

Assignment 2 Solution

The assignment consists of 7 questions.

1. (15%) Prove using mathematical induction that for all $n \geq 1$,

$$1 + 4 + 7 + \dots + (3n - 2) = \frac{n(3n - 1)}{2}.$$

Proof:

(Base) When $n = 1$, $1 = \frac{1(3-1)}{2}$ is true.

(Induction) Assume for an integer $n = k$, the statement holds. It means

$$1 + 4 + 7 + \dots + (3k - 2) = \frac{k(3k - 1)}{2}.$$

When $n = k + 1$, we have

$$\begin{aligned} 1 + 4 + 7 + \dots + (3k - 2) + (3(k + 1) - 2) &= \frac{k(3k - 1)}{2} + (3(k + 1) - 2) \\ &= \frac{3k^2 - k + 6k + 2}{2} = \frac{3k^2 + 2k + 3k + 2}{2} = \frac{(k + 1)(3k + 2)}{2} \\ &= \frac{(k + 1)(3(k + 1) - 1)}{2}. \end{aligned}$$

Then the statement holds when $n = k + 1$.

By the principle of mathematical induction, the statement holds for all $n \geq 1$.

QED

2. (15%) Use contrapositive proof to prove: If x and $y \in \mathbb{Z}$, $x + y + xy$ is even, then x and y have the same parity (either both are even, or both are odd).

Proof:

Since $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$, we use “prove by contrapositive” to prove

For $x, y \in \mathbb{Z}$, if x and y have different parities, then $x + y + xy$ is odd.

If x and y have different parities, there must be an odd number and an even number in x and y .

Without loss of generality, we assume x is even, then y is odd.

It means $x = 2m$, $y = 2n + 1$, where m, n are integers.

$$\begin{aligned} \text{Then we have } x + y + xy &= 2m + (2n + 1) + 2m(2n + 1) \\ &= 4mn + 4m + 2n + 1 \\ &= 2(2mn + 2m + n) + 1. \end{aligned}$$

Here, $2mn + 2m + n$ is an integer, so $x + y + xy$ must be odd.

Then use proof by contrapositive, the original statement is proved.

QED

3. (15%) Suppose $A = \{\text{red, green, blue}\}$, $B = \{\text{red, yellow, orange}\}$, $C = \{\text{red, orange, green, purple}\}$, and the universal set $U = A \cup B \cup C$. Find the following:

- 1) $A \cap (B \cup C)$.
- 2) $(A \cap B) \cup C^c$.
- 3) $(A - B) \cap C$.

Solution:

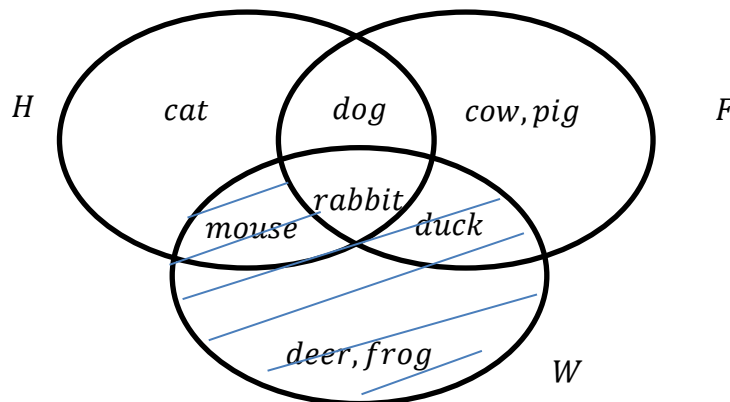
- 1) $A \cap (B \cup C) = \{\text{red, green}\}$.
- 2) $(A \cap B) \cup C^c = \{\text{red, yellow, blue}\}$.
- 3) $(A - B) \cap C = \{\text{green}\}$.

4. (15%) Suppose $A = \{\text{toast, juice}\}$, $B = \{\text{egg, milk}\}$, $H = \{\text{cat, dog, rabbit, mouse}\}$, $F = \{\text{dog, cow, duck, pig, rabbit}\}$, $W = \{\text{duck, rabbit, deer, frog, mouse}\}$.

- 1) Find cartesian product $A \times B$.
- 2) Use Venn diagram to illustrate $(H \cap F)^c \cap W$.

Solution:

- 1) $A \times B = \{(\text{toast, egg}), (\text{toast, milk}), (\text{juice, egg}), (\text{juice, milk})\}$.
- 2) $(H \cap F)^c \cap W = \{\text{mouse, deer, duck, frog}\}$.



5. (15%) Using the properties of sets to prove

$$(A - C) - (C - B) = A - C$$

Start the proof from the Left-hand side. Use algebraic rules to prove it. State the name of the algebraic rule you used in each step. Do **not** skip any steps.

Proof:

$$\begin{aligned} (A - C) - (C - B) &= (A \cap C^c) \cap (C \cap B^c)^c && \text{Difference law} \\ &= (A \cap C^c) \cap (C^c \cup (B^c)^c) && \text{De Morgan's law} \end{aligned}$$

$$\begin{aligned}
&= (A \cap C^c) \cap (C^c \cup B) && \text{Double complement law} \\
&= A \cap (C^c \cap (C^c \cup B)) && \text{Associative law} \\
&= A \cap C^c && \text{Absorption law} \\
&= A - C && \text{Difference law}
\end{aligned}$$

QED

6. (15%) Determine if the follow functions are injective, surjective, and bijective.

- 1) $f: R \rightarrow R, f(x) = -x^2 + 4x + 9$.
- 2) $g: N \rightarrow N, g(x) = x + 3$.
- 3) $h: R \rightarrow (0, +\infty), h(x) = \log(e^{-x} + 1)$.

Solution:

- 1) Not injective: $f(0) = f(4)$.
Not surjective: range of f is $(-\infty, 13] \neq R$.
Not bijective: neither injective nor surjective.
- 2) Injective: $g(x_1) = g(x_2) \rightarrow x_1 = x_2$.
Not surjective: range of g is $\{n | n \in N, n \geq 4\} \neq N$.
Not bijective: injective but not surjective.
- 3) Injective: $h(x_1) = h(x_2) \rightarrow x_1 = x_2$.
Surjective: range of h is $(0, +\infty)$ equals to its codomain.
Bijective: both injective and surjective.

7. (10%) Find the composition function of the following functions:

- 1) $f(x) = 4x + 3, g(x) = -2x^2 + 3x + 5$. Find $(g \circ f)(x)$.
- 2) $f(x) = \frac{3}{5}x + 6, g(x) = 6x^2 - 15x + 20$. Find $(f \circ g)\left(\frac{5}{3}\right)$.

Solution:

$$\begin{aligned}
1) \quad (g \circ f)(x) &= -2(4x + 3)^2 + 3(4x + 3) + 5 \\
&= -2(16x^2 + 24x + 9) + 12x + 14 \\
&= -32x^2 - 36x - 4
\end{aligned}$$

$$\begin{aligned}
2) \quad (f \circ g)(x) &= \frac{3}{5}(6x^2 - 15x + 20) + 6 \\
&= \frac{18}{5}x^2 - 9x + 18
\end{aligned}$$

$$\begin{aligned}
(f \circ g)\left(\frac{5}{3}\right) &= \frac{18}{5} \cdot \left(\frac{5}{3}\right)^2 - 9 \cdot \frac{5}{3} + 18 \\
&= 10 - 15 + 18 \\
&= 13
\end{aligned}$$