u}_3 = e^{0t} \left[\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \text{ is also a solution to } \frac{d\vec{u}}{dt} = A\vec{u}.$$

$$\text{Hence, the general solution to } \vec{u}' = A\vec{u} \text{ is } \vec{u}(t) = c_1 e^{2t} \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \left[\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right].$$

It follows that the general solution of the original system is

$$\begin{aligned} x(t) &= c_1 \frac{e^{2t}}{2} + c_2 + c_3 t, \\ y(t) &= c_1 e^{2t} - c_3. \end{aligned}$$

9. (a) Let $\frac{dx}{dt} = z$, then $\frac{d^2x}{dt^2} = y + e^{-t} \Leftrightarrow \frac{d}{dt} \left(\frac{dx}{dt} \right) = y + e^{-t} \Leftrightarrow \frac{dz}{\frac{dx}{dt} = z} = y + e^{-t}$ and

$$\frac{dy}{dt} = -2x - 5 \frac{dx}{dt} - 4y + e^{-t} \Leftrightarrow \frac{dy}{dt} = -2x - 5z - 4y + e^{-t}.$$

$$\text{So } \begin{cases} \frac{d^2x}{dt^2} = y + e^{-t} \\ \frac{dy}{dt} = -2x - 5\frac{dx}{dt} - 4y + e^{-t} \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} = z \\ \frac{dz}{dt} = y + e^{-t} \\ \frac{dy}{dt} = -2x - 5z - 4y + e^{-t} \end{cases}$$

$$\Rightarrow \begin{pmatrix} \frac{dx}{dt} \\ \frac{dz}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{pmatrix} \begin{pmatrix} x \\ z \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-t} \\ e^{-t} \end{pmatrix}$$

So we have $\frac{d[\vec{u}(t)]}{dt} = A\vec{u}(t) + \vec{f}$, where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{pmatrix}$, $\vec{u}(t) = \begin{bmatrix} x(t) \\ z(t) \\ y(t) \end{bmatrix}$ and

$$\vec{f}(t) = \begin{pmatrix} 0 \\ e^{-t} \\ e^{-t} \end{pmatrix}.$$

(b) $e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ and $e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ are independent solutions to $\frac{d}{dt}[\vec{u}(t)] = A\vec{u}(t)$.

The third independent solution to $\frac{d}{dt}[\vec{u}(t)] = A\vec{u}(t)$ is $e^{-t} \left[\vec{w} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right]$, where

$$(A + I_3)\vec{w} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Solve $(A + I_3)\vec{w} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, i.e.

$$\begin{pmatrix} 1 & 1 & 0 & | & 1 \\ 0 & 1 & 1 & | & -1 \\ -2 & -5 & -3 & | & 1 \end{pmatrix} \xrightarrow{R_3+2R_1} \begin{pmatrix} 1 & 1 & 0 & | & 1 \\ 0 & 1 & 1 & | & -1 \\ 0 & -3 & -3 & | & 3 \end{pmatrix} \xrightarrow{R_3+3R_2} \begin{pmatrix} 1 & 1 & 0 & | & 1 \\ 0 & 1 & 1 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\therefore \begin{cases} w_1 + w_2 = 1 \\ w_2 + w_3 = -1 \end{cases}$$

Choose $w_3 = 0$, then $w_2 = -1$ and $w_1 = 2$. Then $\vec{w} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

$\therefore \vec{u}_3 = e^{-t} \left[\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right]$ is also a solution to $\frac{d\vec{u}}{dt} = A\vec{u}$.

Hence, the general solution to the system $\frac{d}{dt}[\vec{u}(t)] = A\vec{u}(t)$.

$$\vec{u}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 2+t \\ -1-t \\ t \end{pmatrix} = \begin{bmatrix} e^{-2t} & e^{-t} & (2+t)e^{-t} \\ -2e^{-2t} & -e^{-t} & (-1-t)e^{-t} \\ 4e^{-2t} & e^{-t} & te^{-t} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

(c) Let $\vec{u}_p = h_1(t)e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + h_2(t)e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + h_3(t)e^{-t} \begin{pmatrix} 2+t \\ -1-t \\ t \end{pmatrix}.$

Then $h_1'(t)e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + h_2'(t)e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + h_3'(t)e^{-t} \begin{pmatrix} 2+t \\ -1-t \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-t} \\ e^{-t} \end{pmatrix},$

i.e. $\begin{pmatrix} e^{-2t} & e^{-t} & (2+t)e^{-t} \\ -2e^{-2t} & -e^{-t} & (-1-t)e^{-t} \\ 4e^{-2t} & e^{-t} & te^{-t} \end{pmatrix} \begin{pmatrix} h_1'(t) \\ h_2'(t) \\ h_3'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-t} \\ e^{-t} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} h_1'(t) \\ h_2'(t) \\ h_3'(t) \end{pmatrix} = \begin{pmatrix} e^{-2t} & e^{-t} & (2+t)e^{-t} \\ -2e^{-2t} & -e^{-t} & (-1-t)e^{-t} \\ 4e^{-2t} & e^{-t} & te^{-t} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ e^{-t} \\ e^{-t} \end{pmatrix}$

By Gauss-Jordan elimination method,

$$\begin{aligned} & \left(\begin{array}{ccc|ccc} e^{-2t} & e^{-t} & (2+t)e^{-t} & 1 & 0 & 0 \\ -2e^{-2t} & -e^{-t} & (-1-t)e^{-t} & 0 & 1 & 0 \\ 4e^{-2t} & e^{-t} & te^{-t} & 0 & 0 & 1 \end{array} \right) \xrightarrow[r_3-4r_1]{r_2+2r_1} \left(\begin{array}{ccc|ccc} e^{-2t} & e^{-t} & (2+t)e^{-t} & 1 & 0 & 0 \\ 0 & e^{-t} & (3+t)e^{-t} & 2 & 1 & 0 \\ 0 & -3e^{-t} & (-8-3t)e^{-t} & -4 & 0 & 1 \end{array} \right) \\ & \xrightarrow{r_3+3r_2} \left(\begin{array}{ccc|ccc} e^{-2t} & e^{-t} & (2+t)e^{-t} & 1 & 0 & 0 \\ 0 & e^{-t} & (3+t)e^{-t} & 2 & 1 & 0 \\ 0 & 0 & e^{-t} & 2 & 3 & 1 \end{array} \right) \\ & \xrightarrow[r_1-(2+t)r_3]{r_2-(3+t)r_3} \left(\begin{array}{ccc|ccc} e^{-2t} & e^{-t} & 0 & -3-2t & -6-3t & -2-t \\ 0 & e^{-t} & 0 & -4-2t & -8-3t & -3-t \\ 0 & 0 & e^{-t} & 2 & 3 & 1 \end{array} \right) \\ & \xrightarrow{r_1-r_2} \left(\begin{array}{ccc|ccc} e^{-2t} & 0 & 0 & 1 & 2 & 1 \\ 0 & e^{-t} & 0 & -4-2t & -8-3t & -3-t \\ 0 & 0 & e^{-t} & 2 & 3 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & e^{2t} & 2e^{2t} & e^{2t} \\ 0 & 1 & 0 & (-4-2t)e^t & (-8-3t)e^t & (-3-t)e^t \\ 0 & 0 & 1 & 2e^t & 3e^t & e^t \end{array} \right) \end{aligned}$$

Thus, $\begin{bmatrix} e^{-2t} & e^{-t} & (2+t)e^{-t} \\ -2e^{-2t} & -e^{-t} & (-1-t)e^{-t} \\ 4e^{-2t} & e^{-t} & te^{-t} \end{bmatrix}^{-1} = \begin{bmatrix} e^{2t} & 2e^{2t} & e^{2t} \\ (-4-2t)e^t & (-8-3t)e^t & (-3-t)e^t \\ 2e^t & 3e^t & e^t \end{bmatrix}.$

Then

$$\begin{pmatrix} h_1'(t) \\ h_2'(t) \\ h_3'(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & 2e^{2t} & e^{2t} \\ (-4-2t)e^t & (-8-3t)e^t & (-3-t)e^t \\ 2e^t & 3e^t & e^t \end{pmatrix} \begin{pmatrix} 0 \\ e^{-t} \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 3e^t \\ -11-4t \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \end{pmatrix} = \begin{pmatrix} \int 3e^t dt \\ \int (-11 - 4t) dt \\ \int 4 dt \end{pmatrix} = \begin{pmatrix} 3e^t \\ -11t - 2t^2 \\ 4t \end{pmatrix}$$

Thus,

$$\vec{u}_p = \begin{pmatrix} e^{-2t} & e^{-t} & (2+t)e^{-t} \\ -2e^{-2t} & -e^{-t} & (-1-t)e^{-t} \\ 4e^{-2t} & e^{-t} & te^{-t} \end{pmatrix} \begin{pmatrix} 3e^t \\ -11t - 2t^2 \\ 4t \end{pmatrix}$$

(d) It follows that the general solutions to $\begin{cases} \frac{d^2x}{dt^2} = y + e^{-t} \\ \frac{dy}{dt} = -2x - 5\frac{dx}{dt} - 4y + e^{-t} \end{cases}$ is:

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} e^{-2t} & e^{-t} & (2+t)e^{-t} \\ 4e^{-2t} & e^{-t} & te^{-t} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{bmatrix} e^{-2t} & e^{-t} & (2+t)e^{-t} \\ 4e^{-2t} & e^{-t} & te^{-t} \end{bmatrix} \begin{pmatrix} 3e^t \\ -11t - 2t^2 \\ 4t \end{pmatrix} \\ &= \begin{bmatrix} e^{-2t} & e^{-t} & (2+t)e^{-t} \\ 4e^{-2t} & e^{-t} & te^{-t} \end{bmatrix} \begin{pmatrix} c_1 + 3e^t \\ c_2 - 11t - 2t^2 \\ c_3 + 4t \end{pmatrix} \end{aligned}$$

-End-