

EE3121

Differential Equations for Engineers

Lecture 4: Existence, Uniqueness, and Qualitative Analysis



Instructor: Prof. Jie Chen

DEPARTMENT OF ELECTRICAL ENGINEERING

First-Order Differential Equation

- Question: What are the **existence** and **uniqueness** conditions of solution to a given first-order differential equation?
- Question: How to analyze the solution if it is unable to be solved in a closed form?

4.1 Existence and Uniqueness Theorem

- We now answer the first question:

What are the **existence** and **uniqueness** conditions of solution to a given first-order differential equation?

- Consider

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

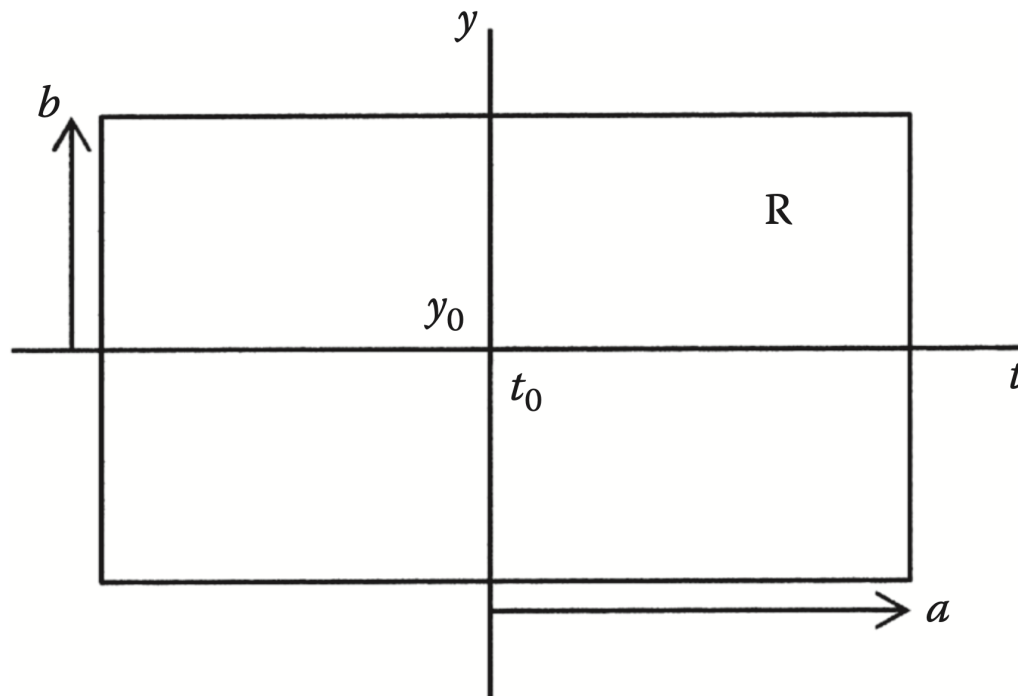
What is $y(t)$?

4.1.1 Lipschitz Condition

■ Consider

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

where $f(t, y)$ is defined on a rectangle R .



4.1.1 Lipschitz Condition

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$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

where $f(t, y)$ is defined on a rectangle R .

- The function $f(t, y)$ is said to satisfy a Lipschitz condition with respect to y on R if there exists a constant $K > 0$ such that $|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|$ for all points (t, y) in R .

4.1.1 Lipschitz Condition: Example

■ Consider

$$\frac{dy}{dt} = f(t, y) = g(t) + y^4,$$

and $R : 0 \leq t \leq 10, -2 \leq y \leq 2$.

$$f(t, y_1) - f(t, y_2) = y_1^4 - y_2^4 = (y_1 - y_2)(y_1 + y_2)(y_1^2 + y_2^2),$$

then

$$\begin{aligned} \frac{|f(t, y_1) - f(t, y_2)|}{|y_1 - y_2|} &= |(y_1 + y_2)(y_1^2 + y_2^2)| \\ &\leq |(y_1 + y_2)(y_1^2 + y_2^2)|_{y_1=y_2=\pm 2} = 4 = K. \end{aligned}$$

4.1.1 Lipschitz Condition: Example

- Note that, in the example, the Lipschitz constant K is given by the point in R where $y_1 = y_2 = \operatorname{argmax}_y \left| \frac{\partial f(t,y)}{\partial y} \right|$.
- More generally, if $\frac{\partial f(t,y)}{\partial y}$ is continuous in R and $\left| \frac{\partial f(t,y)}{\partial y} \right| \leq K$ in R , then $f(t, y)$ satisfies a Lipschitz condition in R with constant K .

4.1.2 Statement of the Existence and Uniqueness Theorem

- Let the function $f(t, y)$ be defined on the rectangle $R : |t - t_0| \leq a, |y - y_0| \leq b$, and in R let $f(t, y)$:
 - Be continuous.
 - Be bounded by M .
 - Satisfy a Lipschitz condition with constant K with respect to y .

Then a unique solution exists to

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

in the interval $I : |t - t_0| \leq \min(a, b/M)$.

- Note that $\min(a, b/M)$ is only the **lower bound** of the time in which a solution is valid.

4.2 Successive Approximations (General Form)

- Successive approximations for *almost linear* system (learned in last lecture)
- Successive approximations for *general* first-order ODE system
 - Solve a difficult ODE $\xrightarrow{\text{approximately}}$ solving a sequence of easier ODEs iteratively
 - Consider

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

What is $y(t)$?

4.2 Successive Approximations (General Form)

- The successive approximations are the solutions $y(t_1), y(t_2), \dots, y(t_n)$

$$\frac{dy_1}{dt} = f(t, y_0), \quad y_1(t_0) = y_0$$

$$\frac{dy_2}{dt} = f(t, y_1), \quad y_2(t_0) = y_0$$

⋮

$$\frac{dy_n}{dt} = f(t, y_{n-1}), \quad y_n(t_0) = y_0$$

- Integral calculus:

$$y_n(t) = y_0 + \int_{t_0}^t f(\tau, y_{n-1}(\tau)) d\tau$$

4.2.1 Successive Approximation Example

- Consider $\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0.$
- $y(t) = \tan(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 \dots$
- Successive approximations $y(t_1), y(t_2), \dots, y(t_n)$ are determined from

$$\frac{dy_n}{dt} = 1 + y_{n-1}^2(t), \quad y_n(0) = 0.$$

4.2.1 Successive Approximations Example

So,

$$y_0(t) = 0 \Rightarrow \frac{dy_1}{dt} = 1 \Rightarrow y_1(t) = t \Rightarrow \frac{dy_2}{dt} = 1 + t^2 \Rightarrow$$

$$y_2(t) = t + \frac{1}{3}t^3 \Rightarrow \frac{dy_3}{dt} = 1 + t^2 + \frac{2}{3}t^4 + \frac{1}{9}t^6 \Rightarrow$$

$$y_3(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{1}{63}t^7 \Rightarrow \frac{dy_4}{dt} = 1 + \left(t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{1}{63}t^7\right)^2 \Rightarrow$$

$$y_4(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 \dots$$

The successive approximations $y_n(t)$ are converging rapidly to the true solution $\tan(t)$.

4.2.2 Convergence of Successive Approximations

■ **Theorem:** The sequence of approximations given by

$$y_0(t) = y(0), y_1(t) = y_0 + \int_{t_0}^t f(\tau, y_0(\tau))d\tau \dots$$

$$y_n(t) = y_0 + \int_{t_0}^t f(\tau, y_{n-1}(\tau))d\tau$$

converges to a unique solution of

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

4.2.2 Convergence of Successive Approximations

- Proof: Consider the sequence $\{z_i(t) = y_i(t) - y_{i-1}(t)\}, i = 1, \dots, n$.
Then

$$y_n(t) = y_0(t) + \sum_{i=1}^n z_i(t).$$

- Show the convergence of sequence $\sum_{i=1}^n z_i(t)$.

$$z_1(t) = y_1(t) - y_0(t) = \int_{t_0}^t f(\tau, y_0(\tau)) d\tau,$$

$$|z_1(t)| = \left| \int_{t_0}^t f(\tau, y_0(\tau)) d\tau \right| \leq \int_{t_0}^t |f(\tau, y_0(\tau))| d\tau \leq \int_{t_0}^t M d\tau = M(t - t_0)$$

4.2.2 Convergence of Successive Approximations

- We have

$$|z_1(t)| \leq M(t - t_0).$$

- Next

$$z_2(t) = y_2(t) - y_1(t) = \int_{t_0}^t (f(\tau, y_1(\tau)) - f(\tau, y_0(\tau))) d\tau,$$

$$\begin{aligned} |z_2(t)| &\leq \int_{t_0}^t |(f(\tau, y_1(\tau)) - f(\tau, y_0(\tau)))| d\tau \leq \int_{t_0}^t K |y_1(\tau) - y_0(\tau)| d\tau \\ &= \int_{t_0}^t K |z_1(\tau)| d\tau \leq \int_{t_0}^t K M(\tau - t_0) d\tau = K M(t - t_0)^2 / 2. \end{aligned}$$

4.2.2 Convergence of Successive Approximations

- We have

$$|z_2(t)| \leq KM(t - t_0)^2/2.$$

- Continuing in this way we find that

$$|z_n(t)| \leq \frac{MK^n(t - t_0)^n}{n!}.$$

- Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} y_n(t) &= y_0(t) + \lim_{n \rightarrow \infty} \sum_{i=1}^n z_i(t) \leq y_0(t) + \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{MK^i(t - t_0)^i}{i!} \\ &= y_0(t) + \frac{M}{K} e^{K(t-t_0)} \end{aligned}$$

4.3 Qualitative Analysis

- We continue to answer the second question:

How to analyze the solution if it is unable to be solved in a closed form?

4.3 Qualitative Analysis

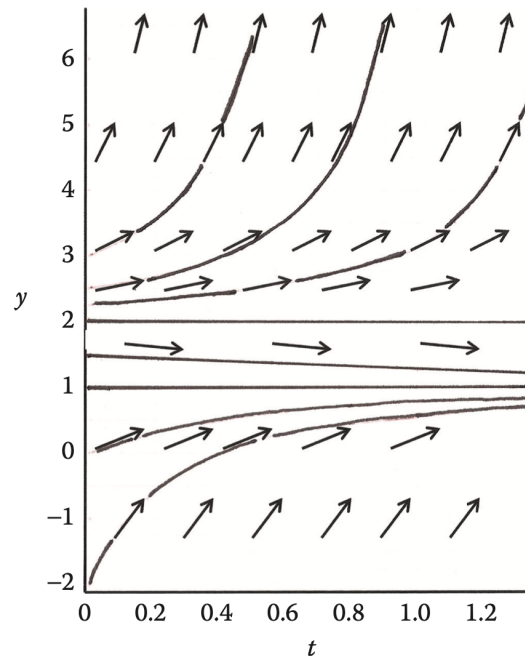
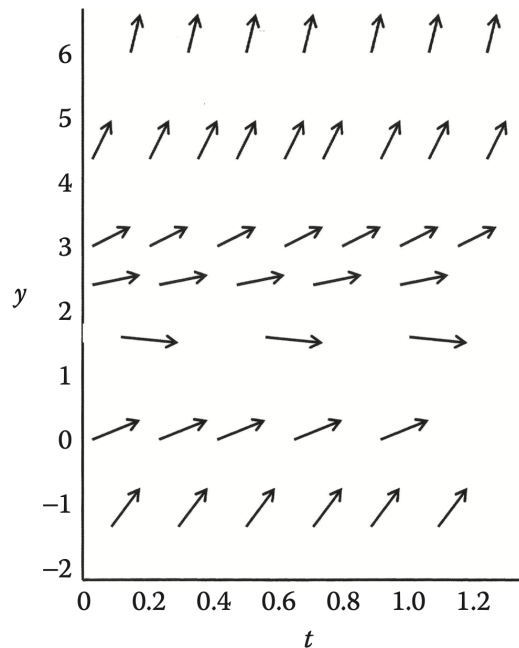
- Consider the ODE

$$\frac{dy}{dt} = (y - 1)(y - 2), \quad y(0) = y_0.$$

- Even if we could not solve it (it can be solved actually), we could gain insights into the behavior of the solution by examining the function $f(t, y) = (y - 1)(y - 2)$.
- We create a direction field with vectors drawn for discrete points in the (y, t) .

4.3 Qualitative Analysis

- The value of $(y-1)(y-2)$ is proportional to the tangent of the angle of the vector, the proportionality constant depending on the graph scale chosen.



- The direction field presents a visual display of the slope of solution trajectories through those points.

4.4 Stability Revisited

- Stability for a linear system in terms of its response to initial conditions (In Lecture 2)
- For a **linear** system, the possible equilibrium point is unique and can be set as the origin.
- For a **nonlinear** system, with their potential for multiple equilibrium points, we need to refine that definition of stability.

4.4 Stability Revisited

- For a **nonlinear** system, with their potential for multiple equilibrium points, we need to refine that definition of stability.
- An equilibrium point y_E is said to be **stable** for $t > t_0$ if for every $\varepsilon > 0$, there is a $\delta > 0$ (which may be a function of t_0 and ε) such that if $|y(t_0) - y_E| < \delta$ then $|y(t) - y_E| < \varepsilon$ for all $t > t_0$.
- If $\lim_{t \rightarrow \infty} y(t) = y_E$, then the equilibrium point is **asymptotically stable**.

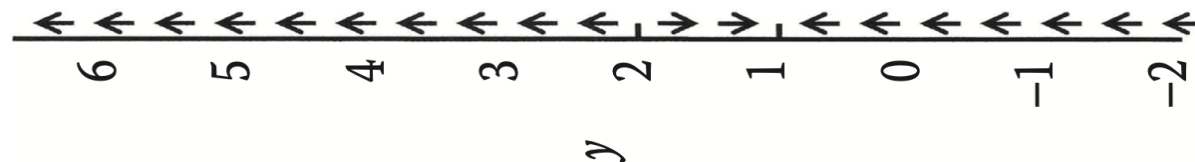
4.4.1 Stability Revisited: Example

- Recall the ODE

$$\frac{dy}{dt} = (y - 1)(y - 2), \quad y(0) = y_0,$$

which has two equilibrium points at $y = 1$ and $y = 2$.

- Since $f(t, y) = (y - 1)(y - 2)$ is independent of time, the direction field can be collapsed into a direction line.



- The line clarifies the ODE solution's equilibrium and stability properties. $y = 1$ is a stable equilibrium point, whereas $y = 2$ is an unstable one.

4.5 Local Solutions

- The existence and uniqueness theorem provides the interval

$$I : |t - t_0| \leq \min(a, b/M)$$

- The lower bound on the length of time $t_E = \min(a, b/M)$ leaves the possibility that the solution of ODE may indeed only be local.

4.5.1 Local Solutions: Example 1

- Consider the ODE

$$\frac{dy}{dt} = \frac{1}{4-y}, \quad y(0) = 0.$$

- The solution $y(t)$ begins at zero and moves in the $+y$ direction.
- Recall: for the function $f(t, y)$ in R
 - $R : |t - t_0| \leq a, |y - y_0| \leq b$
 - Be continuous.
 - Be bounded by M .
 - Satisfy a Lipschitz condition with constant K with respect to y .
- The function $f(t, y) = \frac{1}{4-y}$.

4.5.1 Local Solutions: Example 1

- The function $f(t, y) = \frac{1}{4-y}$
 - $a > 0, 0 < b < 4$
 - $f(t, y) \leq \frac{1}{4-b} = M$
 - $f(t, y)$ satisfies a Lipschitz condition.
- A solution is certain to exist in the interval

$$|t - t_0| = |t| \leq t_E = \min(a, b/M) = \min(a, (4 - b)b).$$

Choose $a > (4 - b)b, \forall b \in (0, 4)$. Then,

$$t_E = (4 - b)b.$$

4.5.1 Local Solutions: Example 1

- t_E is a function of b :

$$t_E = (4 - b)b.$$

- $\max t_E = t_E(b^*) = 4$ with $b^* = 2$
- The highest bound on the time a solution is certain to exist is 4.
- In tutorial, we will reexamine this problem and it can be shown the solution is indeed local, but it actually exists for $t < 8$.
- $\max t_E$ only represents a rough measure (bound) and develops only crude, conservative estimates.

4.5.1 Local Solutions: Example 2

- Consider the three ODEs:

$$\frac{dy_A}{dt} = f(t, y_A) = 1 + y_A^2, \quad y_A(0) = 0$$

$$\frac{dy_B}{dt} = f(t, y_B) = (1 + y_B)^2, \quad y_B(0) = 0$$

$$\frac{dy_C}{dt} = f(t, y_C) = (1 + y_C)(2 + y_C), \quad y_C(0) = 0$$

- Using direction lines: the solution of each ODE starts at zero and moves in the $+y$ direction.

4.5.1 Local Solutions: Example 2

- Given a pair (t, y) ,

$$f(t, y_C) > f(t, y_B) > f(t, y_A) > 0$$

- $y_A(t) = \tan t$ defined over $0 \leq t < \pi/2$ and $\lim_{t \rightarrow \pi/2} y_A(t) = \infty$
- $f(t, y_B) > f(t, y_A) > 0 \implies \lim_{t \rightarrow \pi/2} y_B(t) = \infty$ in a shorter period of time than does $y_A(t)$
- $f(t, y_C) > f(t, y_B) > 0 \implies \lim_{t \rightarrow \pi/2} y_C(t) = \infty$ in a shorter period of time than does $y_B(t)$
- In tutorial, we will study y_B and leave y_C as an exercise.

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