

MA1200 Calculus and Basic Linear Algebra I

Lecture Note **11**

Application of Differentiation **2**

I) L'Hôpital's Rule

II.1) More on Optimization

II.2) Derivatives in Physics

II.3) Newton's Method *

I) L'Hôpital's Rule

A power formula for computing limits: L' Hopital's Rule

- In Chapter 6 (Limits), we learned various techniques to compute the limits.

$$\lim_{x \rightarrow 0} \frac{x^2 - 3x}{x^2 - 5x} = \lim_{x \rightarrow 0} \frac{x - 3}{x - 5} = \frac{3}{5},$$

Handwritten red note: $\frac{0}{0}$

- Each of these techniques is useful in some special types of limits only.
- One cannot compute the following limits using simple algebraic tricks:

$$\lim_{x \rightarrow \infty} \frac{\ln(1+x)}{x}, \quad \text{Handwritten green note: } \frac{\infty}{\infty} \quad \lim_{x \rightarrow 0} \frac{e^{2x^2} - 1}{x^2}, \quad \text{Handwritten green note: } \frac{0}{0}$$

- Observe that these two limits assumes the form $0/0$ (second limits) or ∞/∞ (first limits).
- The following theorem, called L' Hopital Rule, provides an efficient method to compute these kinds of limits.

Theorem 1A (L' Hopital Rule, $\left(\frac{0}{0}\right)$ type)

Let $f(x)$ and $g(x)$ be two differentiable functions. Suppose that

1. $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$
2. $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists $\left(i.e. \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \text{ or } \infty \text{ or } -\infty \right)$

Then, we have

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

Theorem 1B (L' Hopital Rule, $\left(\frac{\infty}{\infty}\right)$ type)

Let $f(x)$ and $g(x)$ be two differentiable functions. Suppose that

1. $\lim_{x \rightarrow a} f(x) = \infty$ or $-\infty$ and $\lim_{x \rightarrow a} g(x) = \infty$ or $-\infty$
2. $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists $\left(i.e. \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \text{ or } \infty \text{ or } -\infty \right)$

Then, we have

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

Example 15

Compute the limits

$$(a) \lim_{x \rightarrow \infty} \frac{\ln(1+x)}{x}, \quad (b) \lim_{x \rightarrow 0} \frac{e^{2x^2} - 1}{x^2}$$

☺Solution:

- (a) Note that $f(x) = \ln(1+x) \rightarrow \infty$ and $g(x) = x \rightarrow \infty$ as $x \rightarrow \infty$, then the limits is of the type $\frac{\infty}{\infty}$. Using the L'Hopital Rule, we get

$$\lim_{x \rightarrow \infty} \frac{\ln(1+x)}{x} \stackrel{\frac{\infty}{\infty} \text{ type}}{\underset{\substack{\uparrow \\ \text{L'Hopital}}}{\cong}} \lim_{x \rightarrow \infty} \frac{\overbrace{\frac{f'(x)}{g'(x)}}^{\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}}}{1} = \lim_{x \rightarrow \infty} \frac{1}{1+x} \stackrel{\rightarrow \frac{1}{\infty}}{\cong} 0$$

- (b) On the other hand, note that $f(x) = e^{2x^2} - 1 \rightarrow 0$ and $g(x) = x^2 \rightarrow 0$ as $x \rightarrow 0$ and the limits is of the type $\frac{0}{0}$. Using L'Hopital Rule, we get

$$\lim_{x \rightarrow 0} \frac{e^{2x^2} - 1}{x^2} \stackrel{\frac{0}{0} \text{ type}}{\underset{\substack{\uparrow \\ \text{L'Hopital}}}{\cong}} \lim_{x \rightarrow 0} \frac{\overbrace{\frac{f'(x)}{g'(x)}}^{\lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)}}}{2x} = \lim_{x \rightarrow 0} \frac{4xe^{2x^2}}{2x} = \lim_{x \rightarrow 0} 2e^{2x^2} = 2e^{2(0)^2} = 2.$$

Example 16A

Compute the limits

$$\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{x^2}$$

☺Solution:

Since $f(x) = 1 - \cos 3x \rightarrow 0$ and $g(x) = x^2 \rightarrow 0$ as $x \rightarrow 0$. Using L' Hopital Rule, we get

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos 3x}{x^2} &\stackrel{\text{L'Hopital}}{=} \lim_{x \rightarrow 0} \frac{\overbrace{3 \sin 3x}^{\rightarrow 0}}{2x} && \stackrel{\text{L'Hopital}}{=} \lim_{x \rightarrow 0} \frac{9 \cos 3x}{2} \\ &= \frac{9 \cos 3(0)}{2} = \frac{9}{2}. && \text{(2nd time)} \end{aligned}$$

Lesson: Sometimes one has to apply l'Hopital's Rule several times.

In fact, L' Hopital provides an “extremely” effective way for us to compute the limits. Consider the Example 16 again,

Before Use

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{1 - \cos 3x}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{\cos 0 - \cos 3x}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{-2 \sin \frac{0 + 3x}{2} \sin \frac{0 - 3x}{2}}{x^2} \\
 &= \lim_{x \rightarrow 0} 2 \frac{\sin^2 \left(\frac{3x}{2} \right)}{x^2} \\
 &= \lim_{x \rightarrow 0} 2 \left(\frac{9}{4} \right) \left(\frac{\sin \frac{3x}{2}}{\frac{3x}{2}} \right)^2 = \frac{9}{2}.
 \end{aligned}$$

After Use

$$\begin{aligned}
 & \lim_{x \rightarrow 0} \frac{1 - \cos 3x}{x^2} \\
 &= \lim_{x \rightarrow 0} \frac{3 \sin 3x}{2x} \quad (\text{L'Hopital's Rule}) \\
 &= \lim_{x \rightarrow 0} \frac{9 \cos 3x}{2} \quad (\text{L'Hopital's Rule}) \\
 &= \frac{9 \cos 3(0)}{2} = \frac{9}{2}.
 \end{aligned}$$

Example 16B

Compute the following limits

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x - \sin x}$$

☺Solution:

Note that both $f(x) = \tan x - x \rightarrow 0$ and $g(x) = x - \sin x \rightarrow 0$ when $x \rightarrow 0$.

Thus, one can use the L'Hopital Rule and get:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x - \sin x} \stackrel{\text{L'Hopital}}{=} \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{1 - \cos x} \quad \left(\frac{0}{0} - \text{type} \right)$$

$$\stackrel{\text{L'Hopital}}{=} \lim_{x \rightarrow 0} \frac{2 \sec x (\sec x \tan x)}{\sin x} = \lim_{x \rightarrow 0} \frac{2}{\cos^3 x} = \frac{2}{\cos^3 0} = 2.$$

Remarks about L' Hopital Rule

1. One can use L'Hopital Rule only when the limits is of the type $\frac{0}{0}$ and $\frac{\infty}{\infty}$. This rule cannot be applied when the limits is not of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. For example:

L'H. (wrong!)

$$\lim_{x \rightarrow 0} \frac{2x+1}{x+1} \stackrel{\rightarrow \frac{1}{1}=1}{\neq} \lim_{x \rightarrow 0} \frac{2}{1} = 2 \quad \left(\text{True limits} = \frac{2(0)+1}{0+1} = 1 \right).$$

2. L'Hoptial Rule can be applied ONLY WHEN the limits $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists (either a real number or infinity). For example: It is INCORRECT to say that

$$\lim_{x \rightarrow \infty} \frac{x + \sin x}{x - 3} \stackrel{\rightarrow \frac{\infty}{\infty}}{=} \lim_{x \rightarrow \infty} \frac{1 + \cos x}{1} = \lim_{x \rightarrow \infty} \left(1 + \overbrace{\cos x}^{\rightarrow ??} \right) = \text{does not exist.}$$

In fact, this limits exists and $= 1$.

3. One has to be careful that the use of L'Hopital Rule in problem related to differentiability. *(Circular conclusion!)*

Example:

Suppose we would like to check the differentiability of $f(x) = \sin 2x$ at $x = \pi$ using first principle, it is INCORRECT to say that

$$\lim_{h \rightarrow 0} \frac{\sin(2(\pi + h)) - \sin 2\pi}{h} = \lim_{h \rightarrow 0} \frac{\sin 2h}{h} = \lim_{h \rightarrow 0} \frac{2 \cos 2h}{1} = 2 \cos 0 = 2$$

since the above argument assumes that $\sin x$ is differentiable at $x = 2\pi$ which is something that you need to show. To obtain the correct solution, one has to use sum-to-product formula and the fact that $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$.

Indeterminate forms

In this sub-section, we will discuss the computation of the following limits:

$$(1) \lim_{x \rightarrow 0} x^x, \quad (\rightarrow 0^0 \text{ as } x \rightarrow 0)$$

$$(2) \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 2x + 2} - \sqrt{x^2 + 1} \right), \quad (\rightarrow \infty - \infty \text{ as } x \rightarrow \infty)$$

$$(3) \lim_{x \rightarrow 0} \frac{1}{x} \tan \frac{x}{2}, \quad (\rightarrow 0 \times \infty \text{ as } x \rightarrow 0)$$

$$(4) \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}}, \quad (\rightarrow 1^\infty \neq 1 \text{ as } x \rightarrow 0)$$

$$(5) \lim_{x \rightarrow 0} \left(\frac{1}{x^2} \right)^x \quad (\rightarrow \infty^0 \text{ as } x \rightarrow 0)$$

Such kinds of limits are said to be of *indeterminate forms*. Those limits can be computed using L'Hopital Rule. In order to apply this rule, one has to rewrite the above limits into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that L'Hopital can be applied.

Type 1: $0 \times \infty$ type

Example 17

Compute the limits

$$\lim_{x \rightarrow 0^+} x \ln x.$$

😊Note:

$x \ln x \rightarrow 0 \times (-\infty)$ as $x \rightarrow 0$.

(😊Note: Here we consider the limits when $x \rightarrow 0^+$ since $\ln x$ is defined for $x > 0$ only)

😊Solution:

By rewriting the limits and using L'Hopital Rule, we have

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\overbrace{\ln x}^{-\infty}}{\frac{1}{x}} \stackrel{\frac{-\infty}{\infty}}{=} \lim_{x \rightarrow 0^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0^+} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} (-x) = 0.$$

Type 2: $\infty - \infty$ type

Example 18

Compute the limits

$$\lim_{x \rightarrow 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$$

☺Note:

Limits $\rightarrow \infty - \infty$ as $x \rightarrow 1^+$.

☺Solution:

One can take the common factor and get

$$\lim_{x \rightarrow 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \lim_{x \rightarrow 1^+} \frac{x \ln x - (x-1)}{(x-1) \ln x}$$

$$x \ln x - (x-1) \rightarrow 0$$

$$(x-1) \ln x \rightarrow 0$$

("= 0/0")

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 1^+} \frac{f'(x)}{g'(x)} \\ &\cong \lim_{x \rightarrow 1^+} \frac{\ln x + x \left(\frac{1}{x} \right) - 1}{\ln x + (x-1) \left(\frac{1}{x} \right)} = \lim_{x \rightarrow 1^+} \frac{\ln x}{\ln x + \frac{x-1}{x}} \end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 1^+} \frac{x \ln x}{x \ln x + x - 1} \quad \left(\frac{0}{0} \text{ type} \right) \\
&\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} \\
&\quad \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 1^+} \frac{\ln x + x \left(\frac{1}{x} \right)}{\ln x + x \left(\frac{1}{x} \right) + 1} \\
&= \lim_{x \rightarrow 1^+} \frac{\ln x + 1}{\ln x + 2} = \frac{\ln 1 + 1}{\ln 1 + 2} = \frac{1}{2}.
\end{aligned}$$

Example 19

Compute the limits

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin^2 x} - \frac{x}{\sin^3 x} \right)$$

(☺Note: Again, this limits is of $\infty - \infty$ type.)

☺Note:

$$\frac{x}{\sin^3 x} = \overbrace{\left(\frac{x}{\sin x} \right)}^{\rightarrow 1} \overbrace{\left(\frac{1}{\sin^2 x} \right)}^{\rightarrow \infty} \rightarrow \infty$$

☺Solution:

Note that

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin^2 x} - \frac{x}{\sin^3 x} \right) = \lim_{x \rightarrow 0} \frac{\sin x - x}{\sin^3 x} \quad \left(\frac{0}{0} \text{ type} \right)$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} \\ &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\cos x - 1}{3 \sin^2 x \cos x} \quad \left(\frac{0}{0} \text{ type} \right) \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} \\ &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{-\sin x}{6 \sin x \cos^2 x - 3 \sin^3 x} \end{aligned}$$

$$= \lim_{x \rightarrow 0} \frac{-1}{6 \cos^2 x - 3 \sin^2 x}$$

$$= \frac{-1}{6 \cos^2 0 - 3 \sin^2 0} = -\frac{1}{6}.$$

Type 3: 0^0 , ∞^0 and 1^∞ type

One can compute these limits (via L'Hopital Rule) by first transforming the limits into the form of $0 \times \infty$ so that one can compute the limits using the method described in Type 1. This can be done by taking natural logarithm, i.e

$$\text{“} y = 0^0 \Rightarrow \ln y = \underbrace{0}_{0} \underbrace{\ln 0}_{-\infty} \text{”}$$

$$\text{“} y = \infty^0 \Rightarrow \ln y = \underbrace{0}_{0} \underbrace{\ln \infty}_{\infty} \text{”}$$

$$\text{“} y = 1^\infty \Rightarrow \ln y = \underbrace{\infty}_{\infty} \underbrace{\ln 1}_{0} \text{”}$$

To compute the limits $\lim_{x \rightarrow a} y$, we first compute $\lim_{x \rightarrow a} \ln y$ and get $\lim_{x \rightarrow a} \ln y = L$. Then the limits can be obtained by

$$\lim_{x \rightarrow a} y = \lim_{x \rightarrow a} e^{\ln y} = e^{\lim_{x \rightarrow a} \ln y} = e^L.$$

$y = y(x)$

Example 20

$$= e^{\ln(1+x)^{\frac{1}{2x}}} = e^{\frac{1}{2x} \ln(1+x)}$$

$$\lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{2x}}$$

$(1+x)^{\frac{1}{2x}} \rightarrow 1^\infty$
 $(1^\infty - \text{type})$

☺Solution

Let $y = (1+x)^{\frac{1}{2x}} \Rightarrow \ln y = \ln(1+x)^{\frac{1}{2x}} = \underbrace{\frac{1}{2x}}_{\rightarrow \infty} \underbrace{\ln(1+x)}_{\rightarrow 0}$.

Taking limits on both sides, we have

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{1}{2x} \ln(1+x) = \lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{2x} \quad \left(\frac{0}{0} \text{ type} \right)$$

$L'H$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x}}{2} = \lim_{x \rightarrow 0^+} \frac{1}{2(1+x)} = \frac{1}{2(1+0)} = \frac{1}{2}.$$

Thus we get

$$\lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{2x}} = \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln y} = e^{\lim_{x \rightarrow 0^+} \ln y} = e^{\frac{1}{2}}.$$

Example 21

Evaluate the limits

$$\lim_{x \rightarrow 0^+} x^{\sin x}.$$

$$x^{\sin x} \rightarrow 0^0$$

$(0^0 - \text{type})$

☺Solution

Let $y = x^{\sin x}$, then $\ln y = \ln(x^{\sin x}) = \sin x \ln x$.

Taking limits on both sides, we get

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \sin x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{\sin x}} \quad \left(\frac{\infty}{\infty} \text{ type} \right)$$

$$\stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{(\sin x)^2} \cos x} = - \lim_{x \rightarrow 0^+} \frac{(\sin x)^2}{x \cos x} = - \lim_{x \rightarrow 0^+} \frac{(\sin x)^2}{x \cos x}$$

$$= - \lim_{x \rightarrow 0^+} \left(\frac{\sin x}{x} \right) \frac{\sin x}{\cos x} = -1 \times 0 = 0$$

$$\Rightarrow \lim_{x \rightarrow 0^+} x^{\sin x} = \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln y} = e^{\lim_{x \rightarrow 0^+} \ln y} = e^0 = 1.$$

Example 22

Compute the limits

$$\lim_{x \rightarrow 0^+} (\cot x)^x$$

$$\begin{aligned} (\cot x)^x &\rightarrow \infty^0 \\ (\infty^0 - \text{type}) \end{aligned}$$

☺Solution:

Let $y = (\cot x)^x$, then $\ln y = x \ln(\cot x)$.

Taking limits on both sides, we get

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} x \ln(\cot x) = \lim_{x \rightarrow 0^+} \frac{\ln(\cot x)}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty} \text{ type} \right)$$

ℳ H.

$$\begin{aligned} &\downarrow \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{-\csc^2 x}{\cot x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{-\frac{\sin x}{\cos x \sin^2 x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{x^2}{\cos x \sin x} = \lim_{x \rightarrow 0^+} \frac{1}{\cos x} \left(\frac{x}{\sin x} \right) x \\ &= 1 \times 1 \times 0 = 0 \end{aligned}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} (\cot x)^x = \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln y} = e^{\lim_{x \rightarrow 0^+} \ln y} = e^0 = 1.$$

Some Harder Examples

Example 23

Let $f(x)$ be a differentiable function with $f(0) = 0$ and $f'(0) = 1$, we consider another function

$$g(x) = \begin{cases} \frac{f(x)}{2 \sin x} & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}.$$

Find the possible a such that the function $g(x)$ is continuous at $x = 0$.

☺Solution:

Using L'Hopital Rule, we get

$$\lim_{x \rightarrow 0} g(x) \stackrel{x \neq 0}{=} \lim_{x \rightarrow 0} \frac{\overbrace{f(x)}^{\rightarrow \frac{f(0)}{2 \sin 0} = \frac{0}{0}}}{2 \sin x} \stackrel{\text{L'H.}}{=} \lim_{x \rightarrow 0} \frac{f'(x)}{2 \cos x} = \frac{f'(0)}{2 \cos 0} = \frac{f'(0)}{2} = \frac{1}{2}.$$

$g(x)$ is continuous at $x = 0$

$$\Leftrightarrow \lim_{x \rightarrow 0} g(x) = g(0).$$

Since $g(x)$ is continuous at $x = 0$, then we have

$$\lim_{x \rightarrow 0} g(x) \stackrel{!}{=} g(0) = a \Rightarrow a = \frac{1}{2}.$$

Example 24

Let $f(x)$ and $g(x)$ be two differentiable functions over $\mathbb{R}$. Suppose that

- $f(0) = 0, f'(0) = 1$
- $g(0) = 0, g'(0) = 0$ and $g''(0) = 1$.

We consider the following ~~the~~ function:

$$P(x) = \begin{cases} f(2x) & \text{if } x \geq 0 \\ \frac{g(x)}{x} & \text{if } x < 0 \end{cases}.$$

- (a) Determine whether the function $P(x)$ is continuous at $x = 0$.
- (b) Determine whether the function $P(x)$ is differentiable at $x = 0$.

☺Solution:

- (a) We need to check whether $\lim_{x \rightarrow 0} P(x) = P(0) = f(0) = 0$. To compute the limits on the left-hand sides, we need to consider left-hand limits and right-hand limits:

$$\lim_{x \rightarrow 0^+} P(x) = \lim_{x \rightarrow 0^+} f(2x) = f(2(0)) = 0$$

$$\lim_{x \rightarrow 0^-} P(x) = \lim_{x \rightarrow 0^-} \frac{g(x)}{x} \stackrel{\frac{g(x)}{x} \rightarrow \frac{g(0)}{0} = \left(\frac{0}{0}\right)}{\cong} \lim_{x \rightarrow 0^-} \frac{g'(x)}{1} = g'(0) = 0.$$

Thus $\lim_{x \rightarrow 0} P(x) = 0$. Since $\lim_{x \rightarrow 0} P(x) = 0 = P(0)$, so $P(x)$ is continuous at $x = 0$.

(b) Using first principle, we consider (put $x = 0$)

$$\lim_{h \rightarrow 0} \frac{P(0 + h) - P(0)}{h} = \lim_{h \rightarrow 0} \frac{P(h) - 0}{h} = \lim_{h \rightarrow 0} \frac{P(h)}{h}.$$

Again, we need to consider the left-hand limits and right-hand limits:

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{P(h)}{h} &= \lim_{h \rightarrow 0^+} \frac{f(2h)}{h} \stackrel{\frac{f(2h)}{h} \rightarrow \frac{f(0)}{0} = \frac{0}{0}}{\cong} \lim_{h \rightarrow 0^+} \frac{\overbrace{2f'(2h)}^{\frac{d}{dh}f(2h) = \frac{df(2h)d(2h)}{d(2h)dh}}}{1} = 2f'(0) \\ &= 2(1) = 2. \end{aligned}$$

$$\begin{aligned}
\lim_{h \rightarrow 0^-} \frac{P(h)}{h} &= \lim_{h \rightarrow 0^-} \frac{\frac{g(h)}{h}}{h} = \lim_{h \rightarrow 0^-} \frac{g(h)}{h^2} \stackrel{\frac{g(h)}{h^2} \rightarrow \frac{g(0)}{0^2} = \frac{0}{0}}{\cong} \lim_{h \rightarrow 0^-} \frac{g'(h)}{2h} \\
&\stackrel{\frac{g'(h)}{2h} \rightarrow \frac{g'(0)}{2(0)} = \frac{0}{0}}{\cong} \lim_{h \rightarrow 0^-} \frac{g''(h)}{2} \\
&= \frac{g''(0)}{2} = \frac{1}{2}
\end{aligned}$$

Since the left-hand limits does not equal to right-hand limits and the limits $\lim_{h \rightarrow 0} \frac{P(0+h)-P(0)}{h} = \lim_{h \rightarrow 0} \frac{P(h)}{h}$ does not exist, thus the function $P(x)$ is not differentiable at $x = 0$.

Remark:

It is OK to apply the L'Hopital Rule to check the differentiability since it just requires the differentiability of $f(x)$ and $g(x)$ which are given in the problem.

Extra Example 1

Compute the limits

$$\lim_{x \rightarrow \infty} \frac{x^2 + 1}{x} \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{\tan^{-1}(3x^2)}{x^3 + x^2}$$

(Note: Here, $\tan^{-1} y$ takes the value between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$)

☺Solution:

The first limit is of the type $\frac{x^2+1}{x} \rightarrow \frac{\infty}{\infty}$ (as $x \rightarrow \infty$). Using L'Hopital Rule, we get

$$\lim_{x \rightarrow \infty} \frac{x^2 + 1}{x} \xrightarrow{\text{L'H.}} \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(x^2 + 1)}{\frac{d}{dx}x} = \lim_{x \rightarrow \infty} \frac{2x}{1} = \lim_{x \rightarrow \infty} 2x = \infty.$$

On the other hand, the second limits is of the type $\frac{\tan^{-1}(3x^2)}{x^3+x^2} \rightarrow \frac{\tan^{-1} 0}{0} = \frac{0}{0}$. By L' Hopital Rule again, we have

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\tan^{-1}(3x^2)}{x^3 + x^2} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \tan^{-1}(3x^2)}{\frac{d}{dx} (x^3 + x^2)} \\
 &= \frac{\frac{d}{dx} \tan^{-1}(3x^2)}{\frac{d(3x^2)}{dx}} \lim_{x \rightarrow 0} \frac{1}{1 + (3x^2)^2} (6x) \\
 &\cong \lim_{x \rightarrow 0} \frac{6x}{3x^2 + 2x} = \lim_{x \rightarrow 0} \frac{6x}{x(3x + 2)(1 + (3x^2)^2)} \\
 &= \lim_{x \rightarrow 0} \frac{6}{(3x + 2)(1 + (3x^2)^2)} \\
 &= \frac{6}{2(1)} = 3.
 \end{aligned}$$

Extra Example 2

Compute the limits

$$\lim_{x \rightarrow \infty} x^2 e^{-3x}.$$

☺Solution:

Note that the limits is of the type $x^2 e^{-3x} \rightarrow \infty^2 e^{-\infty} = \infty \times 0$, one has to transform the limits into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in order to apply L'Hopital Rule.

$$\lim_{x \rightarrow \infty} x^2 e^{-3x} = \lim_{x \rightarrow \infty} \frac{x^2}{e^{3x}} \left(\frac{\infty}{\infty} \text{ type} \right)$$

$$\stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} x^2}{\frac{d}{dx} e^{3x}} = \lim_{x \rightarrow \infty} \frac{2x}{3e^{3x}} \left(\frac{\infty}{\infty} \text{ type} \right)$$

$$\stackrel{\text{L'H.}}{=} \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} 2x}{\frac{d}{dx} 3e^{3x}} = \lim_{x \rightarrow \infty} \frac{2}{9e^{3x}} = \frac{2}{9 \times \infty} = 0.$$

Extra Example 3

Compute the limits

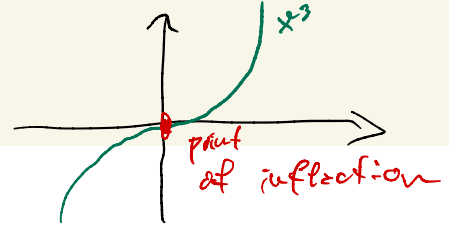
$$\lim_{x \rightarrow 0^+} \left(\frac{1}{\tan x} - \frac{1}{x} \right)$$

Since $\left(\frac{1}{\tan x} - \frac{1}{x} \right) \rightarrow \frac{1}{\tan 0} - \frac{1}{0} = \infty - \infty$ when $x \rightarrow 0^+$, we need to transform the limits into the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in order to apply L'Hopital Rule. Note that

$$\begin{aligned} \lim_{x \rightarrow 0^+} \left(\frac{1}{\tan x} - \frac{1}{x} \right) &= \lim_{x \rightarrow 0^+} \frac{x - \tan x}{x \tan x} \quad \left(\frac{0}{0} - \text{type} \right) \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx} (x - \tan x)}{\frac{d}{dx} x \tan x} = \lim_{x \rightarrow 0^+} \frac{1 - \sec^2 x}{\tan x + x \sec^2 x} \quad \left(\frac{0}{0} - \text{type} \right) \\ &= \lim_{x \rightarrow 0^+} \frac{-2 \sec x (\sec x \tan x)}{\sec^2 x + \sec^2 x + 2x \sec x (\sec x \tan x)} = \frac{0}{1 + 1 + 0} = 0. \end{aligned}$$

II.1) More on Optimization

Amendment on 2nd derivatives:



Theorem F

Let f be a differentiable function in some neighbourhood of a point c such that $f'(c) = 0$ and $f''(c)$ exists.

1. If $f''(c) < 0$, then $x = c$ gives a local maximum point of the curve $y = f(x)$.
2. If $f''(c) > 0$, then $x = c$ gives a local minimum.
3. If $f''(c) = 0$ and $f''(x)$ is positive on one side of $x = c$ and negative on the other, then $x = c$ gives a point of inflexion (inflection).

N.B. A point $(c, f(c))$ on a curve $y = f(x)$ is a point of inflexion if $f''(c) = 0$ and if the graph of $y = f(x)$ is concave upward on one side of $x = c$ and concave downward on the other.

Example 6

Consider the following functions (i) $y = x^3$, (ii) $y = x^4$.

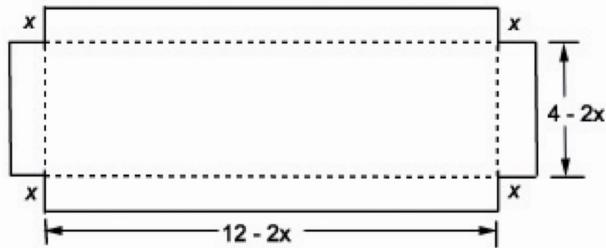
For (i), the curve has a point of inflexion at $x = 0$ since $y'' = 6x$ changes sign there.

For (ii), the curve does not have a point of inflexion at $x = 0$ although $y'' = 0$ there. (Why?)

Example 7

Find the dimensions of a box without cover that will yield the largest volume, if it is to be constructed from a rectangular metal plate, 4 m by 12 m.

Solution:



Let x m be the height of the box. Then $(12 - 2x)$ m is the length of the box and $(4 - 2x)$ m is the width of the box. The volume of the box is given by

$$\begin{aligned} V(x) &= x(4 - 2x)(12 - 2x) \text{ m}^3 \\ &= 4x(2 - x)(6 - x) \text{ m}^3 \\ &= 4(12x - 8x^2 + x^3) \text{ m}^3 \end{aligned}$$

Moreover, $V'(x) = 4(12 - 16x + 3x^2)$. If $V'(x) = 0$, then $x = 0.903, 4.43$ (to 3 sig. fig.).

Because the dimensions are non-negative, x can only take values in $(0, 2)$. However, 4.43 is not in $(0, 2)$.

$V''(x) = 4(-16 + 6x) \Rightarrow V''(0.903) < 0 \Rightarrow V(x)$ has a local maximum at $x = 0.903$

The values $V(0) = 0 = V(2)$ and $V(0.903) \approx 20.2$ reveal that the absolute maximum is at $x = 0.903$.

$\therefore$ The dimensions of the largest box are 10.2 m by 2.2 m by 0.903 m.

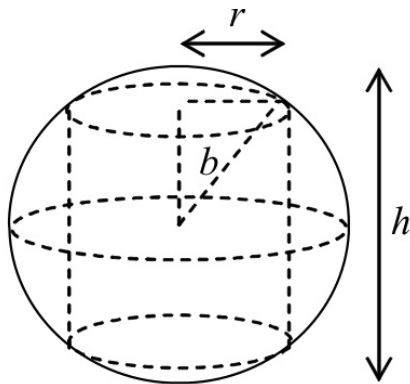
Example 8

Find the greatest volume of a cylinder that can be inscribed in a sphere of radius b .

Solution:

Let h and r be the height and radius of the cylinder respectively.

Step 1:



Step 2:

We want to find the maximum value of V (volume), $V = \pi r^2 h$.

As $b^2 = r^2 + \left(\frac{h}{2}\right)^2$, we have $V = \pi \left(b^2 h - \frac{h^3}{4} \right)$. Now the value V depends on the variable h only and the range of h is $0 \leq h \leq 2b$.

Step 3: $V' = \pi \left(b^2 - \frac{3h^2}{4} \right)$. $V' = 0$ when $h = \frac{2b}{\sqrt{3}}$ (since $h \geq 0$).

As $V' > 0$ when $0 \leq h < \frac{2b}{\sqrt{3}}$ and $V' < 0$ when $\frac{2b}{\sqrt{3}} < h \leq 2b$, we conclude that the greatest volume of

cylinder that can be cut is $V_{\max} = V \left(\frac{2b}{\sqrt{3}} \right) = \pi \left(\frac{2b^3}{\sqrt{3}} - \frac{8b^3}{12\sqrt{3}} \right)$. □

II.2) Derivatives in Physics

1 Dynamical applications (p.306 – p.312, p.348 – p.352)

The concept of derivatives can be applied to solve some physical problems. Consider an object which is moving along a straight line. We let

$x(t)$ = the position of the object at time t ,

$v(t)$ = the velocity of the object at time t ,

$a(t)$ = the acceleration of the object at time t ,

then $v(t) = \frac{dx}{dt}$ and $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$.

Example 1

If the velocity $v(t)$ of a body varies inversely as the square root of the distance $s(t)$, prove that the acceleration $a(t)$ varies as the fourth power of the velocity.

Solution:

Now, we have $v(t) = \frac{k}{\sqrt{s(t)}}$, where k is a constant.

$$a(t) = \frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} \underset{\frac{ds}{dt}=v(t)}{=} v(t) \frac{dv}{ds}. \text{ Observe that } v = \frac{k}{\sqrt{s}} = ks^{-\frac{1}{2}} \Rightarrow \frac{dv}{ds} = -\frac{1}{2} ks^{-\frac{3}{2}}.$$

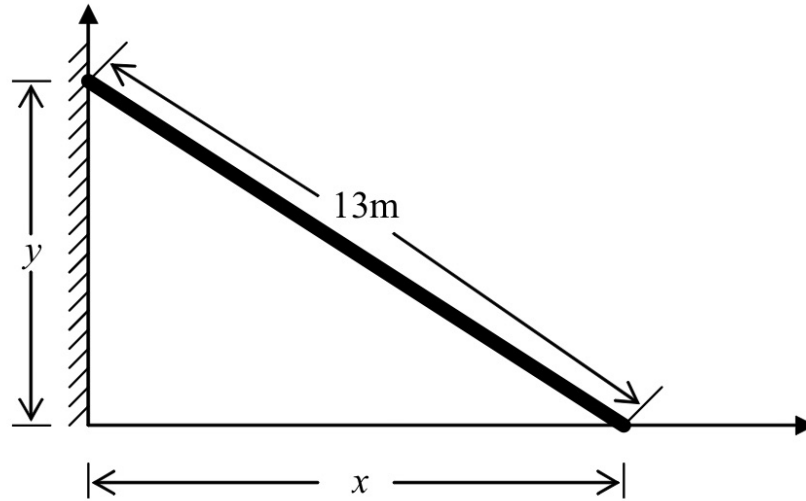
So

$$a(t) = v(t) \left(-\frac{1}{2} k [s(t)]^{-\frac{3}{2}} \right) \underset{\substack{v(t) = \frac{k}{\sqrt{s(t)}} \\ \Rightarrow s(t) = \left[\frac{k}{v(t)} \right]^2}}{=} -\frac{kv(t)}{2} \left(\left[\frac{k}{v(t)} \right]^2 \right)^{-\frac{3}{2}} = -\frac{kv(t)}{2} \left[\frac{k}{v(t)} \right]^{-3} = -\frac{[v(t)]^4}{2k^2}$$

Example 2

A ladder 13m long rests against a wall which is perpendicular to the level ground along which the lower end of the ladder slips at the rate 4m/s. How fast is the upper end of the ladder running down the wall when the lower end is 12m from the wall?

Solution:



Let x = distance of the lower end of the ladder from 0 at time t .

y = height of the upper end of the ladder above the level ground at time t .

$$x^2 + y^2 = 169 \Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0, \text{ when } x = 12, \text{ we have } 2 \times 12 \times 4 + 2 \times 5 \frac{dy}{dt} = 0.$$

$$\text{Thus, } \frac{dy}{dt} = -\frac{48}{5} \text{ m/s.}$$

II.3) Newton's method *

(*): optional

3.1 Numerical solution of nonlinear equation, $f(x) = 0$, using Newton's (or Newton-Raphson) method (p.425 – p.430) (OPTIONAL)

Suppose we have an approximation x_k to a simple real root x^* ($= x_k + h$) of the equation $f(x) = 0$. Then

$$\begin{aligned} f(x^*) &= f(x_k + h) = f(x_k) + hf'(x_k) + \frac{h^2}{2!} f''(x_k) + \dots \\ &= 0 \quad \text{by Taylor's expansion for } f(x_k + h). \end{aligned}$$

If h is small, then $h^2 \ll 1$, we may neglect the second and higher order terms in h ,

$$\begin{aligned} f(x_k) + hf'(x_k) &\approx 0 \\ \Rightarrow \quad h &\approx -\frac{f(x_k)}{f'(x_k)}, \quad \text{provided } f'(x_k) \neq 0, \end{aligned}$$

and so we take the next approximation to x^* as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Using this formula for $k = 0, 1, 2, \dots$, we have a process known as the *Newton* (or *Newton-Raphson method*).

Example 14

Perform 4 iterations of Newton's method to compute the root of $xe^x - 1 = 0$ with an initial approximation $x_0 = 0.5$.

Solution:

$$\text{Let } f(x) := xe^x - 1. \quad \Rightarrow \quad f'(x) = (x+1)e^x$$

Using Newton's method,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k e^{x_k} - 1}{(x_k + 1)e^{x_k}}, \quad k = 0, 1, 2, \dots$$

With an initial approximation, $x_0 = 0.5$, we have

$$x_1 = 0.5 - \frac{0.5e^{0.5} - 1}{(0.5 + 1)e^{0.5}} \approx 0.571020,$$

$$x_2 = 0.57102 - \frac{0.57102e^{0.57102} - 1}{(0.57102 + 1)e^{0.57102}} \approx 0.567156,$$

$$x_3 = 0.567156 - \frac{0.567156e^{0.567156} - 1}{(0.567156 + 1)e^{0.567156}} \approx 0.567143,$$

$$x_4 = 0.567143 - \frac{0.567143e^{0.567143} - 1}{(0.567143 + 1)e^{0.567143}} \approx 0.567143.$$