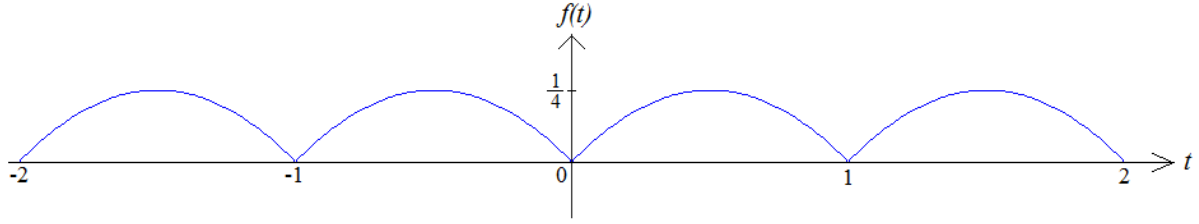


1. (a) $f(t) = \begin{cases} -t - t^2, & \text{if } -1 \leq t < 0 \\ t - t^2, & \text{if } 0 \leq t < 1 \end{cases}$ and $f(t+2) = f(t)$.



Observe that $f(t)$ is an even function, since its graph is symmetrical about the y -axis.

(b) Period $2T = 2 \Rightarrow T = 1$

The Fourier series of $f(t)$ is

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right) \right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi t) + b_n \sin(n\pi t)], \end{aligned}$$

where

$$a_0 = \frac{2}{T} \int_0^T f(t) dt \quad (\text{since } f(t) \text{ is an even function})$$

$$= \frac{2}{1} \int_0^1 (t - t^2) dt = 2 \left[\frac{t^2}{2} - \frac{t^3}{3} \right]_0^1 = 2 \left[\left(\frac{1}{2} - \frac{1}{3} \right) - 0 \right] = \frac{1}{3}$$

and for $n = 1, 2, 3, \dots$,

$$a_n = \frac{2}{T} \int_0^T f(t) \cos\left(\frac{n\pi t}{T}\right) dt \quad (\text{since } f(t) \text{ is an even function})$$

$$= \frac{2}{1} \int_0^1 (t - t^2) \cos(n\pi t) dt$$

$$= 2 \int_0^1 (t - t^2) d\left[\frac{\sin(n\pi t)}{n\pi}\right]$$

$$= 2 \left\{ \left[(t - t^2) \frac{\sin(n\pi t)}{n\pi} \right]_0^1 - \int_0^1 \frac{\sin(n\pi t)}{n\pi} d(t - t^2) \right\}$$

$$= 2 \left\{ (0 - 0) - \frac{1}{n\pi} \int_0^1 \sin(n\pi t) (1 - 2t) dt \right\}$$

$$= \frac{2}{n\pi} \int_0^1 (1 - 2t) d\left[\frac{\cos(n\pi t)}{n\pi}\right]$$

$$= \frac{2}{(n\pi)^2} \left\{ [(1 - 2t) \cos(n\pi t)]_0^1 - \int_0^1 \cos(n\pi t) d(1 - 2t) \right\}$$

$$= \frac{2}{n^2\pi^2} \left\{ (-\cos(n\pi) - 1) + 2 \int_0^1 \cos(n\pi t) dt \right\}$$

$$= \frac{-2}{n^2\pi^2} [1 + \cos(n\pi)] + \frac{4}{n^2\pi^2} \left[\frac{\sin(n\pi t)}{n\pi} \right]_0^1$$

$$= \frac{-2}{n^2\pi^2} [1 + (-1)^n] + \frac{4}{n^2\pi^2} (0 - 0)$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{-4}{n^2\pi^2} & \text{if } n \text{ is even} \end{cases}$$

$b_n = 0$ (since $f(t)$ is an even function).

Hence, the Fourier series of $f(t)$ is

$$f(t) = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{-4}{(2n)^2\pi^2} \cos[(2n)\pi t] = \frac{1}{6} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(2n\pi t)$$

(c) Choose $t \in [-1, 1]$ such that $\cos(2n\pi t) = (-1)^n$, i.e. take $t = \frac{1}{2}$.

Since $f(t)$ is continuous at $t = \frac{1}{2}$, the Fourier series converges to $f\left(\frac{1}{2}\right) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ at $t = \frac{1}{2}$.

Putting $t = \frac{1}{2}$ into the Fourier series:

$$\frac{1}{4} = \frac{1}{6} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi) = \frac{1}{6} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \pi^2 \left(\frac{1}{6} - \frac{1}{4} \right) = -\frac{\pi^2}{12}$$

2.
$$\begin{cases} \frac{\partial^2 u}{\partial y \partial x} = -2 \frac{\partial u}{\partial y} + 4e^x y^3 \\ \frac{\partial u}{\partial y}(0, y) = y^4 - 2 \\ u(x, 0) = e^{3x} \end{cases}$$

$$\frac{\partial^2 u}{\partial y \partial x} = -2 \frac{\partial u}{\partial y} + 4e^x y^3 \Rightarrow \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + 2u \right) = 4e^x y^3$$

Integrating both sides w.r.t. y : $\frac{\partial u}{\partial x} + 2u = e^x y^4 + F(x)$

Integrating factor: $e^{\int 2dx} = e^{2x}$

Then $\frac{\partial}{\partial x} (e^{2x} u) = e^{3x} y^4 + F(x) e^{2x}$

Integrating both sides w.r.t. x : $e^{2x} u = \frac{1}{3} e^{3x} y^4 + \int F(x) e^{2x} dx + G(y)$

$$\Rightarrow u(x, y) = \frac{1}{3} e^x y^4 + e^{-2x} \int F(x) e^{2x} dx + e^{-2x} G(y)$$

$$= \frac{1}{3} e^x y^4 + H(x) + e^{-2x} G(y)$$

where $H(x) = e^{-2x} \int F(x) e^{2x} dx$.

Then $\frac{\partial u}{\partial y} = \frac{4}{3}e^x y^3 + e^{-2x} G'(y)$

From the condition $\frac{\partial u}{\partial y}(0, y) = y^4 - 2$, we have

$$y^4 - 2 = \frac{\partial u}{\partial y}(0, y) = \frac{4}{3}y^3 + G'(y)$$

$$\Rightarrow G'(y) = y^4 - \frac{4}{3}y^3 - 2$$

$$\Rightarrow G(y) = \frac{y^5}{5} - \frac{y^4}{3} - 2y + C$$

$$\therefore u(x, y) = \frac{1}{3}e^x y^4 + H(x) + e^{-2x} \left(\frac{y^5}{5} - \frac{y^4}{3} - 2y + C \right).$$

From the condition $u(x, 0) = e^{3x}$,

$$e^{3x} = u(x, 0) = 0 + H(x) + e^{-2x}(0 - 0 - 0 + C) \Rightarrow H(x) = e^{3x} - C e^{-2x}$$

Hence, the solution is

$$\begin{aligned} u(x, y) &= \frac{1}{3}e^x y^4 + (e^{3x} - C e^{-2x}) + e^{-2x} \left(\frac{y^5}{5} - \frac{y^4}{3} - 2y + C \right) \\ &= \frac{1}{3}e^x y^4 + e^{3x} + e^{-2x} \left(\frac{y^5}{5} - \frac{y^4}{3} - 2y \right) \end{aligned}$$

$$3. \quad \begin{cases} \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + 2U = 0, & 0 < x < 4, \quad t > 0 \\ \frac{\partial U}{\partial x}(0, t) = \frac{\partial U}{\partial x}(4, t) = 0, & t > 0 \\ U(x, 0) = 4 - x, & 0 < x < 4 \end{cases}$$

Let $U(x, t) = X(x)T(t)$ be a non-trivial solution.

$$\text{Then } XT' - X''T + 2XT = 0 \Rightarrow X(T' + 2T) = X''T \Rightarrow \frac{X''}{X} = \frac{T' + 2T}{T}.$$

The equality holds only when $\frac{X''}{X} = \frac{T' + 2T}{T} = c \Rightarrow X'' - cX = 0$ and $T' + (2 - c)T = 0$.

Note that $\frac{\partial U}{\partial x} = X'T$.

From the boundary conditions $\frac{\partial U}{\partial x}(0, t) = \frac{\partial U}{\partial x}(4, t) = 0$,

$$0 = \frac{\partial U}{\partial x}(0, t) = X'(0)T(t)$$

$$0 = \frac{\partial U}{\partial x}(4, t) = X'(4)T(t)$$

Since $T(t)$ is a non-trivial function, we must have $X'(0) = X'(4) = 0$.

Then $X(x)$ satisfies $\begin{cases} X'' - cX = 0 \\ X'(0) = X'(4) = 0 \end{cases} \dots (1).$

Case 1: $c = 0$

Equation (1) becomes $X'' = 0 \Rightarrow X'(x) = c_1 \Rightarrow X(x) = c_1x + c_2$.

From the boundary conditions,

$$0 = X'(0) = X'(4) = c_1 \Rightarrow c_1 = 0$$

$\therefore X(x) = c_2$. Take $c_2 = 1$. Then $X(x) = 1$ is a non-trivial solution of (1).

Then $T' + (2 - c)T = 0$ becomes $T' + 2T = 0 \Rightarrow T(t) = c_3e^{-2t}$.

Take $c_3 = 1$. Then $T(t) = e^{-2t}$ is a non-trivial solution of $T' + 2T = 0$.

$\therefore U(x, t) = X(x)T(t) = 1 \cdot e^{-2t} = e^{-2t}$ is a non-trivial solution of

$$\begin{cases} \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + 2U = 0, & 0 < x < 4, t > 0 \\ \frac{\partial U}{\partial x}(0, t) = \frac{\partial U}{\partial x}(4, t) = 0, & t > 0 \end{cases} \dots (2)$$

Case 2: $c > 0$

$$X'' - cX = 0 \Rightarrow X(x) = c_4e^{\sqrt{c}x} + c_5e^{-\sqrt{c}x}.$$

$$\text{Then } X'(x) = c_4\sqrt{c}e^{\sqrt{c}x} - c_5\sqrt{c}e^{-\sqrt{c}x}.$$

From the boundary conditions,

$$X'(0) = 0 \Rightarrow 0 = c_4\sqrt{c} - c_5\sqrt{c} = \sqrt{c}(c_4 - c_5) \Rightarrow c_4 = c_5$$

$$X'(4) = 0 \Rightarrow 0 = c_4\sqrt{c}e^{4\sqrt{c}} - c_5\sqrt{c}e^{-4\sqrt{c}} = c_4\sqrt{c}(e^{4\sqrt{c}} - e^{-4\sqrt{c}})$$

$$\Rightarrow c_4 = 0 \because e^{4\sqrt{c}} - e^{-4\sqrt{c}} \neq 0 \text{ for all } c > 0.$$

$$\therefore c_5 = c_4 = 0 \Rightarrow X(x) = 0 \text{ (not accepted).}$$

$\therefore$ We cannot have $c > 0$.

Case 3: $c < 0$

$$X'' - cX = 0 \Rightarrow X(x) = c_6 \cos(\sqrt{-c}x) + c_7 \sin(\sqrt{-c}x).$$

$$\text{Then } X'(x) = -c_6\sqrt{-c} \sin(\sqrt{-c}x) + c_7\sqrt{-c} \cos(\sqrt{-c}x).$$

From the boundary conditions,

$$X'(0) = 0 \Rightarrow 0 = c_7\sqrt{-c} \Rightarrow c_7 = 0$$

$$X'(4) = 0 \Rightarrow 0 = -c_6\sqrt{-c} \sin(4\sqrt{-c}) \Rightarrow c_6 = 0 \text{ (rejected) or } \sin(4\sqrt{-c}) = 0$$

$$\therefore \sin(4\sqrt{-c}) = 0 \Rightarrow 4\sqrt{-c} = n\pi, \quad n = 1, 2, 3, \dots \Rightarrow c = -\frac{n^2\pi^2}{16}, \quad n = 1, 2, 3, \dots$$

$$\text{Then } X(x) = c_6 \cos\left(\frac{n\pi x}{4}\right), \quad n = 1, 2, 3, \dots$$

Take $c_6 = 1$, then $X_n(x) = \cos\left(\frac{n\pi x}{4}\right)$, $n = 1, 2, 3, \dots$ are non-trivial solutions of (1).

The corresponding ODE for T is $T' + \left(2 + \frac{n^2\pi^2}{16}\right)T = 0$, which has the general solution

$$T(t) = c_8 e^{-\left(2 + \frac{n^2\pi^2}{16}\right)t}, \quad n = 1, 2, 3, \dots$$

Take $c_8 = 1$, then $T_n(t) = e^{-\left(2 + \frac{n^2\pi^2}{16}\right)t}$, $n = 1, 2, 3, \dots$ are non-trivial solutions of

$$T' + \left(2 + \frac{n^2\pi^2}{16}\right)T = 0.$$

Then $U_n(x, t) = X_n(x)T_n(t) = e^{-\left(2 + \frac{n^2\pi^2}{16}\right)t} \cos\left(\frac{n\pi x}{4}\right)$, $n = 1, 2, 3, \dots$ are non-trivial solutions of (2).

By the principle of superposition,

$$U(x, t) = c_0 e^{-2t} + \sum_{n=1}^{\infty} c_n e^{-\left(2 + \frac{n^2\pi^2}{16}\right)t} \cos\left(\frac{n\pi x}{4}\right) \dots (3)$$

satisfies (2).

From the initial condition $U(x, 0) = 4 - x$, $0 < x < 4$, we have

$$4 - x = u(x, 0) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{4}\right) \quad \text{for } 0 < x < 4 \dots (4).$$

The Half-range Fourier cosine series for $4 - x$ defined on $(0, 4)$ is

$$4 - x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) \dots (5)$$

$$\text{where } a_0 = \frac{2}{4} \int_0^4 (4 - x) dx = \frac{1}{2} \left[4x - \frac{x^2}{2} \right]_0^4 = \frac{1}{2} \left[\left(16 - \frac{16}{2} \right) - 0 \right] = 4,$$

$$\begin{aligned} a_n &= \frac{2}{4} \int_0^4 (4 - x) \cos\left(\frac{n\pi x}{4}\right) dx = \frac{1}{2} \int_0^4 (4 - x) d\left[\frac{\sin\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}}\right] \\ &= \frac{2}{n\pi} \left\{ \left[(4 - x) \sin\left(\frac{n\pi x}{4}\right) \right]_0^4 - \int_0^4 \sin\left(\frac{n\pi x}{4}\right) (-1) dx \right\} = \frac{2}{n\pi} \left\{ (0 - 0) - \left[\frac{\cos\left(\frac{n\pi x}{4}\right)}{\frac{n\pi}{4}} \right]_0^4 \right\} \\ &= \frac{8}{n^2\pi^2} [1 - \cos(n\pi)] = \frac{8}{n^2\pi^2} [1 - (-1)^n] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{16}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Comparing (4) with (5), we have

$$c_0 = \frac{a_0}{2} = \frac{4}{2} = 2 \quad \& \quad c_n = a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{16}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}.$$

Hence, the solution of the given problem is

$$U(x, t) = 2e^{-2t} + \sum_{n=1}^{\infty} \frac{16}{(2n-1)^2\pi^2} e^{-\left[2 + \frac{(2n-1)^2\pi^2}{16}\right]t} \cos\left[\frac{(2n-1)\pi x}{4}\right].$$