

MA3001 Solution to Take-home Assignment 2

$$\begin{aligned}
 1. (a) \quad \mathcal{L}[\sin^2(3t)] &= \frac{1}{2} \mathcal{L}[1 - \cos(6t)] \\
 &= \frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2 + 6^2} \right)
 \end{aligned}$$

By the First Shift Theorem,

$$\mathcal{L}[e^{4t} \sin^2(3t)] = \frac{1}{2} \left[\frac{1}{s-4} - \frac{s-4}{(s-4)^2 + 36} \right]$$

$$(b) \quad g(t) = \begin{cases} 0 & \text{if } t < 1 \\ 5t & \text{if } 1 \leq t < 2 \\ t^2 + 1 & \text{if } t \geq 2 \end{cases}$$

$$= 5t [u_1(t) - u_2(t)] + (t^2 + 1) u_2(t)$$

$$= 5t u_1(t) + (t^2 - 5t + 1) u_2(t)$$

$$\text{Let } f_1(t-1) = 5t \quad \& \quad f_2(t-2) = t^2 - 5t + 1.$$

$$\begin{aligned}
 \text{Then } f_1(t) &= 5(t+1) \quad \& \quad f_2(t) = (t+2)^2 - 5(t+2) + 1 \\
 &= 5t + 5 & & = t^2 + 4t + 4 - 5t - 10 + 1 \\
 & & & = t^2 - t - 5
 \end{aligned}$$

By the Second Shift Theorem,

$$\begin{aligned}
 \mathcal{L}[g(t)] &= e^{-s} \mathcal{L}[5t+5] + e^{-2s} \mathcal{L}[t^2 - t - 5] \\
 &= e^{-s} \left(\frac{5}{s^2} + \frac{5}{s} \right) + e^{-2s} \left(\frac{2!}{s^3} - \frac{1}{s^2} - \frac{5}{s} \right)
 \end{aligned}$$

$$1(c) \quad \mathcal{L}[\sinh(2t)] = \frac{2}{s^2 - 2^2} = \frac{2}{s^2 - 4}$$

$$\begin{aligned} \therefore \mathcal{L}[t^2 \sinh(2t)] &= (-1)^2 \frac{d^2\left(\frac{2}{s^2-4}\right)}{ds^2} \\ &= \frac{d}{ds} \left\{ \frac{d}{ds} [2(s^2-4)^{-1}] \right\} \\ &= \frac{d}{ds} [-4s(s^2-4)^{-2}] \\ &= -4s \cdot (-2)(s^2-4)^{-3} \cdot 2s - 4(s^2-4)^{-2} \\ &= \frac{12s^2+16}{(s^2-4)^3} \end{aligned}$$

$$2(a) \quad \frac{s+5}{s^2+2s-3} = \frac{s+5}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}$$

$$\therefore s+5 = A(s-1) + B(s+3)$$

$$\text{Put } s=1 : 6=4B \Rightarrow B = \frac{3}{2}$$

$$\text{Put } s=-3 : 2=-4A \Rightarrow A = -\frac{1}{2}$$

$$\begin{aligned} \therefore \mathcal{L}^{-1}\left[\frac{s+5}{s^2+2s-3}\right] &= -\frac{1}{2} \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] + \frac{3}{2} \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] \\ &= -\frac{1}{2} e^{-3t} + \frac{3}{2} e^t \end{aligned}$$

$$(b) \quad \mathcal{L}^{-1}\left[\frac{3s-1}{(s+2)^5}\right] = \mathcal{L}^{-1}\left[\frac{3(s+2)-6-1}{(s+2)^5}\right]$$

$$= e^{-2t} \mathcal{L}^{-1}\left[\frac{3s-7}{s^5}\right] \quad \text{by the First Shift Theorem}$$

$$= e^{-2t} \mathcal{L}^{-1}\left[\frac{3}{s^4} - \frac{7}{s^5}\right]$$

$$\begin{aligned}
 &= e^{-2t} \left\{ \frac{3}{3!} \mathcal{L}^{-1} \left[\frac{3!}{s^4} \right] - \frac{7}{4!} \mathcal{L}^{-1} \left[\frac{4!}{s^5} \right] \right\} \\
 &= e^{-2t} \left(\frac{1}{2} t^3 - \frac{7}{24} t^4 \right)
 \end{aligned}$$

$$\begin{aligned}
 2(c) \quad \mathcal{L}^{-1} \left[\frac{1}{s^2(s^2+9)} \right] &= \frac{1}{9} \mathcal{L}^{-1} \left[\frac{(s^2+9)-s^2}{s^2(s^2+9)} \right] \\
 &= \frac{1}{9} \mathcal{L}^{-1} \left[\frac{1}{s^2} - \frac{1}{s^2+9} \right] \\
 &= \frac{1}{9} \left\{ \mathcal{L}^{-1} \left[\frac{1}{s^2} \right] - \frac{1}{3} \mathcal{L}^{-1} \left[\frac{3}{s^2+3^2} \right] \right\} \\
 &= \frac{1}{9} \left[t - \frac{1}{3} \sin(3t) \right]
 \end{aligned}$$

$\therefore$ By the Second Shift Theorem,

$$\mathcal{L}^{-1} \left[\frac{e^{-2s}}{s^2(s^2+9)} \right] = \frac{1}{9} \left\{ (t-2) - \frac{1}{3} \sin[3(t-2)] \right\} u_2(t)$$

$$3. \quad y'' + 6y' + 13y = 20e^t, \quad y(0) = 3, \quad y'(0) = -2$$

Take $\mathcal{L}$ on both sides :

$$\mathcal{L}[y''] + 6 \mathcal{L}[y'] + 13 \mathcal{L}[y] = 20 \mathcal{L}[e^t]$$

$$\Rightarrow [s^2 Y(s) - s y(0) - y'(0)] + 6 [s Y(s) - y(0)] + 13 Y(s) = \frac{20}{s-1}$$

$$\Rightarrow [s^2 Y(s) - 3s - (-2)] + 6 [s Y(s) - 3] + 13 Y(s) = \frac{20}{s-1}$$

$$\Rightarrow (s^2 + 6s + 13) Y(s) = \frac{20}{s-1} + 3s + 16$$

$$\Rightarrow Y(s) = \frac{20}{(s-1)(s^2+6s+13)} + \frac{3s+16}{s^2+6s+13}$$

$$\text{Let } \frac{20}{(s-1)(s^2+6s+13)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+6s+13}$$

$$\text{Then } 20 = A(s^2+6s+13) + (Bs+C)(s-1)$$

$$\text{Put } s=1 : 20 = 20A \Rightarrow A=1$$

$$\text{Compare coeff. of } s^2 : 0 = A+B \Rightarrow B=-1$$

$$\text{Compare constant term : } 20 = 13A - C \Rightarrow C = 13A - 20 = -7$$

$$\begin{aligned} \therefore Y(s) &= \frac{1}{s-1} - \frac{s+7}{s^2+6s+13} + \frac{3s+16}{s^2+6s+13} \\ &= \frac{1}{s-1} + \frac{2s+9}{s^2+6s+13} \end{aligned}$$

Take $\mathcal{L}^{-1}$ on both sides:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{2s+9}{s^2+6s+13}\right] \\ &= e^t + \mathcal{L}^{-1}\left[\frac{2(s+3)+3}{(s+3)^2+4}\right] \\ &= e^t + e^{-3t} \mathcal{L}^{-1}\left[\frac{2s+3}{s^2+4}\right] \\ &= e^t + e^{-3t} \left\{ 2 \mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right] + \frac{3}{2} \mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right] \right\} \\ &= e^t + e^{-3t} \left(2 \cos 2t + \frac{3}{2} \sin 2t \right) \end{aligned}$$

$$4. \begin{cases} \frac{dx}{dt} = x + 2y - 2 \\ \frac{dy}{dt} = -5x - y + 1 \end{cases}$$

$$\frac{d\vec{u}}{dt} = A\vec{u} + \vec{F}, \text{ where } A = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}, \vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \vec{F} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

$$\text{Solve } |A - \lambda I_2| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -5 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-1-\lambda) + 10 = 0$$

$$\Rightarrow \lambda^2 + 9 = 0$$

$$\Rightarrow \lambda = \pm 3i$$

$$\text{For } \lambda = 3i, \text{ solve } (A - \lambda I_2) \vec{v} = \vec{0}.$$

$$\text{i.e. } \left(\begin{array}{cc|c} 1-3i & 2 & 0 \\ -5 & -1-3i & 0 \end{array} \right) \xrightarrow{(1-3i)R_2 + 5R_1} \left(\begin{array}{cc|c} 1-3i & 2 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\therefore (1-3i)v_1 + 2v_2 = 0$$

$$\text{Let } v_2 = (1-3i), \text{ then } v_1 = -2.$$

$$\therefore \vec{v} = \begin{pmatrix} -2 \\ 1-3i \end{pmatrix} \text{ is an eigenvector of } A \text{ corresponding to } \lambda = 3i.$$

$$\therefore \vec{u}_1 = e^{3it} \begin{pmatrix} -2 \\ 1-3i \end{pmatrix} \text{ is a solution.}$$

$$\begin{aligned} \text{Now } e^{3it} \begin{pmatrix} -2 \\ 1-3i \end{pmatrix} &= (\cos 3t + i \sin 3t) \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right] \\ &= \left[\cos 3t \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \sin 3t \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right] + i \left[\sin 3t \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \cos 3t \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right] \end{aligned}$$

∴ The complementary solution is

$$\begin{aligned}\vec{u}_c &= C_1 \left[\cos 3t \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \sin 3t \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right] + C_2 \left[\sin 3t \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \cos 3t \begin{pmatrix} 0 \\ -3 \end{pmatrix} \right] \\ &= C_1 \begin{pmatrix} -2 \cos 3t \\ \cos 3t + 3 \sin 3t \end{pmatrix} + C_2 \begin{pmatrix} -2 \sin 3t \\ \sin 3t - 3 \cos 3t \end{pmatrix}\end{aligned}$$

where C_1, C_2 are arbitrary constants.

$$\text{Let } \vec{u}_p = h_1(t) \begin{pmatrix} -2 \cos 3t \\ \cos 3t + 3 \sin 3t \end{pmatrix} + h_2(t) \begin{pmatrix} -2 \sin 3t \\ \sin 3t - 3 \cos 3t \end{pmatrix}.$$

$$\text{Then } h_1'(t) \begin{pmatrix} -2 \cos 3t \\ \cos 3t + 3 \sin 3t \end{pmatrix} + h_2'(t) \begin{pmatrix} -2 \sin 3t \\ \sin 3t - 3 \cos 3t \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\text{i.e. } \begin{cases} -2 \cos 3t h_1'(t) - 2 \sin 3t h_2'(t) = -2 & \text{--- (1)} \\ (\cos 3t + 3 \sin 3t) h_1'(t) + (\sin 3t - 3 \cos 3t) h_2'(t) = 1 & \text{--- (2)} \end{cases}$$

$$\text{(2)} \times 2 + \text{(1)}:$$

$$6 \sin 3t h_1'(t) - 6 \cos 3t h_2'(t) = 0 \quad \text{--- (3)}$$

$$\cos 3t \times \text{(3)} + 3 \sin 3t \times \text{(1)}:$$

$$-6 (\cos^2(3t) + \sin^2(3t)) h_2'(t) = -6 \sin 3t$$

$$\Rightarrow h_2'(t) = \sin 3t$$

$$\Rightarrow h_2(t) = \int \sin 3t \, dt = -\frac{1}{3} \cos 3t$$

Substitute $h_2'(t) = \sin 3t$ into (3) :

$$6 \sin 3t h_1'(t) - 6 \cos 3t \sin 3t = 0$$

$$\Rightarrow h_1'(t) = \cos 3t$$

$$\Rightarrow h_1(t) = \int \cos 3t \, dt = \frac{1}{3} \sin 3t$$

$$\begin{aligned}\therefore \vec{u}_p &= \frac{1}{3} \sin 3t \begin{pmatrix} -2 \cos 3t \\ \cos 3t + 3 \sin 3t \end{pmatrix} - \frac{1}{3} \cos 3t \begin{pmatrix} -2 \sin 3t \\ \sin 3t - 3 \cos 3t \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\end{aligned}$$

$\therefore$ The general solution is

$$\vec{u} = C_1 \begin{pmatrix} -2 \cos 3t \\ \cos 3t + 3 \sin 3t \end{pmatrix} + C_2 \begin{pmatrix} -2 \sin 3t \\ \sin 3t - 3 \cos 3t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$