

MA2001

MULTI-VARIABLE CALCULUS AND LINEAR ALGEBRA

Chapter 2

Differential Calculus of Multivariable Functions (Part A)

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Chapter 2: Differential Calculus of Multivariable Functions (Part A)

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1 Introduction

Quantities of interest in real problems usually depend on **more than one (independent) variable**. For example,

(i) The height of the ground at map coordinates (x, y) might be expressed mathematically as

$$\text{Height} = h(x, y).$$

(ii) The temperature in a lecture room varies with position and time:

(iii) The volume of a rectangular box of length x , width y and height z is

(iv) The magnitude of the electric potential $u(x, y, z)$ due to a uniformly charged sphere with total charge Q is

where a is the radius of the sphere, ϵ_0 is the permittivity of free space.

These are all **real-valued** functions of real variables.

In general, we have

$$z = f(x_1, x_2, \dots, x_n) \equiv f(\vec{x}^T)$$

where $x_1, x_2, \dots, x_n$ are the independent variables and z is the dependent variable.

$f(\vec{x}^T)$ is a scalar field.

It is possible to visualize a function of two variables (a surface in 3-D space) by drawing its graph. One way is to draw a contour map.

A level curve of $f(x, y)$ is a set of points (x, y) in $\mathbb{R}^2$ for which

$$f(x, y) = c,$$

where c is a constant.

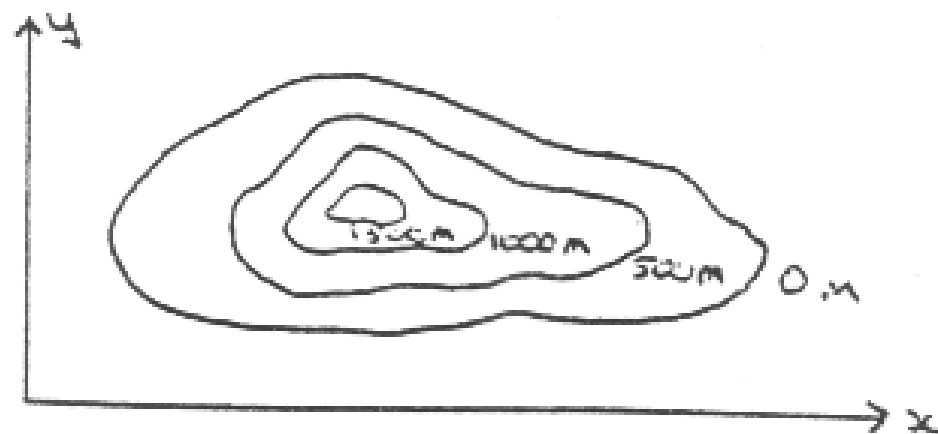
A collection of level curves then forms a contour map (as in the familiar contour maps produced by map makers).

Examples

(i)

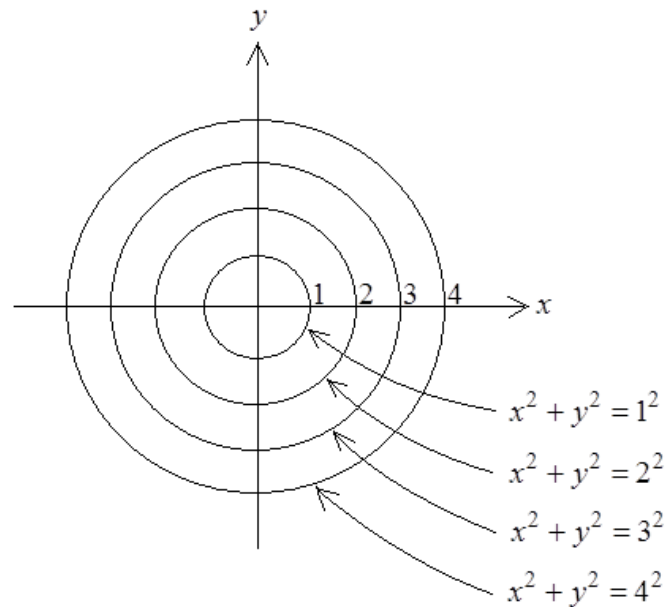


Level curves of a function are defined in the same manner as one obtains the contour lines for a topographical map.



Contour map

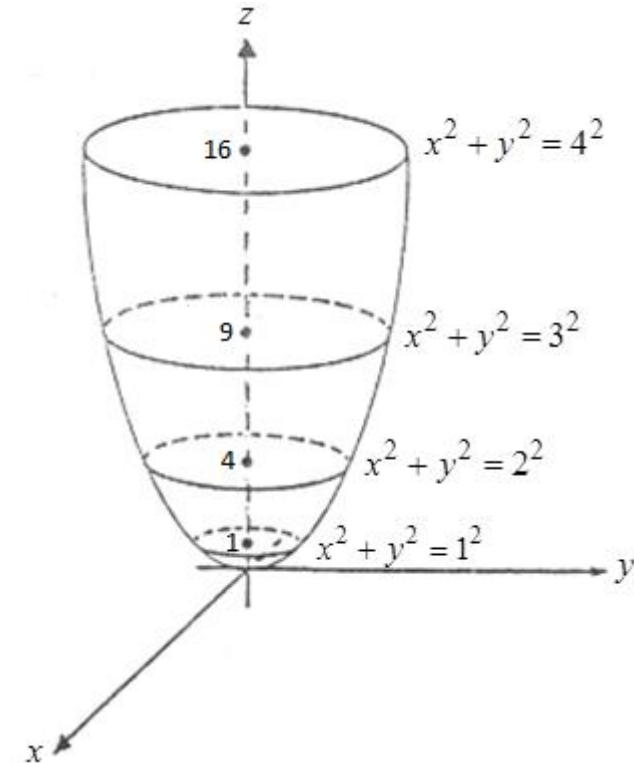
(ii)



Contour map

Some level curves for the function

$$f(x, y) = x^2 + y^2$$

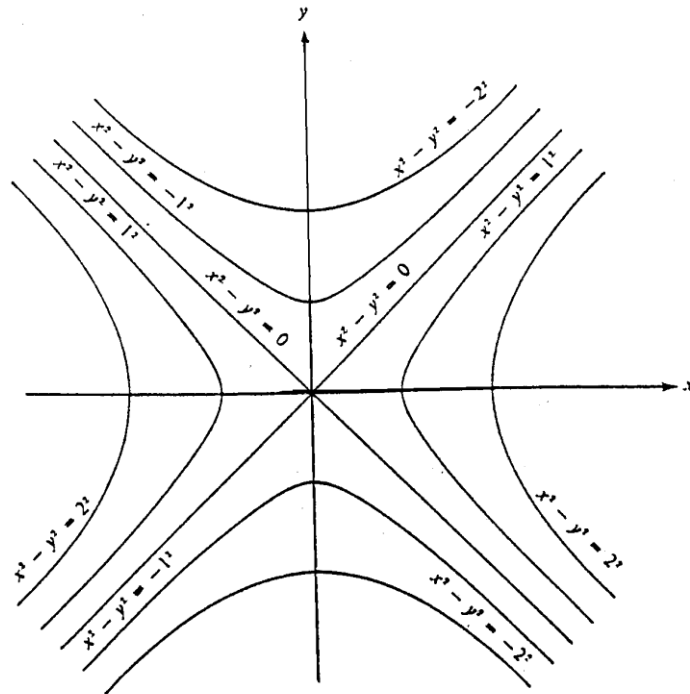


Level curves shown on the graph of

$$z = f(x, y) = x^2 + y^2$$

This is called an **elliptic paraboloid**.

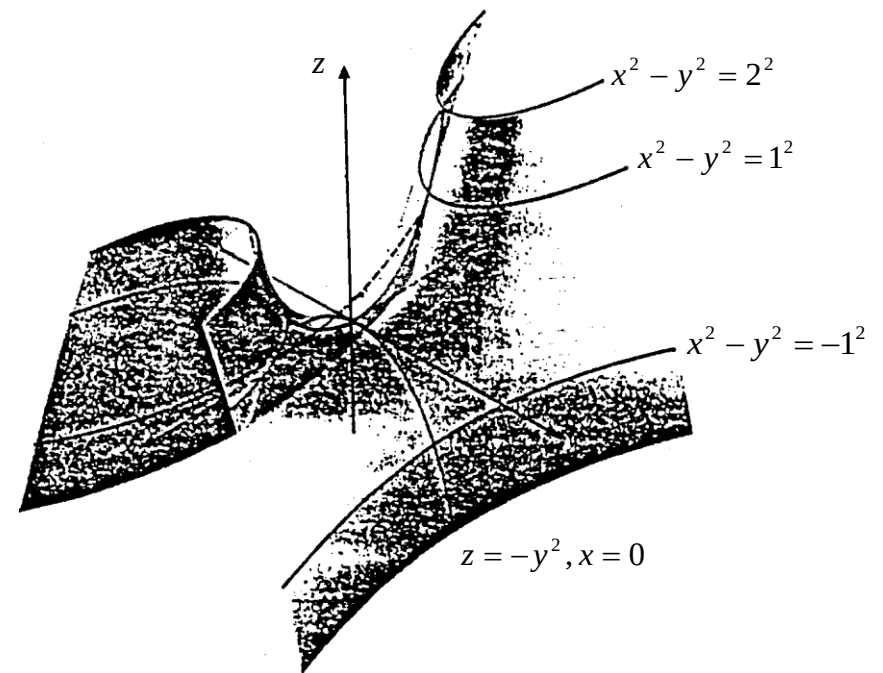
(iii)



Contour map

Level curves for the function

$$f(x, y) = x^2 - y^2$$



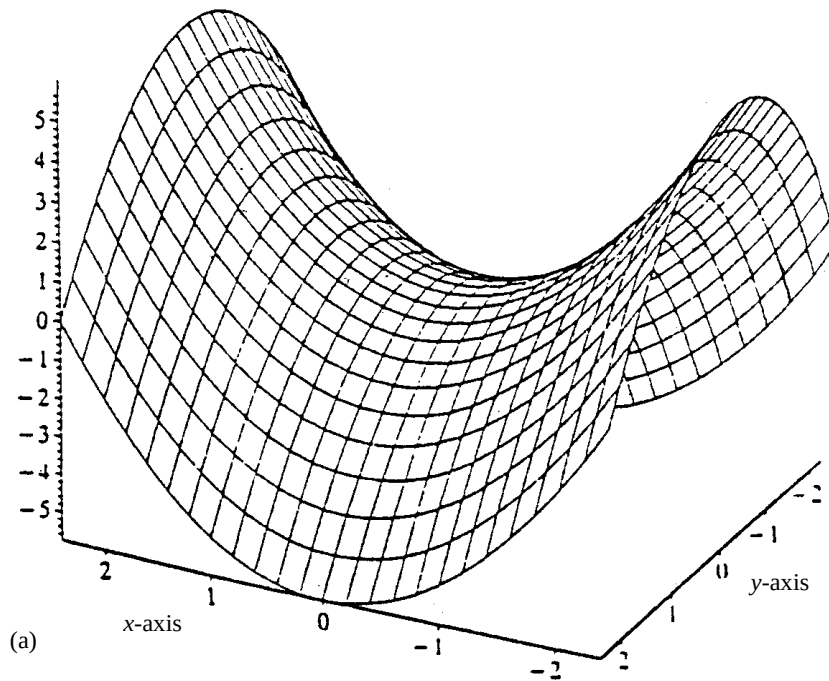
Some level curves on the graph of

$$z = f(x, y) = x^2 - y^2$$

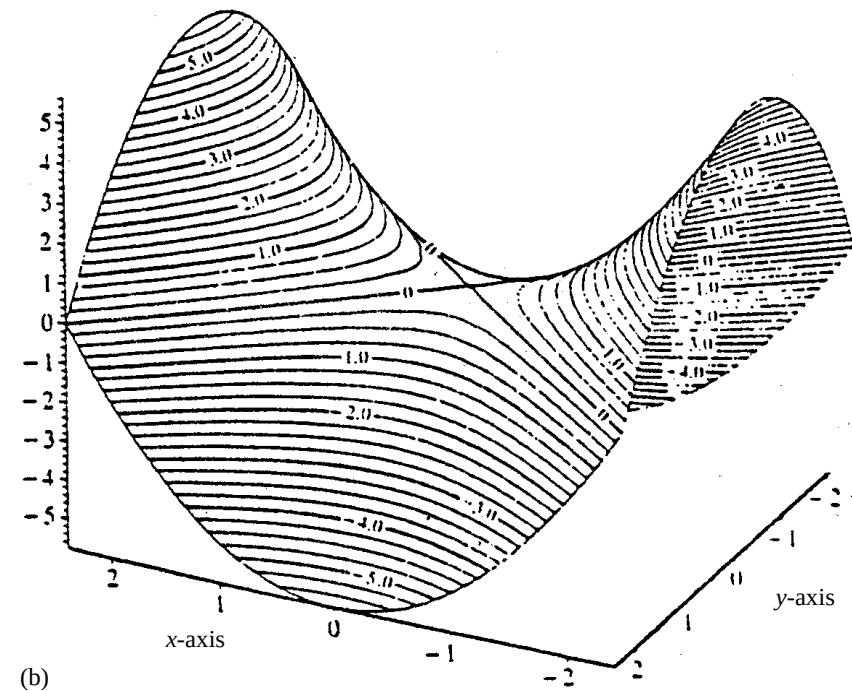
This is called a **hyperbolic paraboloid**.

Computer generated graphs usually show lines for which x is constant and lines for which y is constant in order to give a picture of the surface.

Examples



A computer-generated graph
of $z = x^2 - y^2$.



The graph with level curves
fitted to it.

2 Limit and Continuity

The concepts of limit, continuity and derivative for functions of more than one variable are naturally generalized from the one variable situation. The case for two variables will be considered here.

Definition

A function $f(x, y)$ is said to possess a **limit** L at the point (x_0, y_0) if for any $\varepsilon > 0$ it is possible to find a $\delta > 0$ such that whenever

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

i.e. (x, y) is in a disc of radius δ centered at (x_0, y_0) , we have

$$|f(x, y) - L| < \varepsilon.$$

We write

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L \quad \text{or} \quad \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = L.$$

Intuitively, the above says that when (x, y) is approaching (x_0, y_0) , $f(x, y)$ is approaching L , then the function $f(x, y)$ is said to possess a limit at the point (x_0, y_0) .

Here, L is a real number.

Note that the limit exists only if $f(x, y) \rightarrow L$ regardless of the path along which $(x, y) \rightarrow (x_0, y_0)$. If $f(x, y)$ has different limits along 2 distinct paths as $(x, y) \rightarrow (x_0, y_0)$, then $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ does not exist.

Example 1

Evaluate the following limits, if they exist.

$$(a) \quad \lim_{(x,y) \rightarrow (1,2)} (x^2 + 3y^2 + xy)$$

$$(b) \quad \lim_{(x,y) \rightarrow (1,0)} \frac{x^3 + xy - y^2}{xy^2 - 2xy + 3}$$

$$(c) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

Solution

$$(a) \quad \lim_{(x,y) \rightarrow (1,2)} (x^2 + 3y^2 + xy) = 1^2 + 3(2^2) + 1(2) = 15 \quad (\text{by direct substitution})$$

$$(b) \quad \lim_{(x,y) \rightarrow (1,0)} \frac{x^3 + xy - y^2}{xy^2 - 2xy + 3} = \frac{1^3 + (1)(0) - (0)^2}{(1)(0)^2 - 2(1)(0) + 3} = \frac{1}{3} \quad (\text{by direct substitution})$$

(c) Note that we cannot evaluate the limit by direct substitution because the function

$\frac{\sin(x^2+y^2)}{x^2+y^2}$ is not defined at $(0,0)$.

If we use the result $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$, we can deduce that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1.$$

Remark:

The above example shows that the limit of a function may still exist even when the function is not defined at (x_0, y_0) .



Example 2

Show that $f(x, y) = \frac{x^2}{x^2 + y^2}$ has no limit at the origin.

Solution

First note that the function $f(x, y)$ is not defined at $(0, 0)$.

As $(x, y) \rightarrow (0, 0)$ along the line $y = 0$,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2} \underset{\substack{\text{put} \\ y=0}}{=} \lim_{x \rightarrow 0} \frac{x^2}{\underbrace{x^2 + 0^2}_{f(x,0)}} = \lim_{x \rightarrow 0} (1) = 1.$$

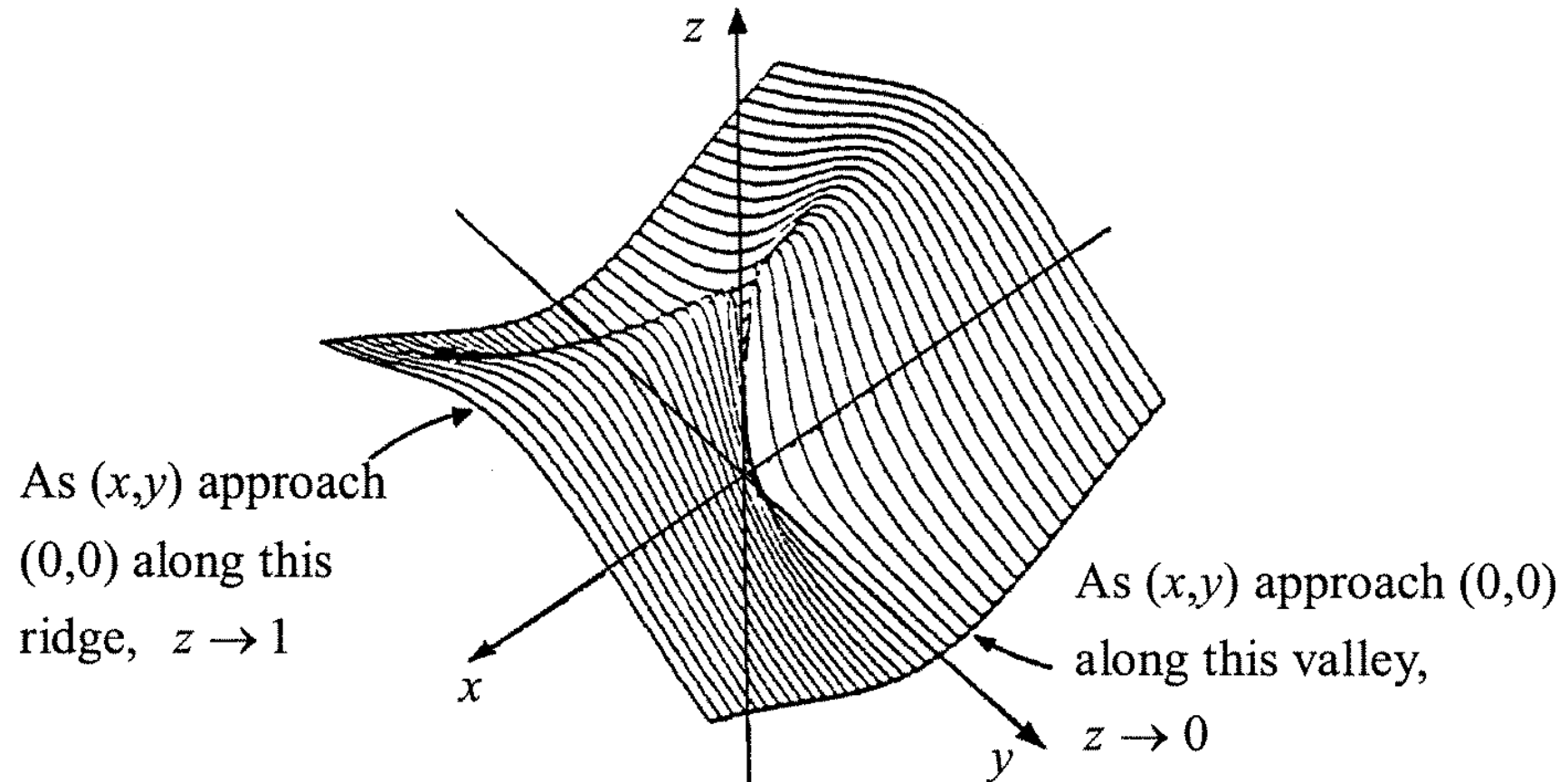
As $(x, y) \rightarrow (0, 0)$ along the line $x = 0$,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2} \underset{\substack{\text{put} \\ x=0}}{=} \lim_{y \rightarrow 0} \frac{0^2}{\underbrace{0^2 + y^2}_{f(0,y)}} = \lim_{y \rightarrow 0} (0) = 0.$$

Since $\frac{x^2}{x^2 + y^2}$ has different limits along 2 different paths as $(x, y) \rightarrow (0, 0)$, the limit

$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2}$ does not exist.

Here is the graph of $z = \frac{x^2}{x^2+y^2}$ for $-1 \leq x \leq 1$, $-1 \leq y \leq 1$:



Example 3

Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} = 0$

Solution

For any point $(x, y) \neq (0, 0)$, we have

$$0 \leq \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}} \leq \frac{x^2 + y^2}{(x^2 + y^2)^{\frac{1}{2}}} = (x^2 + y^2)^{\frac{1}{2}}.$$

Limit of lower bound: $\lim_{(x,y) \rightarrow (0,0)} (0) = 0$

Limit of upper bound: $\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2)^{\frac{1}{2}} = \sqrt{0^2 + 0^2} = 0$

$\therefore$ By the Sandwich Theorem,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}} = 0.$$

□

Continuity

A function $f(x, y)$ is said to be **continuous** at the point (x_0, y_0) if and only if it possesses a limit at (x_0, y_0) and $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$.

In other words, $f(x, y)$ is continuous at (x_0, y_0) if all the following 3 conditions are satisfied:

- (i) f is defined at (x_0, y_0) ,
- (ii) $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ exists, and
- (iii) $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$.

If any of the 3 conditions fails, $f(x, y)$ is discontinuous at (x_0, y_0) .

Example 4

(a) Is $f(x, y) = x^2 + 3y^2 + xy$ continuous at $(1, 2)$?

(b) Is $g(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ continuous at $(0,0)$?

(c) Is $h(x, y) = \frac{x^2}{x^2+y^2}$ continuous at $(0,0)$?

Solution

(a) From Example 1(a),

$$\lim_{(x,y) \rightarrow (1,2)} f(x, y) = \lim_{(x,y) \rightarrow (1,2)} (x^2 + 3y^2 + xy) = 1^2 + 3(2^2) + 1(2) = 15 = f(1, 2).$$

Thus, $f(x, y) = x^2 + 3y^2 + xy$ is **continuous** at $(1, 2)$.

(b) Since $g(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ is not defined at $(0,0)$, $g(x, y)$ is **discontinuous** at $(0,0)$.

(c) Since $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2+y^2}$ does not exist (from Example 2), $h(x, y) = \frac{x^2}{x^2+y^2}$ is **discontinuous** at $(0,0)$. □

We say that a function $f(x, y)$ is continuous everywhere if it is continuous at every point (x_0, y_0) in $\mathbb{R}^2$.

Example 5

(a) Is $f(x, y) = x^2 + 3y^2 + xy$ continuous everywhere?

(b) Is $h(x, y) = \frac{x^2}{x^2 + y^2}$ continuous everywhere?

(c) Find the value of A such that $g(x, y) = \begin{cases} \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}} & \text{for } (x, y) \neq (0, 0) \\ A & \text{for } (x, y) = (0, 0) \end{cases}$ is continuous everywhere.

Solution

(a) For any point (x_0, y_0) in $\mathbb{R}^2$,

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = \lim_{(x,y) \rightarrow (x_0,y_0)} (x^2 + 3y^2 + xy) = x_0^2 + 3y_0^2 + x_0y_0 = f(x_0, y_0)$$

Thus, $f(x, y)$ is continuous everywhere.

(b) First note that $h(x, y) = \frac{x^2}{x^2 + y^2}$ is not continuous at $(0,0)$ (from Example 4(c)), since

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2} \text{ does not exist.}$$

For $(x_0, y_0) \neq (0, 0)$, $h(x, y)$ is defined at (x_0, y_0) and

$$\lim_{(x,y) \rightarrow (x_0,y_0)} h(x,y) = \lim_{(x,y) \rightarrow (x_0,y_0)} \frac{x^2}{x^2 + y^2} = \frac{x_0^2}{x_0^2 + y_0^2} = h(x_0, y_0)$$

Thus, $h(x, y)$ is continuous at (x_0, y_0) , where $(x_0, y_0) \neq (0, 0)$.

Hence, $h(x, y)$ is continuous everywhere except at the origin $(0, 0)$.

$$(c) \quad g(x, y) = \begin{cases} \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} & \text{for } (x, y) \neq (0, 0) \\ A & \text{for } (x, y) = (0, 0) \end{cases} \quad \text{is continuous at every point } (x_0, y_0) \neq (0, 0).$$

This is because when $(x_0, y_0) \neq (0, 0)$,

$$\lim_{(x,y) \rightarrow (x_0,y_0)} g(x, y) = \lim_{(x,y) \rightarrow (x_0,y_0)} \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{x_0^2}{(x_0^2 + y_0^2)^{\frac{1}{2}}} = g(x_0, y_0).$$

$g(x, y)$ is continuous at $(0, 0)$ iff $\lim_{(x,y) \rightarrow (0,0)} g(x, y) = g(0, 0)$.

From Example 3, $\lim_{(x,y) \rightarrow (0,0)} g(x, y) = 0$.

Also, $g(0, 0) = A$.

Hence, $g(x, y)$ is continuous everywhere iff $A = 0$.

□

Example 6

Consider the function $f(x, y) = \begin{cases} \frac{2xy^2}{x^2+y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$.

Show that $f(x, y)$ has the same limit as $f(0, 0)$ when (x, y) approaches $(0, 0)$ along any straight line passing through the origin, however, the function is not continuous at $(0, 0)$.

Solution

Suppose (x, y) is approaching $(0, 0)$ along the (non-vertical) straight line $y = mx$, where m is a real number.

Then

$$\begin{aligned} \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=mx}} f(x, y) &= \lim_{x \rightarrow 0} f(x, mx) = \lim_{x \rightarrow 0} \frac{2x(mx)^2}{x^2 + (mx)^4} \quad \left(\frac{0}{0} \text{ form} \right) \\ &= \lim_{x \rightarrow 0} \frac{\frac{2m^2 x^3}{x^2}}{\frac{x^2 + m^4 x^4}{x^2}} = \lim_{x \rightarrow 0} \frac{2m^2 x}{1 + m^4 x^2} = \frac{2m^2(0)}{1 + m^4(0)^2} = \frac{0}{1} = 0 = f(0, 0) \end{aligned}$$

If the line is vertical and passing through the origin, we have $x = 0$ and this is the case that (x, y) is approaching $(0, 0)$ along the y -axis. In this case,

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ x=0}} f(x, y) = \lim_{\substack{y \rightarrow 0 \\ x=0}} f(0, y) = \lim_{y \rightarrow 0} \frac{2(0)y^2}{0^2 + y^4} = \lim_{y \rightarrow 0} 0 = 0 = f(0, 0).$$

Hence, when $(x, y) \rightarrow (0, 0)$ along any straight line passing through the origin,

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = f(0, 0).$$

However, if $(x, y) \rightarrow (0, 0)$ along the curve $x = t^2$, $y = t$, where $t \in \mathbb{R}$, we have

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ x=t^2 \\ y=t}} f(x, y) = \lim_{t \rightarrow 0} f(t^2, t) = \lim_{t \rightarrow 0} \frac{2(t^2)(t)^2}{(t^2)^2 + (t)^4} = \lim_{t \rightarrow 0} \frac{2t^4}{2t^4} = \lim_{t \rightarrow 0} (1) = 1 \neq 0 = f(0, 0).$$

Since $f(x, y)$ has different limits along different paths as $(x, y) \rightarrow (0, 0)$, the limit

$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

Hence, $f(x, y)$ is not continuous at $(0, 0)$. □

3 Partial Differentiation

Recall that for a function $f(x)$ of one variable,

$$\frac{df}{dx}(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

or equivalently,

$$\frac{df}{dx}(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Now consider a function $f(x, y)$ of two variables. If y is held constant, say $y = y_0$, then $f(x, y_0)$ becomes a function of one variable, x , and we may define its derivative at $x = x_0$, called the **partial derivative of f with respect to x** at the point (x_0, y_0) , as

$$\frac{\partial f}{\partial x}(x_0, y_0) = \left(\frac{d}{dx} f(x, y_0) \right) \Big|_{x=x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

or equivalently,

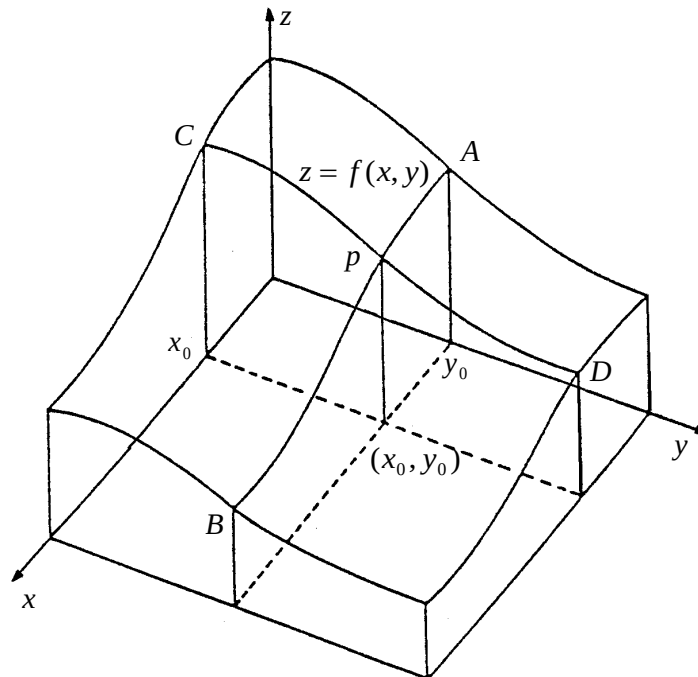
$$\frac{\partial f}{\partial x}(x_0, y_0) = \left(\frac{d}{dx} f(x, y_0) \right) \Big|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}.$$

Similarly, the **partial derivative of f with respect to y** at the point (x_0, y_0) is

$$\frac{\partial f}{\partial y}(x_0, y_0) = \left(\frac{d}{dy} f(x_0, y) \right) \bigg|_{y=y_0} = \lim_{k \rightarrow 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k},$$

or equivalently,

$$\frac{\partial f}{\partial y}(x_0, y_0) = \left(\frac{d}{dy} f(x_0, y) \right) \bigg|_{y=y_0} = \lim_{y \rightarrow y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}.$$



AB – curve of $f(x, y_0)$

CD – curve of $f(x_0, y)$

Remarks:

- (a) A partial derivative is just an ordinary derivative with respect to the variable which is not held constant.
- (b) For $z = f(x, y)$, alternative notations are:

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \frac{\partial z}{\partial x}(x, y) = z_x(x, y).$$

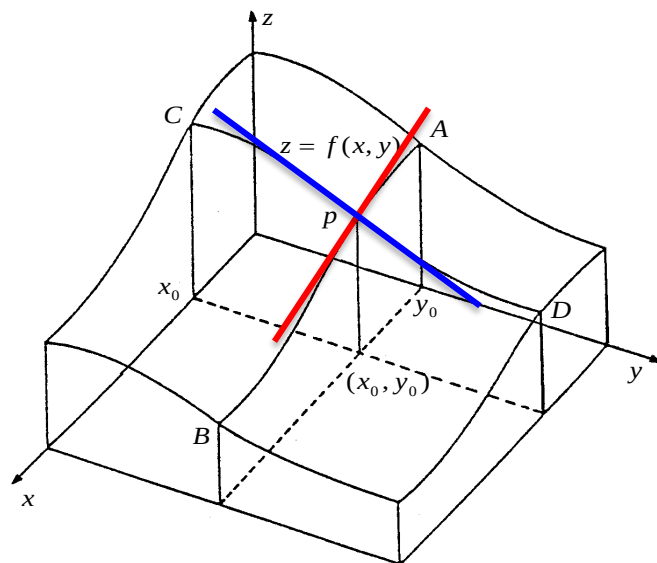
Similarly,

$$\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \frac{\partial z}{\partial y}(x, y) = z_y(x, y).$$

- (c) The definition of a partial derivative only requires the idea of a limit of a function of a single variable.

Interpretations of $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$:

- $\frac{\partial f}{\partial x}(x_0, y_0)$ is the slope of the tangent line to the curve $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ **in the plane $y = y_0$** . It is the rate of change of f at (x_0, y_0) in the **x -direction**.
- $\frac{\partial f}{\partial y}(x_0, y_0)$ is the slope of the tangent line to the curve $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ **in the plane $x = x_0$** . It is the rate of change of f at (x_0, y_0) in the **y -direction**.



Example 7

Find the first partial derivatives of

$$f(x, y) = x^2y + y^3 \sin x.$$

Solution

$$\frac{\partial f}{\partial x} = 2xy + y^3 \cos x$$

$$\frac{\partial f}{\partial y} = x^2 + 3y^2 \sin x$$

Example 8

Evaluate $\frac{\partial f}{\partial x}(1, 2)$ and $\frac{\partial f}{\partial y}(1, 2)$ for

$$f(x, y) = x^2y + y^3 \sin x.$$

Solution

Method 1:

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ first. Then substitute $x = 1$, $y = 2$ into $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = 2xy + y^3 \cos x$$

$$\frac{\partial f}{\partial y} = x^2 + 3y^2 \sin x$$

$$\therefore \frac{\partial f}{\partial x}(1, 2) = (2xy + y^3 \cos x)|_{\substack{x=1 \\ y=2}} = 2(1)(2) + 2^3 \cos 1 = 4 + 8 \cos 1$$

$$\frac{\partial f}{\partial y}(1, 2) = (x^2 + 3y^2 \sin x)|_{\substack{x=1 \\ y=2}} = 1^2 + 3(2)^2 \sin 1 = 1 + 12 \sin 1$$

Method 2:

To find $\frac{\partial f}{\partial x}(1, 2)$, regard y as a constant ($y = 2$).

Consider $f(x, 2) = x^2(2) + 2^3 \sin x = 2x^2 + 8 \sin x$ ($\leftarrow$ **a function of x**).

Then

$$\frac{\partial f}{\partial x}(1, 2) = \frac{d[f(x, 2)]}{dx} \Big|_{x=1} = \left[\frac{d}{dx} (2x^2 + 8 \sin x) \right] \Big|_{x=1} = (4x + 8 \cos x) \Big|_{x=1} = 4 + 8 \cos 1$$

To find $\frac{\partial f}{\partial y}(1, 2)$, regard x as a constant ($x = 1$).

Consider $f(1, y) = (1)^2 y + y^3 \sin 1 = y + y^3 \sin 1$ ($\leftarrow$ **a function of y**)

Then

$$\frac{\partial f}{\partial y}(1, 2) = \frac{d[f(1, y)]}{dy} \Big|_{y=2} = \left[\frac{d}{dy} (y + y^3 \sin 1) \right] \Big|_{y=2} = (1 + 3y^2 \sin 1) \Big|_{y=2} = 1 + 12 \sin 1$$

□

Example 9

Find the first partial derivatives of

$$f(x, y, z) = e^{2z} \cos(xy).$$

Solution

$$\frac{\partial f}{\partial x} = -e^{2z} y \sin(xy)$$

$$\frac{\partial f}{\partial y} = -e^{2z} x \sin(xy)$$

$$\frac{\partial f}{\partial z} = 2e^{2z} \cos(xy)$$

Note:

The number of first order partial derivatives is equal to the number of independent variables in the multivariable function.

Higher order partial derivatives

We may differentiate the first order partial derivatives of a function again to obtain the **second order partial derivatives**.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$$

} Mixed second order
partial derivatives

f_{xy} gives the rate of change of the slope in the x -direction as one moves into the y -direction.

The other second order partial derivatives can be interpreted similarly.

Higher order partial derivatives can be defined similarly.

For example,

$$f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial y \partial x} \right) = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \right)$$

$$f_{xxxy} = \frac{\partial^4 f}{\partial y \partial x^3} = \frac{\partial}{\partial y} \left(\frac{\partial^3 f}{\partial x^3} \right) = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} \left(\frac{\partial^2 f}{\partial x^2} \right) \right) = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \right) \right)$$

Example 10

Let $f(x, y) = \ln\left[\sqrt{x^2 + y^2}\right]$, find its partial derivatives up to order two.

Solution

First note that $f(x, y) = \ln\left[\sqrt{x^2 + y^2}\right] = \ln\left[(x^2 + y^2)^{\frac{1}{2}}\right] = \frac{1}{2}\ln(x^2 + y^2)$.

First order partial derivatives:

$$f_x = \frac{\partial f}{\partial x} = \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{2} \cdot \frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$$

Second order partial derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) = \frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) = \frac{(x^2 + y^2) \cdot 0 - x \cdot 2y}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$
$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{y}{x^2 + y^2} \right) = \frac{(x^2 + y^2) \cdot 0 - y \cdot 2x}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$



Note that in the above example, $f_{xy} = f_{yx}$, which is the case for most functions which are useful in engineering and science. However, it is not always true. The following theorem tells us the conditions for which $f_{xy} = f_{yx}$.

Theorem

If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ exist at (x_0, y_0) and its neighborhood, and $f_{xy}(x, y)$ & $f_{yx}(x, y)$ are both continuous at (x_0, y_0) , then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

Example 11

Let

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- (a) If $(x, y) \neq (0, 0)$, find $f_{xy}(x, y)$, $f_{yx}(x, y)$ and verify that they are equal.
- (b) Show that $f_x(0, 0) = f_y(0, 0)$.
- (c) Show that $f_{xy}(0, 0)$, $f_{yx}(0, 0)$ exist but they are not equal.

Solution

(a) If $(x, y) \neq (0, 0)$,

$$\begin{aligned} f_{xy}(x, y) &= \frac{\partial}{\partial y} \left[\frac{\partial}{\partial x} \left(\frac{xy(x^2 - y^2)}{x^2 + y^2} \right) \right] \\ &= \frac{\partial}{\partial y} \left[\frac{(x^2 + y^2)(3x^2y - y^3) - xy(x^2 - y^2)(2x)}{(x^2 + y^2)^2} \right] \\ &= \frac{\partial}{\partial y} \left[\frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} \right] \\ &= \frac{(x^2 + y^2)^2(x^4 + 12x^2y^2 - 5y^4) - (x^4y + 4x^2y^3 - y^5)2(x^2 + y^2)2y}{(x^2 + y^2)^4} \\ &= \frac{(x^2 + y^2)(x^4 + 12x^2y^2 - 5y^4) - 4y(x^4y + 4x^2y^3 - y^5)}{(x^2 + y^2)^3} \\ &= \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3} \end{aligned}$$

$$\begin{aligned}f_{yx}(x, y) &= \frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} \left(\frac{xy(x^2 - y^2)}{x^2 + y^2} \right) \right] \\&= \frac{\partial}{\partial x} \left[\frac{(x^2 + y^2)(x^3 - 3xy^2) - xy(x^2 - y^2)(2y)}{(x^2 + y^2)^2} \right] \\&= \frac{\partial}{\partial x} \left[\frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2} \right] \\&= \frac{(x^2 + y^2)^2(5x^4 - 12x^2y^2 - y^4) - (x^5 - 4x^3y^2 - xy^4)2(x^2 + y^2)2x}{(x^2 + y^2)^4} \\&= \frac{(x^2 + y^2)(5x^4 - 12x^2y^2 - y^4) - 4x(x^5 - 4x^3y^2 - xy^4)}{(x^2 + y^2)^3} \\&= \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3}\end{aligned}$$

Hence, $f_{xy}(x, y) = f_{yx}(x, y)$ if $(x, y) \neq (0, 0)$.

$$(b) \quad f_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\frac{x(0)(x^2 - 0^2)}{x^2 + 0^2} - 0}{x} = \lim_{x \rightarrow 0} \frac{0 - 0}{x} = \lim_{x \rightarrow 0} 0 = 0$$

$$f_y(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{\frac{(0)y(0^2 - y^2)}{0^2 + y^2} - 0}{y} = \lim_{y \rightarrow 0} \frac{0 - 0}{y} = \lim_{y \rightarrow 0} 0 = 0$$

(c) If $y \neq 0$,

$$f_x(0, y) = \lim_{x \rightarrow 0} \frac{f(x, y) - f(0, y)}{x - 0} = \lim_{x \rightarrow 0} \frac{\frac{xy(x^2 - y^2)}{x^2 + y^2} - 0}{x} = \lim_{x \rightarrow 0} \frac{y(x^2 - y^2)}{x^2 + y^2} = -y.$$

$$\text{Thus, } f_x(0, y) = \begin{cases} -y & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases} = -y \text{ for all } y \in \mathbb{R}.$$

$$f_{xy}(0, 0) = \lim_{y \rightarrow 0} \frac{f_x(0, y) - f_x(0, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{-y - 0}{y} = \lim_{y \rightarrow 0} (-1) = -1$$

Similarly, if $x \neq 0$,

$$f_y(x, 0) = \lim_{y \rightarrow 0} \frac{f(x, y) - f(x, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{\frac{xy(x^2 - y^2)}{x^2 + y^2} - 0}{y} = \lim_{y \rightarrow 0} \frac{x(x^2 - y^2)}{x^2 + y^2} = x.$$

$$\text{Thus, } f_y(x, 0) = \begin{cases} x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} = x \text{ for all } x \in \mathbb{R}.$$

$$f_{yx}(0, 0) = \lim_{x \rightarrow 0} \frac{f_y(x, 0) - f_y(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x - 0}{x} = \lim_{x \rightarrow 0} (1) = 1$$

We find that $f_{xy}(0,0)$ and $f_{yx}(0,0)$ exist but $f_{xy}(0,0) \neq f_{yx}(0,0)$. Why does this happen?

For $f_{xy}(x,y) = \begin{cases} \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3} & \text{if } (x,y) \neq (0,0) \\ -1 & \text{if } (x,y) = (0,0) \end{cases}$, we observe that when (x,y)

approaches $(0,0)$ along the line $y = 0$,

$$\lim_{(x,y) \rightarrow (0,0)} f_{xy}(x,y) \underset{\substack{= \\ \text{put } y=0}}{\quad} \lim_{x \rightarrow 0} f_{xy}(x,0) = \lim_{x \rightarrow 0} \frac{x^6 + 0 - 0 - 0}{(x^2 + 0)^3} = \lim_{x \rightarrow 0} (1) = 1$$

However, when (x,y) approaches $(0,0)$ along the line $x = 0$,

$$\lim_{(x,y) \rightarrow (0,0)} f_{xy}(x,y) \underset{\substack{= \\ \text{put } x=0}}{\quad} \lim_{y \rightarrow 0} f_{xy}(0,y) = \lim_{y \rightarrow 0} \frac{0 + 0 - 0 - y^6}{(0 + y^2)^3} = \lim_{y \rightarrow 0} (-1) = -1$$

It concludes that $f_{xy}(x,y)$ is not continuous at $(0,0)$.



Example 12

Find all the first and second order partial derivatives of $f(x, y, z) = ze^{xy^2}$.

Solution**First order partial derivatives:**

$$f_x = \frac{\partial f}{\partial x} = ze^{xy^2} \cdot y^2 = zy^2 e^{xy^2}$$

$$f_y = \frac{\partial f}{\partial y} = ze^{xy^2} \cdot 2xy = 2xyze^{xy^2}$$

$$f_z = \frac{\partial f}{\partial z} = e^{xy^2}$$

Second order partial derivatives:

$$f_{xx} = \frac{\partial}{\partial x} (zy^2 e^{xy^2}) = zy^2 e^{xy^2} \cdot y^2 = zy^4 e^{xy^2}$$

$$f_{yy} = \frac{\partial}{\partial y} (2xyze^{xy^2}) = 2xze^{xy^2} + 2xyze^{xy^2} \cdot 2xy = 2xz(1 + 2xy^2)e^{xy^2}$$

$$f_{zz} = \frac{\partial}{\partial z}(e^{xy^2}) = 0$$

$$f_{xy} = \frac{\partial}{\partial y}(zy^2e^{xy^2}) = z(2y)e^{xy^2} + zy^2e^{xy^2} \cdot 2xy = 2yz(1 + xy^2)e^{xy^2} = f_{yx}$$

$$f_{xz} = \frac{\partial}{\partial z}(zy^2e^{xy^2}) = y^2e^{xy^2} = f_{zx}$$

$$f_{yz} = \frac{\partial}{\partial z}(2xyze^{xy^2}) = 2xye^{xy^2} = f_{zy}$$

Note: f has 3 independent variables, so it has 3 first order partial derivatives and $3^2 = 9$ second order partial derivatives.

Class Exercise:

For the function $f(x, y, z) = ze^{xy^2}$ in Example 12, find $f_{xyz}(1, 2, 3)$ and $f_{yzx}(1, 2, 3)$, and show that they are equal.

4 The Chain Rule

For functions of one variable, if $y = f(x)$ and $x = x(t)$, then $y = f(x(t))$ and

$$\frac{dy}{dt} = \frac{df}{dx} \frac{dx}{dt}.$$

Now consider functions of two variables.

If $f = f(u)$ and $u = u(x, y)$, then

$$\frac{\partial f}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{df}{du} \frac{\partial u}{\partial y}.$$

Example 13

If $f = f(u) = \ln u$ and $u = (x^2 + y^2)^{\frac{1}{2}}$, evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution

$$\frac{\partial f}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} = \frac{1}{u} \cdot \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \cdot 2x = \frac{1}{(x^2 + y^2)^{\frac{1}{2}}} \cdot \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{x}{x^2 + y^2}.$$

$$\frac{\partial f}{\partial y} = \frac{df}{du} \frac{\partial u}{\partial y} = \frac{1}{u} \cdot \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \cdot 2y = \frac{1}{(x^2 + y^2)^{\frac{1}{2}}} \cdot \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{y}{x^2 + y^2}.$$

Alternative method: We write f as $f(x, y) = \ln \left[(x^2 + y^2)^{\frac{1}{2}} \right]$. Then

$$\frac{\partial f}{\partial x} = \frac{1}{(x^2 + y^2)^{\frac{1}{2}}} \cdot \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{(x^2 + y^2)^{\frac{1}{2}}} \cdot \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \cdot 2y = \frac{y}{x^2 + y^2}$$

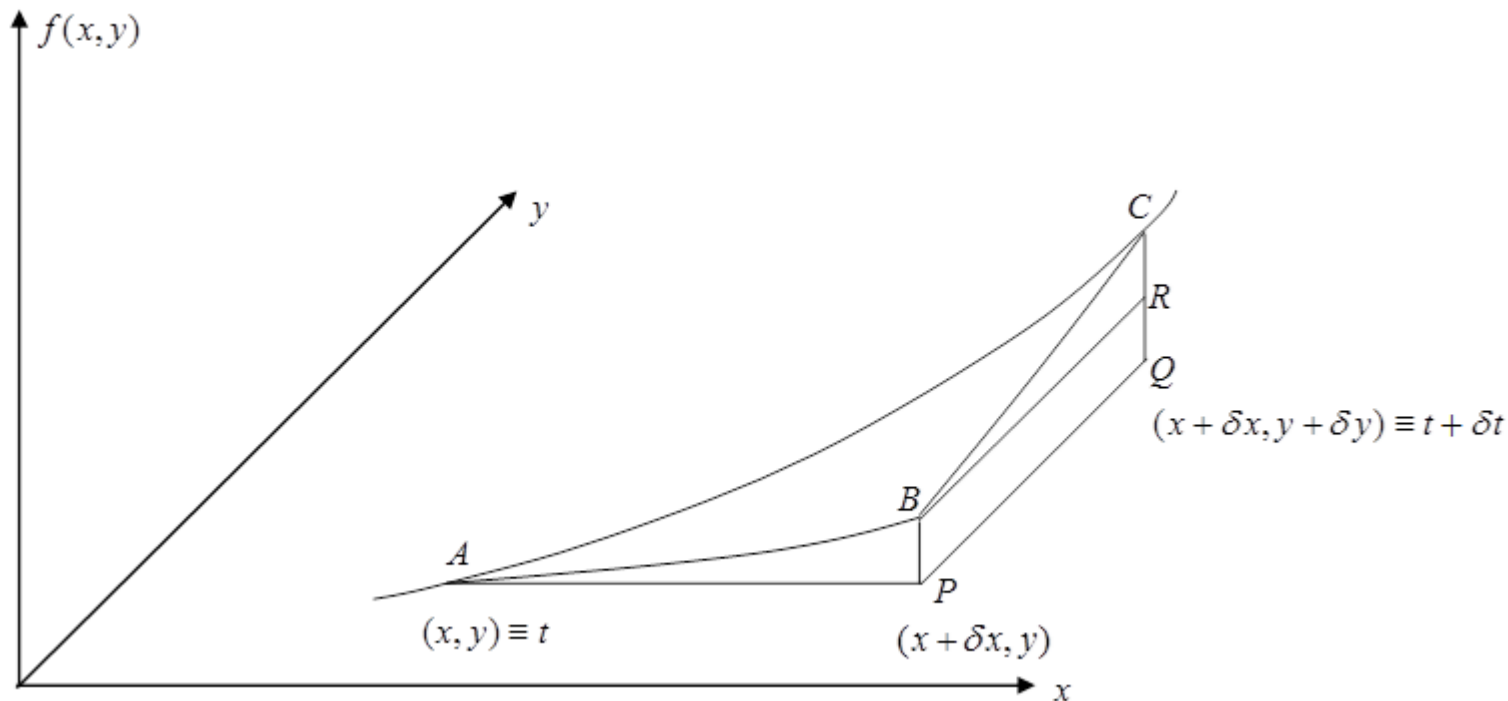
For functions of two variables, we have the following result:

Theorem

If $z = f(x, y)$ where $x = x(t)$ and $y = y(t)$, then

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}}.$$

Proof: (for your reference)



$$\frac{dz}{dt} = \lim_{\delta t \rightarrow 0} \frac{z(t + \delta t) - z(t)}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{f(x(t + \delta t), y(t + \delta t)) - f(x(t), y(t))}{\delta t}$$

Now as t changes to $t + \delta t$, x and y change by δx and δy respectively, where

$$\delta x = x(t + \delta t) - x(t), \quad \delta y = y(t + \delta t) - y(t).$$

Thus the change in f may be expressed as

$$\begin{aligned} & f(x(t + \delta t), y(t + \delta t)) - f(x(t), y(t)) \\ &= f(x + \delta x, y + \delta y) - f(x, y) = QC = QR + RC = PB + RC \\ &= [f(x + \delta x, y) - f(x, y)] + [f(x + \delta x, y + \delta y) - f(x + \delta x, y)] \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{dz}{dt} &= \lim_{\delta t \rightarrow 0} \left[\frac{f(x + \delta x, y) - f(x, y)}{\delta t} + \frac{f(x + \delta x, y + \delta y) - f(x + \delta x, y)}{\delta t} \right] \\ &= \lim_{\delta t \rightarrow 0} \left[\frac{f(x + \delta x, y) - f(x, y)}{\delta x} \right] \lim_{\delta t \rightarrow 0} \left[\frac{\delta x}{\delta t} \right] + \lim_{\delta t \rightarrow 0} \left[\frac{f(x + \delta x, y + \delta y) - f(x + \delta x, y)}{\delta y} \right] \lim_{\delta t \rightarrow 0} \left[\frac{\delta y}{\delta t} \right] \end{aligned}$$

$$\therefore \boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}}$$

□

Example 14

Let $z = f(x, y) = x^2 + 2xy$, where $x = \sin t$ and $y = \cos t$. Find $\frac{dz}{dt}$.

Solution

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= (2x + 2y) \cos t + (2x)(-\sin t) \\ &= 2(\sin t + \cos t) \cos t - 2 \sin^2 t \\ &= 2 \sin t \cos t + 2(\cos^2 t - \sin^2 t) \\ &= \sin 2t + 2 \cos 2t\end{aligned}$$

This can be confirmed by direct substitution:

$$z = \sin^2 t + 2 \sin t \cos t = \sin^2 t + \sin 2t$$

Then

$$\frac{dz}{dt} = 2 \sin t \cos t + 2 \cos 2t = \sin 2t + 2 \cos 2t.$$



A special case is when $z = f(x, y)$ and $y = y(x)$ (i.e. $x = x$) and hence z is a function of x only. Then

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} \underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial f}{\partial y} \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}.$$

Illustration

Let $z = f(x, y) = \tan^{-1} \left(\frac{x}{y} \right)$, where $y = \sin x$. Then

$$\begin{aligned} \frac{dz}{dx} &= \frac{\partial}{\partial x} \left[\tan^{-1} \left(\frac{x}{y} \right) \right] \cdot \frac{dx}{dx} + \frac{\partial}{\partial y} \left[\tan^{-1} \left(\frac{x}{y} \right) \right] \cdot \frac{d(\sin x)}{dx} \\ &= \frac{1}{1 + \left(\frac{x}{y} \right)^2} \cdot \frac{1}{y} \cdot 1 + \frac{1}{1 + \left(\frac{x}{y} \right)^2} \cdot \left(-\frac{x}{y^2} \right) \cdot \cos x \\ &= \frac{y}{x^2 + y^2} - \frac{x \cos x}{x^2 + y^2} \\ &= \frac{\sin x - x \cos x}{x^2 + \sin^2 x} \end{aligned}$$

Example 15 (Chain rule involving second derivatives)

Let $z = f(x, y)$, where $x = x(t)$, $y = y(t)$. Evaluate $\frac{d^2 z}{dt^2}$.

Solution

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Then

$$\begin{aligned} \frac{d^2 z}{dt^2} &= \frac{d}{dt} \left(\frac{dz}{dt} \right) = \frac{d}{dt} \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right) \\ &= \frac{d}{dt} \left(\frac{\partial f}{\partial x} \frac{dx}{dt} \right) + \frac{d}{dt} \left(\frac{\partial f}{\partial y} \frac{dy}{dt} \right) \\ &= \underbrace{\frac{\partial f}{\partial x} \cdot \frac{d}{dt} \left(\frac{dx}{dt} \right) + \frac{d}{dt} \left(\frac{\partial f}{\partial x} \right) \cdot \frac{dx}{dt}}_{\text{by product rule}} + \underbrace{\frac{\partial f}{\partial y} \cdot \frac{d}{dt} \left(\frac{dy}{dt} \right) + \frac{d}{dt} \left(\frac{\partial f}{\partial y} \right) \cdot \frac{dy}{dt}}_{\text{by product rule}} \\ &= \frac{\partial f}{\partial x} \cdot \frac{d^2 x}{dt^2} + \frac{d}{dt} \left(\frac{\partial f}{\partial x} \right) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{d^2 y}{dt^2} + \frac{d}{dt} \left(\frac{\partial f}{\partial y} \right) \cdot \frac{dy}{dt} \end{aligned}$$

We observe that $\frac{\partial f}{\partial x}$ is depending on x and y directly, depending indirectly on t through x

and y respectively. To evaluate $\frac{d}{dt} \left(\frac{\partial f}{\partial x} \right)$, we need to apply the Chain Rule:

$$\frac{d}{dt} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \cdot \frac{dx}{dt} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \cdot \frac{dy}{dt} = \frac{\partial^2 f}{\partial x^2} \cdot \frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{dy}{dt}$$

For the same reason, we apply the Chain Rule to evaluate $\frac{d}{dt} \left(\frac{\partial f}{\partial y} \right)$:

$$\frac{d}{dt} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \cdot \frac{dx}{dt} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \cdot \frac{dy}{dt} = \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{dy}{dt}$$

Then it follows that

$$\begin{aligned} \frac{d^2 z}{dt^2} &= \frac{\partial f}{\partial x} \cdot \frac{d^2 x}{dt^2} + \frac{d}{dt} \left(\frac{\partial f}{\partial x} \right) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{d^2 y}{dt^2} + \frac{d}{dt} \left(\frac{\partial f}{\partial y} \right) \cdot \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} \cdot \frac{d^2 x}{dt^2} + \left[\frac{\partial^2 f}{\partial x^2} \cdot \frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{dy}{dt} \right] \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{d^2 y}{dt^2} + \left[\frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{dy}{dt} \right] \cdot \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} \cdot \frac{d^2 x}{dt^2} + \frac{\partial^2 f}{\partial x^2} \cdot \left(\frac{dx}{dt} \right)^2 + \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{dy}{dt} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{d^2 y}{dt^2} + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dx}{dt} \cdot \frac{dy}{dt} + \frac{\partial^2 f}{\partial y^2} \cdot \left(\frac{dy}{dt} \right)^2 \end{aligned}$$

Example 16

Let $z = f(x, y)$, where $x = t^2$, $y = t$. Suppose

$$\frac{\partial f}{\partial x}(1, 1) = \frac{\partial f}{\partial y}(1, 1) = \frac{\partial^2 f}{\partial x^2}(1, 1) = \frac{\partial^2 f}{\partial y^2}(1, 1) = e ,$$

$$\frac{\partial^2 f}{\partial x \partial y}(1, 1) = \frac{\partial^2 f}{\partial y \partial x}(1, 1) = 2e .$$

Evaluate $\frac{d^2 z}{dt^2}$ for $t = 1$.

Solution

$$x = t^2 \Rightarrow \frac{dx}{dt} = 2t$$

$$y = t \Rightarrow \frac{dy}{dt} = 1$$

Then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 2t \cdot \frac{\partial f}{\partial x} + 1 \cdot \frac{\partial f}{\partial y} = 2t \cdot \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

and

$$\begin{aligned}
\frac{d^2 z}{dt^2} &= \frac{d}{dt} \left(\frac{dz}{dt} \right) = \frac{d}{dt} \left(2t \cdot \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right) = \underbrace{2t \cdot \frac{d}{dt} \left(\frac{\partial f}{\partial x} \right) + \frac{d}{dt} (2t) \cdot \frac{\partial f}{\partial x}}_{\text{by product rule}} + \frac{d}{dt} \left(\frac{\partial f}{\partial y} \right) \\
&= 2t \left[\underbrace{\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \cdot \frac{dx}{dt} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \cdot \frac{dy}{dt}}_{\text{by Chain rule}} \right] + 2 \cdot \frac{\partial f}{\partial x} + \left[\underbrace{\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \cdot \frac{dx}{dt} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \cdot \frac{dy}{dt}}_{\text{by Chain rule}} \right] \\
&= 2t \left[\frac{\partial^2 f}{\partial x^2} \cdot 2t + \frac{\partial^2 f}{\partial y \partial x} \cdot 1 \right] + 2 \cdot \frac{\partial f}{\partial x} + \left[\frac{\partial^2 f}{\partial x \partial y} \cdot 2t + \frac{\partial^2 f}{\partial y^2} \cdot 1 \right] \\
&= 4t^2 \frac{\partial^2 f}{\partial x^2} + 2t \frac{\partial^2 f}{\partial y \partial x} + 2 \frac{\partial f}{\partial x} + 2t \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

If $t = 1$, then $x = 1^2 = 1$ and $y = 1$. Hence,

$$\begin{aligned}
\left. \frac{d^2 z}{dt^2} \right|_{t=1} &= 4(t^2|_{t=1}) \left[\frac{\partial^2 f}{\partial x^2} (1, 1) \right] + 2(t|_{t=1}) \left[\frac{\partial^2 f}{\partial y \partial x} (1, 1) \right] + 2 \left[\frac{\partial f}{\partial x} (1, 1) \right] \\
&\quad + 2(t|_{t=1}) \left[\frac{\partial^2 f}{\partial x \partial y} (1, 1) \right] + \left[\frac{\partial^2 f}{\partial y^2} (1, 1) \right] \\
&= 4 \cdot 1^2 \cdot e + 2 \cdot 1 \cdot 2e + 2 \cdot e + 2 \cdot 1 \cdot 2e + e = 15e
\end{aligned}$$

□

Theorem (Change of Variables)

If $z = f(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$, then

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}}$$

and

$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}}.$$

Example 17

It is given that $z = f(x, y)$, where $f(x, y) = e^{2xy}$.

With the change of variables $x = r \cos \theta$ and $y = r \sin \theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ in terms of r and θ .

Solution

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} \\&= 2ye^{2xy} \cdot \cos \theta + 2xe^{2xy} \cdot \sin \theta \\&= 2r \sin \theta e^{2(r \cos \theta)(r \sin \theta)} \cdot \cos \theta + 2r \cos \theta e^{2(r \cos \theta)(r \sin \theta)} \cdot \sin \theta \\&= 2r(2 \sin \theta \cos \theta) e^{2(r \cos \theta)(r \sin \theta)} \\&= 2r(\sin 2\theta) e^{r^2 \sin 2\theta}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial \theta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \\&= 2ye^{2xy} \cdot (-r \sin \theta) + 2xe^{2xy} \cdot (r \cos \theta) \\&= 2r \sin \theta e^{2(r \cos \theta)(r \sin \theta)} \cdot (-r \sin \theta) + 2r \cos \theta e^{2(r \cos \theta)(r \sin \theta)} \cdot (r \cos \theta) \\&= 2r^2(\cos^2 \theta - \sin^2 \theta)e^{2(r \cos \theta)(r \sin \theta)} \\&= 2r^2(\cos 2\theta)e^{r^2 \sin 2\theta}\end{aligned}$$

Remark:

These results can be checked by substitution and direct partial differentiation.



Example 18 (Chain rule involving second partial derivatives)

Let $z = f(x, y)$, where $x = x(s, t)$ and $y = y(s, t)$. Assume that the first and second order partial derivatives of f exist, and both f_{xy} and f_{yx} are continuous. Evaluate $\frac{\partial^2 z}{\partial t^2}$.

Solution

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \quad (\text{by Chain rule})$$

Then

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial z}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \right) \\ &= \underbrace{\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \right) \frac{\partial x}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial}{\partial t} \left(\frac{\partial x}{\partial t} \right)}_{\text{by product rule}} + \underbrace{\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial t} \right)}_{\text{by product rule}} \end{aligned}$$

We observe that both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are depending on x and y directly, depending indirectly on t through x and y respectively. To evaluate $\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \right)$ and $\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \right)$, we apply the Chain

Rule and we get

$$\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \frac{\partial x}{\partial t} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \frac{\partial y}{\partial t} = \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial t}$$

and

$$\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \frac{\partial x}{\partial t} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial t} = \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial t}.$$

It follows that

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \right) \frac{\partial x}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial}{\partial t} \left(\frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial t} \right) \\ &= \left(\frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial t} \right) \frac{\partial x}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial^2 x}{\partial t^2} + \left(\frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial t} \right) \frac{\partial y}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial t^2} \\ &= \frac{\partial^2 f}{\partial x^2} \left(\frac{\partial x}{\partial t} \right)^2 + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial t} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 f}{\partial y^2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial t^2} \\ &= \frac{\partial^2 f}{\partial x^2} \left(\frac{\partial x}{\partial t} \right)^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 f}{\partial y^2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{\partial f}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial t^2} \end{aligned}$$

□

Remark:

The above result can be easily obtained with the use of tree diagrams.

Example 19

Consider $z = f(x, y)$, suppose $x = r \cos \theta$ and $y = r \sin \theta$.

Assume that the first and second order partial derivatives of f exist, and both f_{xy} and f_{yx} are continuous. Show that

$$\frac{\partial z}{\partial r} = \cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y}$$

and

$$\frac{\partial^2 z}{\partial r^2} = \cos^2 \theta \frac{\partial^2 f}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 f}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 f}{\partial y^2}.$$

Solution:

$$x = r \cos \theta \Rightarrow \frac{\partial x}{\partial r} = \cos \theta$$

$$y = r \sin \theta \Rightarrow \frac{\partial y}{\partial r} = \sin \theta$$

Then

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} = \cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y}$$

and

$$\begin{aligned}
 \frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial r} \right) \\
 &= \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \right) \\
 &= \cos \theta \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial x} \right) + \sin \theta \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial y} \right) \\
 &= \cos \theta \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \frac{\partial y}{\partial r} \right] + \sin \theta \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial r} \right] \\
 &= \cos \theta \left(\frac{\partial^2 f}{\partial x^2} \cos \theta + \frac{\partial^2 f}{\partial y \partial x} \sin \theta \right) + \sin \theta \left(\frac{\partial^2 f}{\partial x \partial y} \cos \theta + \frac{\partial^2 f}{\partial y^2} \sin \theta \right) \\
 &= \cos^2 \theta \frac{\partial^2 f}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 f}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 f}{\partial y^2}
 \end{aligned}$$

□

Example 20

If $w = xy + \ln z$, where $x = \frac{v^2}{u}$, $y = u + v$ and $z = \tan^{-1} u$, find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ (in terms of u and v) using the Chain rule.

Solution

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \\&= y \cdot \left(-\frac{v^2}{u^2}\right) + x \cdot 1 + \left(\frac{1}{z}\right) \cdot \left(\frac{1}{1+u^2}\right) \\&= (u+v) \left(-\frac{v^2}{u^2}\right) + \frac{v^2}{u} + \left(\frac{1}{\tan^{-1} u}\right) \left(\frac{1}{1+u^2}\right) \\&= -\frac{v^3}{u^2} + \left(\frac{1}{\tan^{-1} u}\right) \left(\frac{1}{1+u^2}\right)\end{aligned}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} = y \cdot \left(\frac{2v}{u}\right) + x \cdot 1 = (u+v) \left(\frac{2v}{u}\right) + \frac{v^2}{u} = 2v + \frac{3v^2}{u}$$



Example 21

Show that in polar coordinates (r, θ) in $\mathbb{R}^2$, the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} = 0.$$

Solution

We know (from MA1200) that the relations between Cartesian coordinates (x, y) and polar coordinates (r, θ) are $x = r \cos \theta$ and $y = r \sin \theta$.

$$x = r \cos \theta \Rightarrow \frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$y = r \sin \theta \Rightarrow \frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta$$

Let $u = u(x, y)$, where $x = r \cos \theta$ and $y = r \sin \theta$. Then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x} (-r \sin \theta) + \frac{\partial u}{\partial y} (r \cos \theta).$$

For simplicity, we use the notations $\frac{\partial u}{\partial r} = u_r$, $\frac{\partial u}{\partial \theta} = u_\theta$, $\frac{\partial u}{\partial x} = u_x$, $\frac{\partial u}{\partial y} = u_y$, etc.

Then we write

$$u_r = u_x \cos \theta + u_y \sin \theta \quad \cdots (1),$$

$$u_\theta = u_x (-r \sin \theta) + u_y (r \cos \theta) \quad \cdots (1').$$

Method 1: (Express u_{xx} and u_{yy} in terms of r and θ . Then substitute these into $u_{xx} + u_{yy} = 0$.)

We first express u_x and u_y in terms of r , θ .

(1) $\times r \cos \theta - (1') \times \sin \theta$:

$$r \cos \theta u_r - \sin \theta u_\theta = r(\cos^2 \theta + \sin^2 \theta) u_x \Rightarrow u_x = \cos \theta u_r - \frac{\sin \theta}{r} u_\theta \quad \cdots (*).$$

(1) $\times r \sin \theta + (1') \times \cos \theta$:

$$r \sin \theta u_r + \cos \theta u_\theta = r(\sin^2 \theta + \cos^2 \theta) u_y \Rightarrow u_y = \sin \theta u_r + \frac{\cos \theta}{r} u_\theta \quad \cdots (**).$$

From (*), we observe that the u on the LHS is a function of x and y , however, the u on the RHS after x, y being replaced by $r \cos \theta$ and $r \sin \theta$, respectively, is a function of r, θ .

Note that u_x is a function of x, y .

If we replace u with u_x in (*), we have

$$\begin{aligned}
 u_{xx} &= \cos \theta \frac{\partial(u_x)}{\partial r} - \frac{\sin \theta}{r} \frac{\partial(u_x)}{\partial \theta} \\
 &= \cos \theta \underbrace{\frac{\partial}{\partial r} \left(\cos \theta u_r - \frac{\sin \theta}{r} u_\theta \right)}_{\text{by (*)}} - \frac{\sin \theta}{r} \underbrace{\frac{\partial}{\partial \theta} \left(\cos \theta u_r - \frac{\sin \theta}{r} u_\theta \right)}_{\text{by (*)}} \\
 &= \cos \theta \left(\cos \theta u_{rr} + \frac{\sin \theta}{r^2} u_\theta - \frac{\sin \theta}{r} u_{\theta r} \right) \\
 &\quad - \frac{1}{r} \sin \theta \left(-\sin \theta u_r + \cos \theta u_{r\theta} - \frac{\sin \theta}{r} u_{\theta\theta} - \frac{\cos \theta}{r} u_\theta \right) \\
 &= \cos^2 \theta u_{rr} + \frac{\sin^2 \theta}{r^2} u_{\theta\theta} - \frac{\cos \theta \sin \theta}{r} (u_{\theta r} + u_{r\theta}) + \frac{\sin^2 \theta}{r} u_r + \frac{2 \cos \theta \sin \theta}{r^2} u_\theta
 \end{aligned}$$

Similarly, if we replace u with u_y in (**), we have

$$\begin{aligned}
 u_{yy} &= \sin \theta \frac{\partial(u_y)}{\partial r} + \frac{\cos \theta}{r} \frac{\partial(u_y)}{\partial \theta} \\
 &= \sin \theta \underbrace{\frac{\partial}{\partial r} \left(\sin \theta u_r + \frac{\cos \theta}{r} u_\theta \right)}_{\text{by (**)}} + \frac{1}{r} \cos \theta \underbrace{\frac{\partial}{\partial \theta} \left(\sin \theta u_r + \frac{\cos \theta}{r} u_\theta \right)}_{\text{by (**)}} \\
 &= \sin \theta \left(\sin \theta u_{rr} - \frac{\cos \theta}{r^2} u_\theta + \frac{\cos \theta}{r} u_{\theta r} \right) \\
 &\quad + \frac{1}{r} \cos \theta \left(\cos \theta u_r + \sin \theta u_{r\theta} + \frac{\cos \theta}{r} u_{\theta\theta} - \frac{\sin \theta}{r} u_\theta \right) \\
 &= \sin^2 \theta u_{rr} + \frac{\cos^2 \theta}{r^2} u_{\theta\theta} + \frac{\cos \theta \sin \theta}{r} (u_{\theta r} + u_{r\theta}) + \frac{\cos^2 \theta}{r} u_r - \frac{2 \cos \theta \sin \theta}{r^2} u_\theta
 \end{aligned}$$

Putting these expressions of u_{xx} and u_{yy} into the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, we get

$$u_{xx} + u_{yy} = 0$$

$$\Rightarrow \left[\cos^2 \theta u_{rr} + \frac{\sin^2 \theta}{r^2} u_{\theta\theta} - \frac{\cos \theta \sin \theta}{r} (u_{\theta r} + u_{r\theta}) + \frac{\sin^2 \theta}{r} u_r + \frac{2 \cos \theta \sin \theta}{r^2} u_\theta \right]$$

$$+ \left[\sin^2 \theta u_{rr} + \frac{\cos^2 \theta}{r^2} u_{\theta\theta} + \frac{\cos \theta \sin \theta}{r} (u_{\theta r} + u_{r\theta}) + \frac{\cos^2 \theta}{r} u_r - \frac{2 \cos \theta \sin \theta}{r^2} u_\theta \right] = 0$$

$$\Rightarrow (\cos^2 \theta + \sin^2 \theta) u_{rr} + \frac{(\sin^2 \theta + \cos^2 \theta)}{r^2} u_{\theta\theta} + \frac{(\sin^2 \theta + \cos^2 \theta)}{r} u_r = 0$$

$$\Rightarrow u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = 0$$

$\therefore$ The result is proved as required.

Method 2: (Find the expressions for u_r , u_{rr} and $u_{\theta\theta}$. Then substitute these into $u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r = 0$.)

Partially differentiate both sides of $u_r = u_x \cos \theta + u_y \sin \theta \cdots (1)$ w.r.t. r :

$$\begin{aligned}
 u_{rr} &= \frac{\partial u_r}{\partial r} = \frac{\partial}{\partial r} (u_x \cos \theta + u_y \sin \theta) = \cos \theta \frac{\partial u_x}{\partial r} + \sin \theta \frac{\partial u_y}{\partial r} \\
 &= \cos \theta \left(\frac{\partial u_x}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u_x}{\partial y} \frac{\partial y}{\partial r} \right) + \sin \theta \left(\frac{\partial u_y}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u_y}{\partial y} \frac{\partial y}{\partial r} \right) \\
 &= \cos \theta (u_{xx} \cos \theta + u_{xy} \sin \theta) + \sin \theta (u_{yx} \cos \theta + u_{yy} \sin \theta) \\
 &= \cos^2 \theta u_{xx} + \cos \theta \sin \theta (u_{xy} + u_{yx}) + \sin^2 \theta u_{yy} \cdots (2)
 \end{aligned}$$

Partially differentiate both sides of $u_\theta = u_x (-r \sin \theta) + u_y (r \cos \theta) \cdots (1')$ w.r.t. θ :

$$\begin{aligned}
 u_{\theta\theta} &= \frac{\partial u_\theta}{\partial \theta} = \frac{\partial}{\partial \theta} [u_x (-r \sin \theta) + u_y (r \cos \theta)] \\
 &= -r \left(\sin \theta \frac{\partial u_x}{\partial \theta} + u_x \cos \theta \right) + r \left(\cos \theta \frac{\partial u_y}{\partial \theta} + u_y (-\sin \theta) \right) \\
 &= -r \left[\sin \theta \left(\frac{\partial u_x}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u_x}{\partial y} \frac{\partial y}{\partial \theta} \right) + u_x \cos \theta \right] + r \left[\cos \theta \left(\frac{\partial u_y}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u_y}{\partial y} \frac{\partial y}{\partial \theta} \right) - \sin \theta u_y \right]
 \end{aligned}$$

$$\begin{aligned}
&= -r \left[\sin \theta \left(\frac{\partial u_x}{\partial x} (-r \sin \theta) + \frac{\partial u_x}{\partial y} (r \cos \theta) \right) + u_x \cos \theta \right] \\
&\quad + r \left[\cos \theta \left(\frac{\partial u_y}{\partial x} (-r \sin \theta) + \frac{\partial u_y}{\partial y} (r \cos \theta) \right) - \sin \theta u_y \right] \\
&= r^2 (\sin^2 \theta u_{xx} - \sin \theta \cos \theta u_{xy}) - r \cos \theta u_x + r^2 (-\cos \theta \sin \theta u_{yx} + \cos^2 \theta u_{yy}) \\
&\quad - r \sin \theta u_y \\
&= r^2 [\sin^2 \theta u_{xx} - \sin \theta \cos \theta (u_{xy} + u_{yx}) + \cos^2 \theta u_{yy}] - r (\cos \theta u_x + \sin \theta u_y) \cdots (3)
\end{aligned}$$

Putting (1), (2), (3) into $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} = 0$, we have

$$\begin{aligned}
&[\cos^2 \theta u_{xx} + \cos \theta \sin \theta (u_{xy} + u_{yx}) + \sin^2 \theta u_{yy}] \\
&+ [\sin^2 \theta u_{xx} - \sin \theta \cos \theta (u_{xy} + u_{yx}) + \cos^2 \theta u_{yy}] - \frac{1}{r} (\cos \theta u_x + \sin \theta u_y) \\
&+ \frac{1}{r} (u_x \cos \theta + u_y \sin \theta) = 0 \\
\Rightarrow &(\cos^2 \theta + \sin^2 \theta) u_{xx} + (\sin^2 \theta + \cos^2 \theta) u_{yy} = 0 \\
\Rightarrow &u_{xx} + u_{yy} = 0
\end{aligned}$$

□

5 Implicit Functions

Let $f(x, y)$ be a function of two independent variables and suppose that $f_x(x, y)$ & $f_y(x, y)$ exist and are continuous at (x_0, y_0) and its neighborhood. In addition, suppose that $f(x_0, y_0) = 0$ and $f_y(x_0, y_0) \neq 0$. The **Implicit Function Theorem** says that in a neighborhood of (x_0, y_0) , the equation $f(x, y) = 0$ determines an implicit function $y = y(x)$ about x_0 with $y_0 = y(x_0)$ and $y = y(x)$ being differentiable at x_0 .

The above is also applicable to system of equations.

The system of m equations
$$\begin{cases} f_1(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \\ \vdots \\ f_m(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \end{cases}$$
 determine locally $y_1, \dots, y_m$ as implicit functions of $x_1, \dots, x_n$ and the partial derivatives of $y_1, \dots, y_m$ with respect to x_i , $1 \leq i \leq n$, can be found.

These will be demonstrated in the following examples.

Example 22

The equation $x^3 - xy^2 + yz^2 - z^3 = 5$ determine z as a function of x, y . Find $\frac{\partial z}{\partial x}$.

Solution

Partially differentiating both sides of $x^3 - xy^2 + yz^2 - z^3 = 5$ with respect to x (while treating y as constant) gives

$$\frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial x}(yz^2) - \frac{\partial}{\partial x}(z^3) = \frac{\partial}{\partial x}(5)$$

$$\Rightarrow 3x^2 - y^2 + y(2z)\frac{\partial z}{\partial x} - 3z^2\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow (2yz - 3z^2)\frac{\partial z}{\partial x} = y^2 - 3x^2$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{y^2 - 3x^2}{2yz - 3z^2} \quad \text{if } 2yz - 3z^2 \neq 0.$$

Example 23 (Implicit differentiation)

The equation

$$xz + y \ln z - z^2 + 4 = 0 \dots\dots (*)$$

defines z as an implicit function of x and y .

Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x, y, z) = (-3, 2, 1)$.

Solution

Partially differentiate both sides of $(*)$ w.r.t. x :

$$\frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial x}(y \ln z) - \frac{\partial}{\partial x}(z^2) + \frac{\partial}{\partial x}(4) = \frac{\partial}{\partial x}(0)$$

$$\Rightarrow \underbrace{\left[x \cdot \frac{\partial z}{\partial x} + z \cdot 1 \right]}_{\text{by product rule}} + y \cdot \frac{1}{z} \frac{\partial z}{\partial x} - 2z \frac{\partial z}{\partial x} + 0 = 0$$

$$\Rightarrow \left(x + \frac{y}{z} - 2z \right) \frac{\partial z}{\partial x} = -z$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-z}{x + \frac{y}{z} - 2z}$$

At $(x, y, z) = (-3, 2, 1)$,

$$\left. \frac{\partial z}{\partial x} \right|_{(x,y,z)=(-3,2,1)} = \left. \frac{-z}{x + \frac{y}{z} - 2z} \right|_{(x,y,z)=(-3,2,1)} = \frac{-1}{-3 + \frac{2}{1} - 2(1)} = \frac{1}{3}$$

Partially differentiate both sides of (*) w.r.t. y :

$$\frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial y}(y \ln z) - \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial y}(4) = \frac{\partial}{\partial y}(0)$$

$$\Rightarrow x \frac{\partial z}{\partial y} + \underbrace{\left[y \cdot \frac{1}{z} \frac{\partial z}{\partial y} + \ln z \cdot 1 \right]}_{\text{by product rule}} - 2z \frac{\partial z}{\partial y} + 0 = 0$$

$$\Rightarrow \left(x + \frac{y}{z} - 2z \right) \frac{\partial z}{\partial y} = -\ln z \Rightarrow \frac{\partial z}{\partial y} = \frac{-\ln z}{x + \frac{y}{z} - 2z}$$

At $(x, y, z) = (-3, 2, 1)$,

$$\left. \frac{\partial z}{\partial y} \right|_{(x,y,z)=(-3,2,1)} = \left. \frac{-\ln z}{x + \frac{y}{z} - 2z} \right|_{(x,y,z)=(-3,2,1)} = \frac{-\ln 1}{-3 + \frac{2}{1} - 2(1)} = \frac{0}{3} = 0$$

Example 24

Find the gradient of the tangent $\frac{dy}{dx}$ at a point (x, y) of the conic

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Solution

y is defined as an implicit function of x by the equation

$$z = f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

In this case

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{2ax + 2hy + 2g}{2hx + 2by + 2f}$$

whenever $2hx + 2by + 2f \neq 0$.

Example 25

If $z = e^x \cos y$, while x and y are implicit functions of t defined by the equations

$$\begin{cases} x^3 + e^x - t^2 - t = 1 \\ yt^2 + y^2t - t + y = 0 \end{cases}$$

then find $\frac{dz}{dt}$ for $t = 0$.

Solution

Differentiate $z = e^x \cos y$ with respect to t , we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = e^x \cos y \frac{dx}{dt} - e^x \sin y \frac{dy}{dt}.$$

The equation $x^3 + e^x - t^2 - t = 1$ defines a function $x = x(t)$ implicitly.

Differentiate both sides of $x^3 + e^x - t^2 - t = 1$ with respect to t , we have

$$3x^2 \frac{dx}{dt} + e^x \frac{dx}{dt} - 2t - 1 = 0 \Rightarrow \frac{dx}{dt} = \frac{2t + 1}{3x^2 + e^x}.$$

From $x^3 + e^x - t^2 - t = 1$, we observe that if $t = 0$, we have $x = 0$.

$$\therefore \left. \frac{dx}{dt} \right|_{t=0} = \left. \frac{2t+1}{3x^2+e^x} \right|_{\substack{t=0 \\ x=0}} = 1.$$

Similarly, the equation $yt^2 + y^2t - t + y = 0$ defines a function $y = y(t)$ implicitly.

Differentiate both sides of $yt^2 + y^2t - t + y = 0$ with respect to t , we have

$$t^2 \frac{dy}{dt} + 2yt + 2yt \frac{dy}{dt} + y^2 - 1 + \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{1 - y^2 - 2yt}{1 + t^2 + 2yt}.$$

From $yt^2 + y^2t - t + y = 0$, we observe that if $t = 0$, we have $y = 0$.

$$\therefore \left. \frac{dy}{dt} \right|_{t=0} = \left. \frac{1 - y^2 - 2yt}{1 + t^2 + 2yt} \right|_{\substack{t=0 \\ y=0}} = 1.$$

Hence,

$$\left. \frac{dz}{dt} \right|_{t=0} = \left(e^x \cos y \Big|_{\substack{x=0 \\ y=0}} \right) \left(\left. \frac{dx}{dt} \right|_{t=0} \right) - \left(e^x \sin y \Big|_{\substack{x=0 \\ y=0}} \right) \left(\left. \frac{dy}{dt} \right|_{t=0} \right) = 1 \times 1 - 0 \times 1 = 1.$$

□

Example 26

Suppose we solve $\begin{cases} x^2 + y^2 = \frac{1}{2}z^2 \\ x + y + z = 2 \end{cases}$ uniquely for x, y as functions of z near $x = 1,$

$y = -1$. Find $\frac{dx}{dz}$, $\frac{dy}{dz}$, $\frac{d^2x}{dz^2}$, $\frac{d^2y}{dz^2}$ when $x = 1, y = -1, z = 2$.

Solution

$$\begin{cases} x^2 + y^2 = \frac{1}{2}z^2 \\ x + y + z = 2 \end{cases}$$

Differentiate both sides of each equation w.r.t. z , we get

$$\begin{cases} 2x \frac{dx}{dz} + 2y \frac{dy}{dz} = z & \dots (1) \\ \frac{dx}{dz} + \frac{dy}{dz} + 1 = 0 & \dots (2) \end{cases}$$

Differentiate both sides w.r.t. z again, we get

$$\begin{cases} 2x \frac{d^2x}{dz^2} + 2 \left(\frac{dx}{dz} \right)^2 + 2y \frac{d^2y}{dz^2} + 2 \left(\frac{dy}{dz} \right)^2 = 1 & \dots (3) \\ \frac{d^2x}{dz^2} + \frac{d^2y}{dz^2} = 0 & \dots (4) \end{cases}$$

Put $x = 1$, $y = -1$, $z = 2$ into (1) and (2), we have

$$\begin{cases} 2 \frac{dx}{dz} - 2 \frac{dy}{dz} = 2 \\ \frac{dx}{dz} + \frac{dy}{dz} + 1 = 0 \end{cases} \Rightarrow \begin{cases} \frac{dx}{dz} - \frac{dy}{dz} = 1 \\ \frac{dx}{dz} + \frac{dy}{dz} = -1 \end{cases} \Rightarrow \begin{cases} \frac{dx}{dz} = 0 \\ \frac{dy}{dz} = -1 \end{cases}$$

Put $x = 1$, $y = -1$, $z = 2$, $\frac{dx}{dz} = 0$, $\frac{dy}{dz} = -1$ into (3) and (4), we have

$$\begin{cases} 2 \frac{d^2x}{dz^2} + 2(0)^2 - 2 \frac{d^2y}{dz^2} + 2(-1)^2 = 1 \\ \frac{d^2x}{dz^2} + \frac{d^2y}{dz^2} = 0 \end{cases} \Rightarrow \begin{cases} \frac{d^2x}{dz^2} - \frac{d^2y}{dz^2} = -\frac{1}{2} \\ \frac{d^2x}{dz^2} + \frac{d^2y}{dz^2} = 0 \end{cases} \Rightarrow \begin{cases} \frac{d^2x}{dz^2} = -\frac{1}{4} \\ \frac{d^2y}{dz^2} = \frac{1}{4} \end{cases}$$

$$\therefore \frac{dx}{dz} = 0, \quad \frac{dy}{dz} = -1, \quad \frac{d^2x}{dz^2} = -\frac{1}{4}, \quad \frac{d^2y}{dz^2} = \frac{1}{4} \quad \text{when } x = 1, \quad y = -1, \quad z = 2.$$

□

Example 27

Suppose that near the point $(x, y, u, v) = (1, 1, 1, 1)$ we can solve $\begin{cases} xu + yvu^2 = 2 \\ xu^3 + y^2v^4 = 2 \end{cases}$ uniquely for u and v as functions $u(x, y)$, $v(x, y)$ of x and y . Compute $\frac{\partial u}{\partial x}(1, 1)$.

Solution

Partially differentiate both equations $\begin{cases} xu + yvu^2 = 2 \\ xu^3 + y^2v^4 = 2 \end{cases}$ with respect to x , we have

$$\begin{cases} u + x \frac{\partial u}{\partial x} + yv(2u) \frac{\partial u}{\partial x} + yu^2 \frac{\partial v}{\partial x} = 0 & \dots (1) \\ u^3 + x(3u^2) \frac{\partial u}{\partial x} + y^2(4v^3) \frac{\partial v}{\partial x} = 0 & \dots (2) \end{cases}.$$

Put $x = 1$, $y = 1$, $u = 1$, $v = 1$ into (1) and (2), we have

$$\begin{cases} 1 + \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0 \\ 1 + 3 \frac{\partial u}{\partial x} + 4 \frac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} 3 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = -1 \\ 3 \frac{\partial u}{\partial x} + 4 \frac{\partial v}{\partial x} = -1 \end{cases} \Rightarrow \frac{\partial u}{\partial x} = -\frac{1}{3} \Rightarrow \frac{\partial u}{\partial x}(1, 1) = -\frac{1}{3}.$$

Definition

Let $y_1, \dots, y_n$ be functions such that each of its first order partial derivatives (with respect to $x_1, \dots, x_n$) exist.

The determinant

$$J \equiv \frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)} \equiv \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

is called the **Jacobian** of $y_1, \dots, y_n$ with respect to $x_1, \dots, x_n$.

Here, the $(i, j)^{\text{th}}$ entry of the matrix $\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$ is

$$\frac{\partial y_i}{\partial x_j}.$$

Example 28 (for reference only (p.81-89))

Functions $x = x(\rho, \phi, \theta)$, $y = y(\rho, \phi, \theta)$, $z = z(\rho, \phi, \theta)$ can be regarded as equations involving $x, y, z, \rho, \phi, \theta$. Assume $x = x(\rho, \phi, \theta)$, $y = y(\rho, \phi, \theta)$, $z = z(\rho, \phi, \theta)$ are differentiable functions of ρ, ϕ, θ and we may define ρ, ϕ, θ as implicit functions of x, y, z .

(a) Using Cramer's Rule, deduce that

$$\frac{\partial \rho}{\partial x} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\phi, \theta)}, \quad \frac{\partial \phi}{\partial x} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\theta, \rho)}, \quad \frac{\partial \theta}{\partial x} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\rho, \phi)},$$

where

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} \equiv \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}.$$

(b) Guess $\frac{\partial \rho}{\partial y}$ and $\frac{\partial \rho}{\partial z}$, then prove your guess for $\frac{\partial \rho}{\partial y}$.

(c) Let

$$\nabla \rho \equiv \frac{\partial \rho}{\partial x} \vec{i} + \frac{\partial \rho}{\partial y} \vec{j} + \frac{\partial \rho}{\partial z} \vec{k}$$

and

$$\frac{\partial \vec{r}}{\partial \phi} \equiv \frac{\partial x}{\partial \phi} \vec{i} + \frac{\partial y}{\partial \phi} \vec{j} + \frac{\partial z}{\partial \phi} \vec{k}, \quad \frac{\partial \vec{r}}{\partial \theta} \equiv \frac{\partial x}{\partial \theta} \vec{i} + \frac{\partial y}{\partial \theta} \vec{j} + \frac{\partial z}{\partial \theta} \vec{k},$$

show that

$$\nabla \rho = \frac{\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}}{J}.$$

(d) Let $x = x(\rho, \phi, \theta) = \rho \sin \phi \cos \theta$, $y = y(\rho, \phi, \theta) = \rho \sin \phi \sin \theta$ and $z = z(\rho, \phi, \theta) = \rho \cos \phi$.

Evaluate

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}, \quad \frac{\partial \rho}{\partial x}, \quad \frac{\partial \rho}{\partial y}, \quad \frac{\partial \rho}{\partial z}.$$

Solution

$$(a) \quad \begin{cases} x = x(\rho, \phi, \theta) \\ y = y(\rho, \phi, \theta) \\ z = z(\rho, \phi, \theta) \end{cases}$$

Partially differentiating both sides with respect to x , we have:

$$\begin{cases} \underbrace{\frac{\partial x}{\partial x}}_{=1} = \frac{\partial x}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial x}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial x}{\partial \theta} \frac{\partial \theta}{\partial x} \\ \underbrace{\frac{\partial y}{\partial x}}_{=0} = \frac{\partial y}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial y}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial \theta}{\partial x} \\ \underbrace{\frac{\partial z}{\partial x}}_{=0} = \frac{\partial z}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial z}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial x} \end{cases} .$$

(Note that $\frac{\partial y}{\partial x} = 0$ and $\frac{\partial z}{\partial x} = 0$ because x, y, z are independent variables.)

Then we use the Cramer's Rule to solve for $\frac{\partial \rho}{\partial x}$, $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \theta}{\partial x}$ (the unknowns):

$$\frac{\partial \rho}{\partial x} = \frac{\begin{vmatrix} 1 & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ 0 & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ 0 & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}} = \frac{\begin{vmatrix} \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}} = \frac{\frac{\partial(y, z)}{\partial(\phi, \theta)}}{\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}} = \frac{\frac{\partial(y, z)}{\partial(\phi, \theta)}}{J} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\phi, \theta)},$$

$$\frac{\partial \phi}{\partial x} = \frac{\begin{vmatrix} \frac{\partial x}{\partial \rho} & 1 & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & 0 & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & 0 & \frac{\partial z}{\partial \theta} \end{vmatrix}}{J} = \frac{-\begin{vmatrix} \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{J} = \frac{-\frac{\partial(y, z)}{\partial(\rho, \theta)}}{J} = \frac{\frac{\partial(y, z)}{\partial(\theta, \rho)}}{J} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\theta, \rho)},$$

$$\frac{\frac{\partial \theta}{\partial x}}{\frac{\partial x}{\partial \rho} \quad \frac{\partial x}{\partial \phi} \quad 1} = \frac{\frac{\partial y}{\partial \rho} \quad \frac{\partial y}{\partial \phi} \quad 0}{\frac{\partial z}{\partial \rho} \quad \frac{\partial z}{\partial \phi} \quad 0} = \frac{\frac{\partial y}{\partial \rho} \quad \frac{\partial y}{\partial \phi}}{\frac{\partial z}{\partial \rho} \quad \frac{\partial z}{\partial \phi}} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\rho, \phi)},$$

(b) Using (a), we guess that

$$\frac{\partial \rho}{\partial y} = \frac{1}{J} \frac{\partial(z, x)}{\partial(\phi, \theta)}, \quad \frac{\partial \rho}{\partial z} = \frac{1}{J} \frac{\partial(x, y)}{\partial(\phi, \theta)}.$$

To check that $\frac{\partial \rho}{\partial y} = \frac{1}{J} \frac{\partial(z, x)}{\partial(\phi, \theta)}$, we partially differentiate the equations $\begin{cases} x = x(\rho, \phi, \theta) \\ y = y(\rho, \phi, \theta) \\ z = z(\rho, \phi, \theta) \end{cases}$

with respect to y :

$$\begin{cases} 0 = \frac{\partial x}{\partial \rho} \frac{\partial \rho}{\partial y} + \frac{\partial x}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial x}{\partial \theta} \frac{\partial \theta}{\partial y} \\ 1 = \frac{\partial y}{\partial \rho} \frac{\partial \rho}{\partial y} + \frac{\partial y}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial y}{\partial \theta} \frac{\partial \theta}{\partial y} \\ 0 = \frac{\partial z}{\partial \rho} \frac{\partial \rho}{\partial y} + \frac{\partial z}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial y} \end{cases}$$

$$\xRightarrow{\text{Cramer's Rule}} \frac{\partial \rho}{\partial y} = \frac{\begin{vmatrix} 0 & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ 1 & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ 0 & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}} = \frac{- \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{J} = \frac{- \frac{\partial(x, z)}{\partial(\phi, \theta)}}{J} = \frac{1}{J} \frac{\partial(z, x)}{\partial(\phi, \theta)}$$

(c)

$$\begin{aligned}
\nabla \rho &\equiv \frac{\partial \rho}{\partial x} \vec{i} + \frac{\partial \rho}{\partial y} \vec{j} + \frac{\partial \rho}{\partial z} \vec{k} \underset{\text{by } \vec{\omega}_{(a),(b)}}{=} \frac{1}{J} \frac{\partial(y, z)}{\partial(\phi, \theta)} \vec{i} + \frac{1}{J} \frac{\partial(z, x)}{\partial(\phi, \theta)} \vec{j} + \frac{1}{J} \frac{\partial(x, y)}{\partial(\phi, \theta)} \vec{k} \\
&= \frac{\frac{\partial(y, z)}{\partial(\phi, \theta)} \vec{i} + \frac{\partial(z, x)}{\partial(\phi, \theta)} \vec{j} + \frac{\partial(x, y)}{\partial(\phi, \theta)} \vec{k}}{J} = \frac{\begin{vmatrix} \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} \vec{i} + \begin{vmatrix} \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \end{vmatrix} \vec{k}}{J} \\
&= \frac{\begin{vmatrix} \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} \vec{k}}{J} = \frac{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{J} \\
&= \frac{\left(\frac{\partial x}{\partial \phi} \vec{i} + \frac{\partial y}{\partial \phi} \vec{j} + \frac{\partial z}{\partial \phi} \vec{k} \right) \times \left(\frac{\partial x}{\partial \theta} \vec{i} + \frac{\partial y}{\partial \theta} \vec{j} + \frac{\partial z}{\partial \theta} \vec{k} \right)}{J} = \frac{\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}}{J}
\end{aligned}$$

(d) $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$. Then

$$\begin{aligned}
 J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix} \\
 &\stackrel{\substack{= \\ \text{expand} \\ \text{along} \\ \text{row 3}}}{=} \cos \phi \begin{vmatrix} \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} - (-\rho \sin \phi) \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} \\
 &= \cos \phi (\rho^2 \cos \phi \sin \phi \cos^2 \theta + \rho^2 \cos \phi \sin \phi \sin^2 \theta) \\
 &\quad + \rho \sin \phi (\rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta) \\
 &= \rho^2 \cos^2 \phi \sin \phi (\cos^2 \theta + \sin^2 \theta) + \rho^2 \sin^3 \phi (\cos^2 \theta + \sin^2 \theta) \\
 &= \rho^2 \cos^2 \phi \sin \phi + \rho^2 \sin^3 \phi \\
 &= \rho^2 \sin \phi
 \end{aligned}$$

$$\frac{\partial \rho}{\partial x} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\phi, \theta)} = \frac{\begin{vmatrix} \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{\rho^2 \sin \phi} = \frac{\begin{vmatrix} \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ -\rho \sin \phi & 0 \end{vmatrix}}{\rho^2 \sin \phi} = \frac{\rho^2 \sin^2 \phi \cos \theta}{\rho^2 \sin \phi} = \sin \phi \cos \theta$$

$$\frac{\partial \rho}{\partial y} = \frac{1}{J} \frac{\partial(z, x)}{\partial(\phi, \theta)} = \frac{\begin{vmatrix} \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \end{vmatrix}}{\rho^2 \sin \phi} = \frac{\begin{vmatrix} -\rho \sin \phi & 0 \\ \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \end{vmatrix}}{\rho^2 \sin \phi} = \frac{\rho^2 \sin^2 \phi \sin \theta}{\rho^2 \sin \phi} = \sin \phi \sin \theta$$

$$\begin{aligned} \frac{\partial \rho}{\partial z} &= \frac{1}{J} \frac{\partial(x, y)}{\partial(\phi, \theta)} = \frac{\begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \end{vmatrix}}{\rho^2 \sin \phi} = \frac{\begin{vmatrix} \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix}}{\rho^2 \sin \phi} \\ &= \frac{\rho^2 \cos \phi \sin \phi \cos^2 \theta + \rho^2 \cos \phi \sin \phi \sin^2 \theta}{\rho^2 \sin \phi} = \frac{\rho^2 \cos \phi \sin \phi}{\rho^2 \sin \phi} \\ &= \cos \phi \end{aligned}$$

□