

Practice Problem Set 4 Solutions

Question 1: The function f is given by

$$f(x, y) = xy \ln \left(\frac{x}{y} \right)$$

for $x, y > 0$. Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Verify that $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f(x, y)$.

We have

$$\begin{aligned} \frac{\partial f}{\partial x} &= y \ln \left(\frac{x}{y} \right) + xy \left(\frac{y}{x} \right) \left(\frac{1}{y} \right) = y \ln \left(\frac{x}{y} \right) + y \\ \frac{\partial f}{\partial y} &= x \ln \left(\frac{x}{y} \right) + xy \left(\frac{y}{x} \right) \left(-\frac{x}{y^2} \right) = x \ln \left(\frac{x}{y} \right) - x \end{aligned}$$

Then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = x \left(y \ln \left(\frac{x}{y} \right) + y \right) + y \left(x \ln \left(\frac{x}{y} \right) - x \right) = 2xy \ln \left(\frac{x}{y} \right) = 2f(x, y)$$

Question 2: Show that the function $f(x, y) = (x^2 + y^2)^{3/2} x^{1/2} y^{1/2}$ is homogeneous of degree 4, and verify that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 4f$$

To show that the function is homogeneous of degree 4, we observe that

$$\begin{aligned} f(cx, cy) &= \left((cx)^2 + (cy)^2 \right)^{3/2} (cx)^{1/2} (cy)^{1/2} = (c^2)^{3/2} (x^2 + y^2)^{3/2} c^{1/2} x^{1/2} c^{1/2} y^{1/2} \\ &= c^4 (x^2 + y^2)^{3/2} x^{1/2} y^{1/2} = c^4 f(x, y) \end{aligned}$$

Next,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \left(\frac{3}{2} \right) 2x (x^2 + y^2)^{1/2} (x^{1/2} y^{1/2}) + (x^2 + y^2)^{3/2} \left(\frac{1}{2} x^{-1/2} y^{1/2} \right) \\ \frac{\partial f}{\partial y} &= \left(\frac{3}{2} \right) 2y (x^2 + y^2)^{1/2} (x^{1/2} y^{1/2}) + (x^2 + y^2)^{3/2} \left(\frac{1}{2} x^{1/2} y^{-1/2} \right) \end{aligned}$$

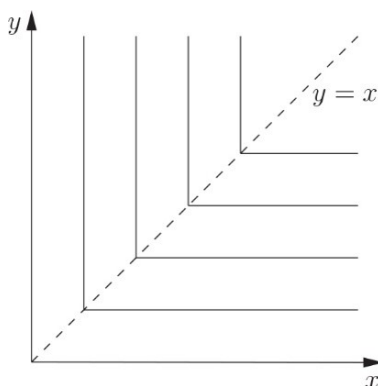
It follows that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = (3x^2 + 3y^2) (x^2 + y^2)^{1/2} (x^{1/2} y^{1/2}) + (x^{1/2} y^{1/2}) (x^2 + y^2)^{3/2}$$

$$= 4(x^2 + y^2)^{3/2}(x^{1/2}y^{1/2}) = 4f(x, y)$$

Question 3: The notation $\min(x, y)$ stands for ‘the smaller of the values of x and y ’. Sketch typical contours for the function $m(x, y) = \min(x, y)$ in $\mathbb{R}_+^2$.

A contour for m has equation $\min(x, y) = c$. Now, if $x < y$, $\min(x, y) = x$. This means in the region where $x < y$ (the part of $\mathbb{R}_+^2$ which lies above the line $x = y$), a contour is just a line $x = c$. Similarly, if $x > y$ (the region below the line $x = y$), a contour is a line $y = c$. The contours, therefore, have the form shown below:



Question 4: The function f is given, for $x, y \neq 0$ by

$$f(x, y) = \frac{4}{x} - \frac{2x}{y} + y$$

Show that f has one critical point and determine what type of critical point this is.

The first-order conditions are

$$f_1(x, y) = -\frac{4}{x^2} - \frac{2}{y} = 0 \qquad f_2(x, y) = \frac{2x}{y^2} + 1 = 0$$

$$\implies y = -(1/2)x^2 \quad \& \quad 2x = -y^2$$

$$\implies x = -2, y = -2$$

Therefore, f has one critical point $(-2, -2)$.

The second derivatives are

$$f_{11}(x, y) = 8/x^3, \quad f_{12}(x, y) = 2/y^2 \quad f_{22}(x, y) = -4x/y^3$$

At $(-2, -2)$, we have $f_{11} = -1$, $f_{12} = 1/2$, $f_{22} = -1$, and

$$D = f_{11}f_{22} - f_{12}^2 = (-1)(-1) - (1/2)^2 > 0 \quad f_{11} = -1 < 0$$

Therefore, the critical point is a maximum.

Question 5: A monopoly manufactures two commodities, X and Y , the markets for which interact, such that the demand functions are given by (x and y are quantities of X and Y , respectively, and p^X and p^Y are prices of X and Y , respectively)

$$x = 800(p^Y - p^X) \quad y = 400(9 + 2p^X - 4p^Y)$$

How much of each commodity should be manufactured to maximize the profit, given that it costs \$1 to produce one unit of X and \$1.50 to produce one unit of Y ?

Writing p^X and p^Y in terms of x and y , we get

$$p^X = \frac{3600 - 2x - y}{800} \quad p^Y = \frac{3600 - x - y}{800}$$

The cost function is $x + \frac{3}{2}y$ and so the profit function is

$$\begin{aligned} \Pi(x, y) &= xp^X + yp^Y - C(x, y) \\ &= \frac{1}{800} \left(3600x - 2x^2 - xy + 3600y - xy - y^2 \right) - x - \frac{3}{2}y \\ &= \frac{1}{800} \left(2800x - 2x^2 - 2xy + 2400y - y^2 \right) \end{aligned}$$

The first-order conditions for a critical point are

$$\Pi_1(x, y) = \frac{2800 - 4x - 2y}{800} = 0 \quad \Pi_2(x, y) = \frac{2400 - 2y - 2x}{800} = 0$$

Thus, we have

$$2x + y = 1400 \quad x + y = 1200$$

Solving these yields $x = 200$ and $y = 1000$. The second partial derivatives are

$$\Pi_{11}(x, y) = -1/200, \quad \Pi_{12}(x, y) = -1/400, \quad \Pi_{22}(x, y) = -1/400$$

So $\Pi_{11} < 0$ and $\Pi_{11}\Pi_{22} - \Pi_{12}^2 > 0$. Therefore, the critical point is a maximum.