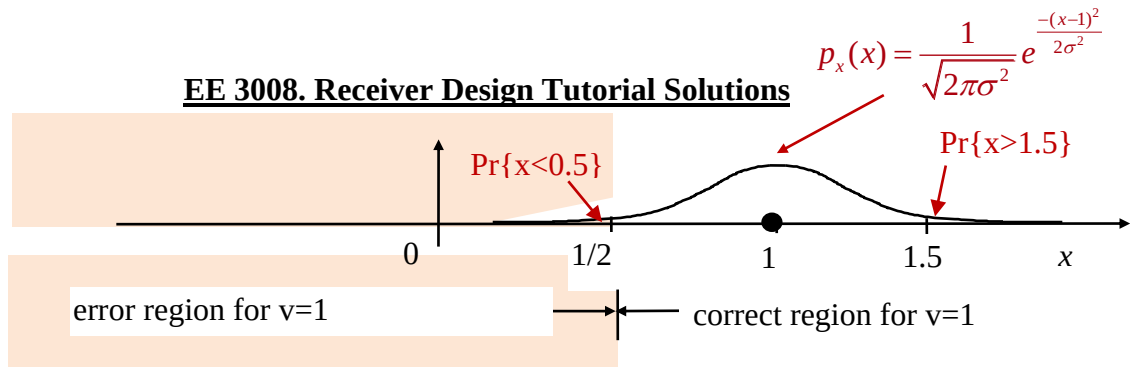


EE 3008. Receiver Design Tutorial Solutions

Q1)



$$\Pr\{1 \rightarrow 0\} = \int_{-\infty}^{0.5} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-1)^2}{2\sigma^2}} dx \stackrel{y=\frac{x-1}{\sigma}}{=} \int_{-\infty}^{-0.5/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \int_{0.5/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = Q(0.5/\sigma)$$

This can also be obtained directly using Q-function as follows.

$$\Pr\{1 \rightarrow 0\} = \Pr\{x < 0.5\} = \Pr\{x > 1.5\} = Q((x' - m)/\sigma)$$

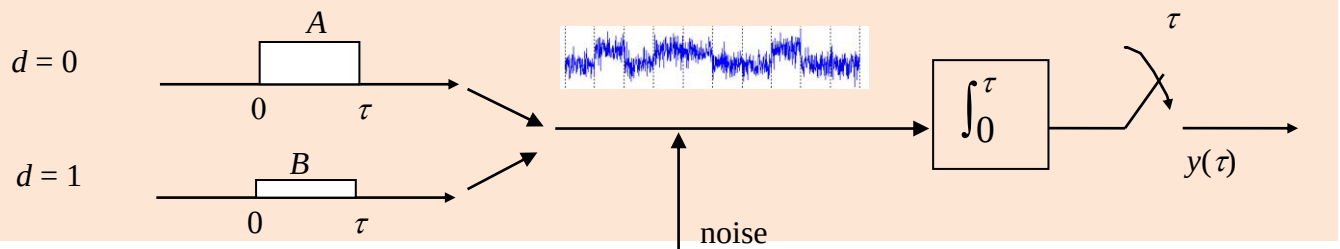
Now $x' = 1.5$ and $m = 1$, so $\Pr\{1 \rightarrow 0\} = Q((1.5 - 1)/\sigma) = Q(0.5/\sigma)$. Similarly we can obtain

$$\Pr\{0 \rightarrow 1\} = \Pr\{1 \rightarrow 0\} = Q(0.5/\sigma)$$

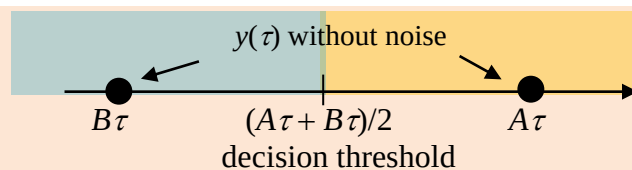
Q2) Refer to lecture notes: $E_b = (10^{-2})^2 10^{-4} = 10^{-8}$. Old $BER = Q(\sqrt{2E_b/N_0}) = Q(\sqrt{5}) \approx 0.013$

Average power = $E_b \times (\text{bit rate})$. After power adjustment, new $BER = Q(\sqrt{2.5}) \approx 0.057$.

Q3) System Model

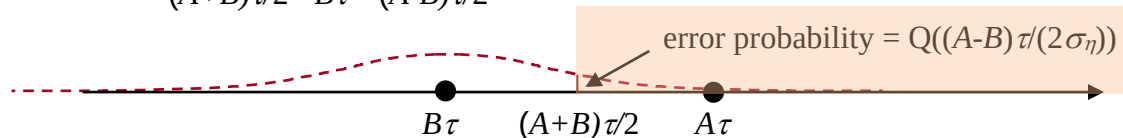


Decision mechanism.

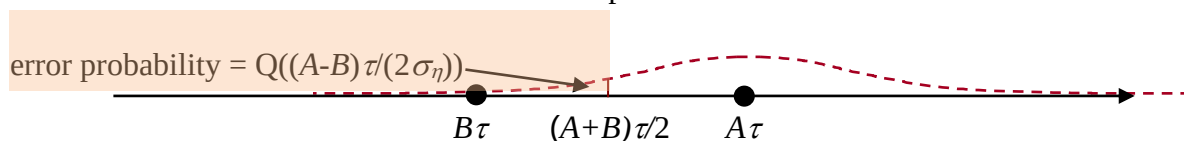


(a) This is the situation when the noise free correlator output is $B\tau$. Note that

$$(A+B)\tau/2 - B\tau = (A-B)\tau/2$$



This is the situation when the noise free correlator output is $A\tau$.



(b) We want to minimize the average power

$$P = (A^2 + B^2)/2. \quad (2)$$

Substituting the condition $A-B=C$ into (2), we have

$$P = (A^2 + (A-C)^2)/2$$

The minimum of P is achieved at $A = C/2$. This is because, when $A = C/2$,

$$\frac{dP}{dA} = \frac{d(A^2 + (A-C)^2)/2}{dA} = A + (A-C) = 0$$

Substituting $A = C/2$ into (2) and using the condition $A-B=C$, we have

$$B = -C/2.$$

Substituting $A = C/2$ and $B = -C/2$ into (2), we have

$$P = (A^2 + B^2)/2 = C^2/4.$$

Therefore $C = 2\sqrt{P}$. Thus $A = \sqrt{P}$ and $B = -\sqrt{P}$.

Thus bipolar is optimal for minimizing power.

(c) Average power $P = (A^2 + B^2)/2$. Thus $B^2 = 2P - A^2$. Thus

$$B = \pm\sqrt{2P - A^2}. \quad (3)$$

Denote $x = (A-B)\tau$. (4)

From the results in (a), to minimize BER, we need to maximize x . Substituting (3) into (4),

$$x = (A-B)\tau = \left(A \pm \sqrt{2P - A^2}\right)\tau$$

Taking derivative and setting it to zero, we have

$$\frac{dx}{dA} = \frac{d\left(A \pm \sqrt{2P - A^2}\right)\tau}{dA} = \left(1 \pm \frac{1}{2} \cdot \frac{(-2A)}{\sqrt{2P - A^2}}\right)\tau = \left(1 \mp \frac{A}{\sqrt{2P - A^2}}\right)\tau = 0$$

This means that $1 \mp \frac{A}{\sqrt{2P - A^2}} = 0 \Rightarrow \sqrt{2P - A^2} = \pm A \Rightarrow 2P - A^2 = A^2$.

Thus $A = \pm\sqrt{P}$ (5)

Substituting the above into (3), we have

$$B = \pm\sqrt{2P - A^2} = \pm\sqrt{P} \quad (6)$$

Combining (5) and (6) and recalling the condition $A > B$, the optimal solution is

$$A = \sqrt{P} \text{ and } B = -\sqrt{P}.$$

Thus bipolar is also optimal for minimizing BER.

From (b) and (c), $A = \sqrt{P}$ and $B = -\sqrt{P}$ is the optimal solution for both (i) minimizing average power for fixed BER and (ii) minimizing BER power for fixed average power.

Q4 (a) From (Q3), (noting that $\sigma_\eta^2 = \sigma^2 \tau$)

$$BER = Q\left(\frac{(A-B)\tau}{2\sigma_\eta}\right) = Q\left(\frac{(A-B)\tau}{2\sigma\sqrt{\tau}}\right) = Q\left(\sqrt{\frac{(A-B)^2 \tau}{4\sigma^2}}\right)$$

(b) When $A = -B$, the system reduces to bipolar. In this case, we have: $BER = Q\left(\sqrt{A^2 \tau / \sigma^2}\right)$

Note that $E_b = A^2 \tau$ for bipolar and $\sigma^2 = N_0/2$. Thus we have: $BER = Q\left(\sqrt{\frac{E_b}{N_0/2}}\right) = Q\left(\sqrt{2E_b / N_0}\right)$

This is consistent with lecture notes.

(c) Let $B = 0$, the system reduces to on-off. In this case, we have

$$BER = Q\left(\sqrt{\frac{A^2\tau}{4\sigma^2}}\right)$$

Note that $E_b = A^2\tau/2$ for on-off and $\sigma^2 = N_0/2$. Thus we have

$$BER = Q\left(\sqrt{E_b / N_0}\right)$$

This is consistent with lecture notes.

Note: Q3 and Q4 are similar. Q4 is on the similarity and difference between bipolar and on-off.

For on-off,
transmitting
zero does not
cost power.

Q5) $\tau = \frac{1}{2.5 \times 10^3} = 4 \times 10^{-4} \text{ s.}$

(a) Using the formula $BER = Q\left(\sqrt{2E_b / N_0}\right)$ and $Q(2)=0.023$, we have $\sqrt{2E_b / N_0} = 2$,

$$\Rightarrow 2E_b / N_0 = 4, \Rightarrow E_b = 2N_0 \Rightarrow A^2\tau = 2N_0 \Rightarrow A^2 = 2N_0 / \tau$$

Therefore $P = A^2 = 2N_0 / \tau = 0.5 \times 10^4 \times N_0 = 0.5 \times 10^{-3} \text{ W}$

(b) Using the formula $BER = Q\left(\sqrt{E_b / N_0}\right)$ and $Q(2)=0.023$, we have $\sqrt{E_b / N_0} = 2$,

$$\Rightarrow E_b / N_0 = 4, \Rightarrow E_b = 4N_0 \Rightarrow A^2\tau / 2 = 4N_0 \Rightarrow A^2 = 8N_0 / \tau$$

error
probability

Therefore $P = A^2 / 2 = 4N_0 / \tau = 10^4 \times N_0 = 10^{-3} \text{ W}$

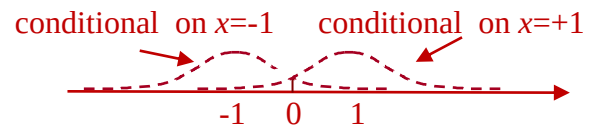
(b) We can see that on-off requires twice power compared with bipolar.

Q6) (a) Compared $f_X(w) = Ke^{-\frac{w^2}{4}}$ with a standard Gaussian distribution $\left(1/\sqrt{2\pi\sigma^2}\right)e^{-\frac{(w-m)^2}{2\sigma^2}}$, we can see that $m=0$ and $\sigma^2 = 2$ (i.e., $\sigma = \sqrt{2}$) here. Therefore $K = 1/\sqrt{2\pi\sigma^2}$.

(b) $\Pr(2.2 \leq w \leq 5) = 1 - \left(1 - Q\left(\frac{2.2-0}{\sqrt{2}}\right)\right) - Q\left(\frac{5-0}{\sqrt{2}}\right) = Q(1.1 \times \sqrt{2}) - Q(2.5 \times \sqrt{2})$.

(c) $p(y) = 0.5 \times p(y|x=+1) + 0.5 \times p(y|x=-1)$

$$= 0.5 \times \left(1/\sqrt{2\pi\sigma^2}\right) \left(e^{-\frac{(w-1)^2}{2\sigma^2}} + e^{-\frac{(w-(-1))^2}{2\sigma^2}}\right)$$



$$\Pr(y > 5) = 0.5 \times Q\left(\frac{5-1}{\sqrt{2}}\right) + 0.5 \times Q\left(\frac{5-(-1)}{\sqrt{2}}\right) = 0.5 \times Q\left(\frac{4}{\sqrt{2}}\right) + 0.5 \times Q\left(\frac{6}{\sqrt{2}}\right)$$

$$\Pr(y \leq -0.5) = 1 - \left(0.5 \times Q\left(\frac{-0.5-1}{\sqrt{2}}\right) + 0.5 \times Q\left(\frac{-0.5-(-1)}{\sqrt{2}}\right)\right) = 1 - \left(0.5 \times Q\left(\frac{-1.5}{\sqrt{2}}\right) + 0.5 \times Q\left(\frac{0.5}{\sqrt{2}}\right)\right)$$

(d) $BER = Q(1/\sqrt{2})$.

(e) When $h > 0$, the decoding rule is: $\hat{x} = +1$ if $y > 0$ and $\hat{x} = -1$ otherwise.

When $h < 0$, the decoding rule is: $\hat{x} = +1$ if $y < 0$ and $\hat{x} = -1$ otherwise.

(How about $h=0$?)

The signal; model $y = hx + w$ is equivalent to the model $y = Cb + \eta$ in Chapter 4.1 except the difference of variable names. Using the formula in Chapter 4.1, $BER = Q(|h|/\sqrt{2})$.