

Complex Dynamical Networks:

Lecture 2: Network Topologies

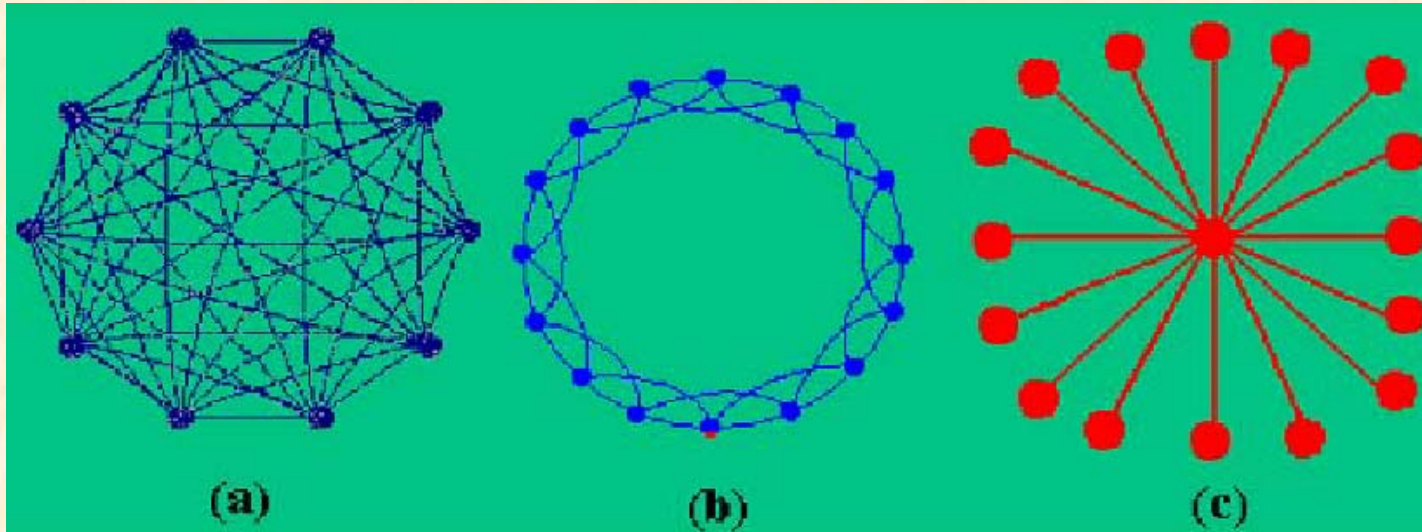
-- Basic Models and Properties

EE 6605

Instructor: G Ron Chen

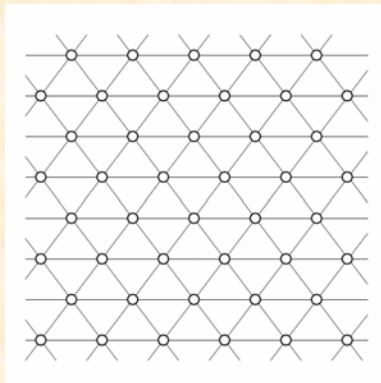
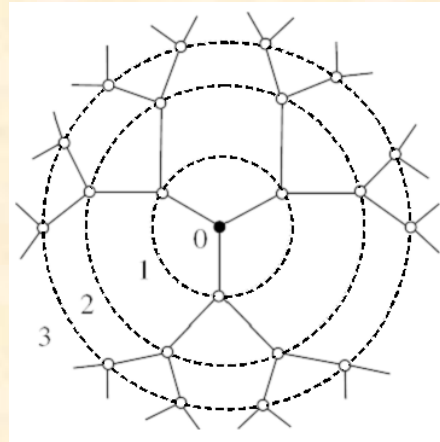
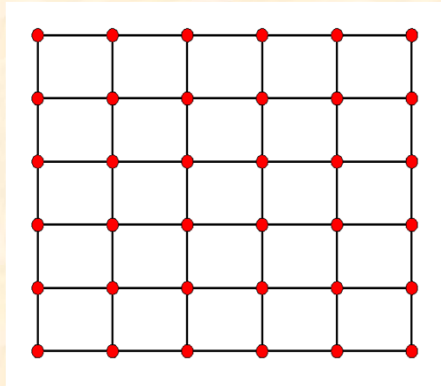


Regular Networks



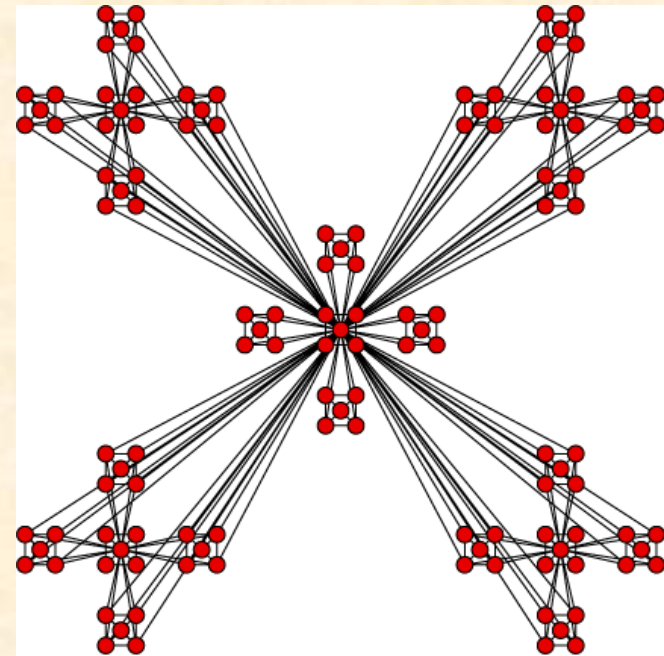
- (a) Fully-connected network
- (b) Ring-shaped coupled network
- (c) Star-shape coupled network

Regular Networks



Fractal Network

Lattice



Basic Properties of Regular Networks:

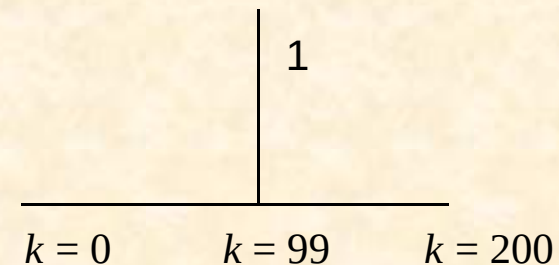
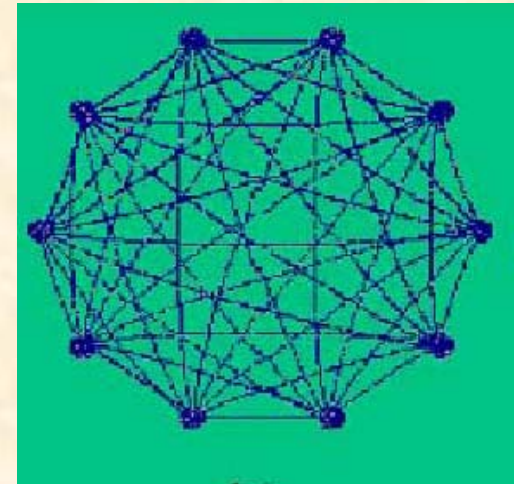
(a) Fully-connected networks:

- Average path length:

$$L_{full} = 1$$

- Network clustering coefficient:

- Degree distribution:
 $C_{full} = 1$
 δ



Example: $N = 100$

Basic Properties of Regular Networks:

(b) Ring-shaped coupled networks

■ Average path length:

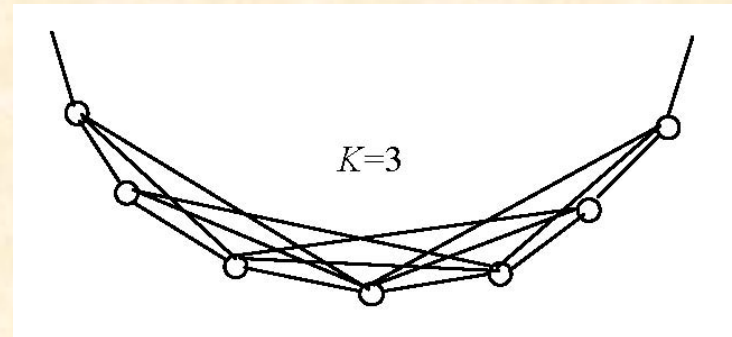
M – number of edges,

K – number of neighboring connections

$$L_{ring} \sim \frac{M(M+1) - 2(K-1)(M-K+1)}{2M} \rightarrow \infty \quad (M \rightarrow \infty)$$

■ Clustering coefficient:

$$C_{ring} = \frac{3(K-1)}{2(2K-1)} \rightarrow \frac{3}{4} \quad (K \rightarrow \infty)$$



Basic Properties of Regular Networks:

(c) Star-shaped coupled networks:

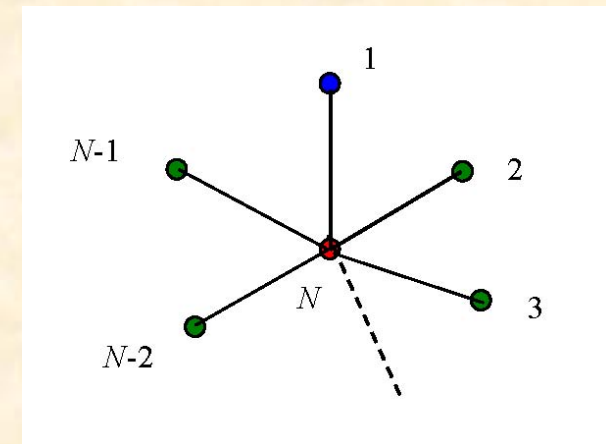
- **Average path length:**

$$L_{star} = 2 - \frac{2}{N} \rightarrow 2 \quad (N \rightarrow \infty)$$

Verifying:

$$\frac{(N-1) \times 1 + [(N-2) + (N-3) + \dots + 2 + 1] \times 2}{(N-1) + [(N-2) + (N-3) + \dots + 2 + 1]} = 2 - \frac{2}{N}$$

- **Clustering coefficient:** $C_{star} = 0$
- **Degree Distribution:** delta-like



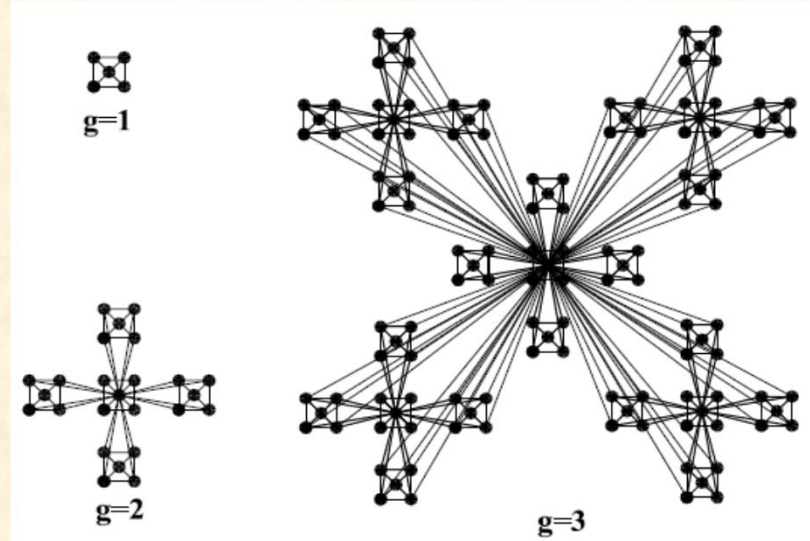
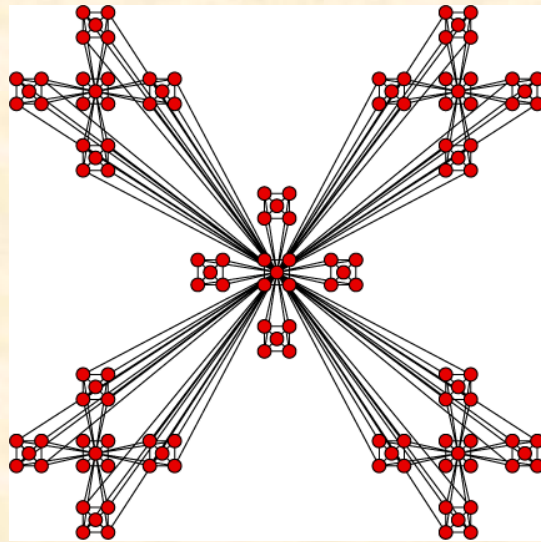
Example: $N = 3$ 

There are 3 different paths with lengths 1, 1, 2

So $L = (1+1+2) / 3 = 4 / 3$ or $L = 2 - [2 / (N=3)] = 4 / 3$

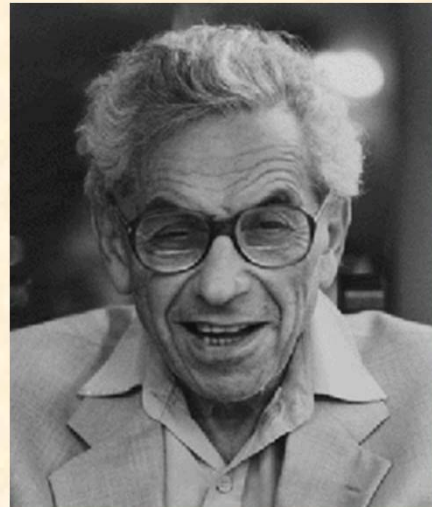
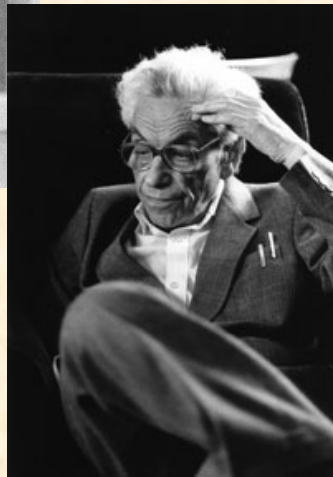
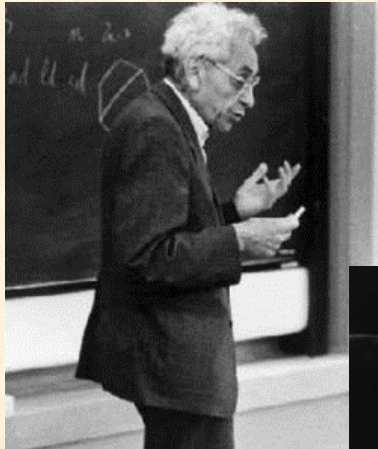
Fractal Networks

Example:



Step $g =$	1	2	3	N
Maximum degree $d_{\max}$	4	$4 + 4 \times 4 = 20$	$4 + 4 \times 4 + 4 \times 4 \times 4 = 84$	$4 + 4 \times 4 + \dots + 4 \times 4 \times 4 + \dots + 4^N = \frac{4}{3}(4^N - 1)$
Minimum degree $d_{\min}$	3	3	3	3
How many nodes with $d_{\min}$	4	4	$4 + 4 \times 4 = 20$	$4 + 4 \times 4 + \dots + 4 \times 4 \times 4 + \dots + 4^{N-1} = \frac{4}{3}(4^{N-1} - 1) \quad (N \geq 2)$

Random Graph Theory



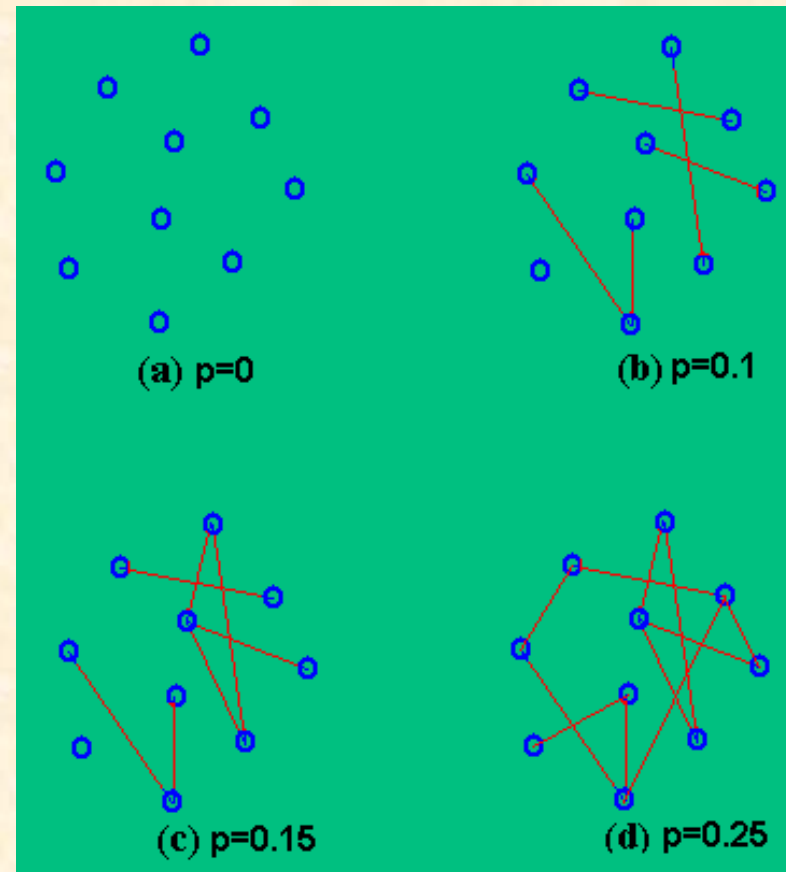
Paul Erdős
(1913-1996)



Alfred Rényi
(1921-1970)

Erdős-Rényi Random Graph Theory

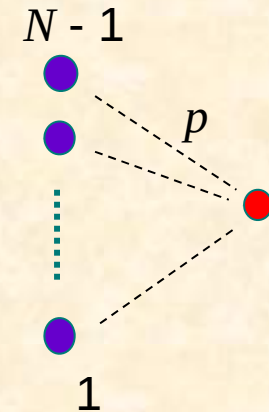
- ❖ Given N isolated nodes
- ❖ Add an edge between two nodes with probability p
- ❖ Statistically, $pN(N-1)/2$ edges can be expected to exist



Random-Graph Networks

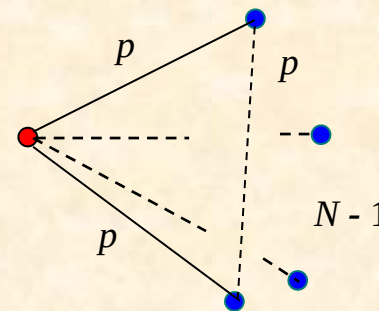
Theorem:

- **Average Degree:** $\langle k \rangle = p(N-1) \approx pN$
- **Clustering Coefficient:** $C = p = \langle k \rangle / N$
- **Degree Distribution:** $P(k) = \frac{\mu^k}{k!} e^{-\mu}$ (Poisson)
- **Average Path Length:** $L_{ER} \sim \ln N / \ln \langle k \rangle$



Verifying Clustering Coefficient:

$$C = \frac{p \cdot p^2 \cdot \binom{N-1}{2}}{p^2 \cdot \binom{N-1}{2}} = p = \langle k \rangle / N$$



Random-Graph Networks

Theorem:

- **Average Degree:** $\langle k \rangle = p(N-1) \approx pN$
- **Clustering Coefficient:** $C = p = \langle k \rangle / N$
- **Degree Distribution:** $P(k) = \frac{\mu^k}{k!} e^{-\mu}$ (Poisson)
- **Average Path Length:** $L \sim \ln N / \ln \langle k \rangle$

Verifying: For any node, it can connect to roughly $n_1 \sim \langle k \rangle$ nodes. Its next node can connect to roughly $n_2 \sim \langle k \rangle n_1 = \langle k \rangle^2$ nodes, and so on. Thus, because the average distance between each pair of nodes is L there are roughly $n \sim \langle k \rangle^L$ nodes in the network, yielding

$$L \sim \ln N / \ln \langle k \rangle$$

Poisson Distribution: $P(k) = \frac{\mu^k}{k!} e^{-\mu}$

- A node of degree k has k edges from the other $N-1$ nodes, each pair of which are connected with probability

$$P(k | N') = \binom{N'}{k} p^k (1-p)^{N'-k}$$

where $N' = N - 1$

- Let $\mu = pN'$ (referred to as **expectation value**) and fix it as constant.

- Thus, $P_{\mu}(k | N') = \frac{N'!}{k!(N'-k)!} \left(\frac{\mu}{N'}\right)^k \left(1 - \frac{\mu}{N'}\right)^{N'-k}$

$$P(k) = \lim_{N' \rightarrow \infty} P_{\mu}(k | N')$$

$$= \lim_{N' \rightarrow \infty} \frac{N'(N'-1)\cdots(N'-k+1)}{N'^k} \frac{\mu^k}{k!} \left(1 - \frac{\mu}{N'}\right)^{N'} \left(1 - \frac{\mu}{N'}\right)^{-k}$$

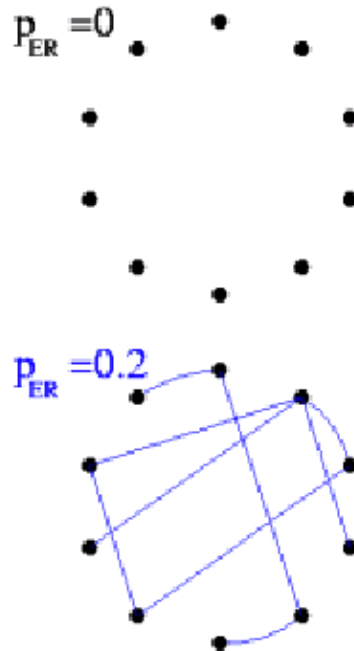
$$= 1 \cdot \frac{\mu^k}{k!} \cdot e^{-\mu} \cdot 1$$

ER Random Graph Models

Erdős-Rényi

(Publ. Math. Inst. Hung. Acad. Sci. 5, 17 (1960))

N nodes, each pair of node is connected with probability p



Features:

- ❖ **Connectivity**
node degree distribution - **Poisson**
- ❖ **Homogeneity**
all nodes have about the same number of edges
- ❖ **Non-growing**

Random Graph and Poisson Degree Distribution

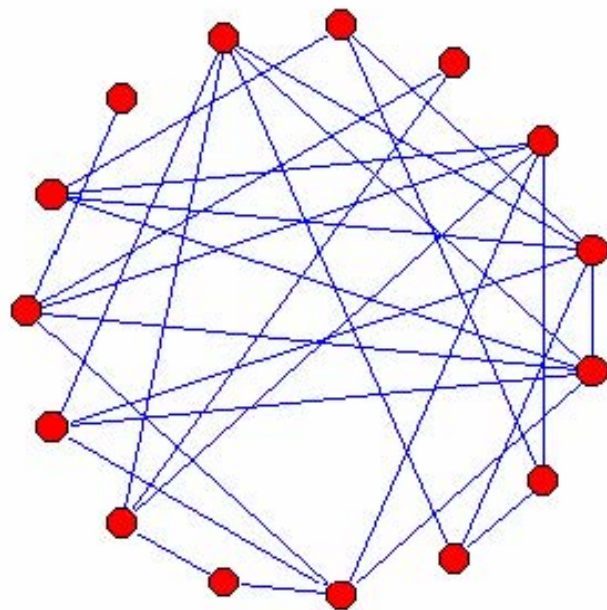
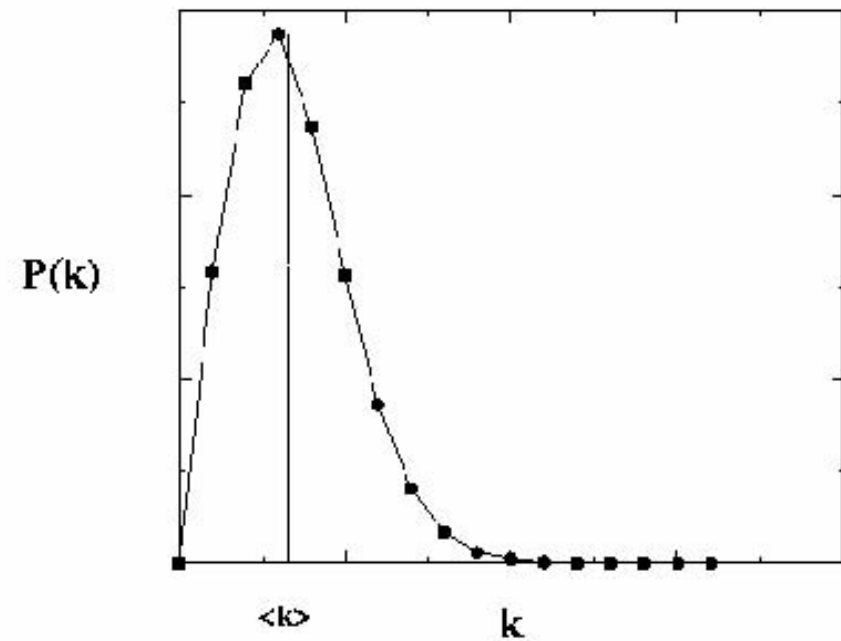


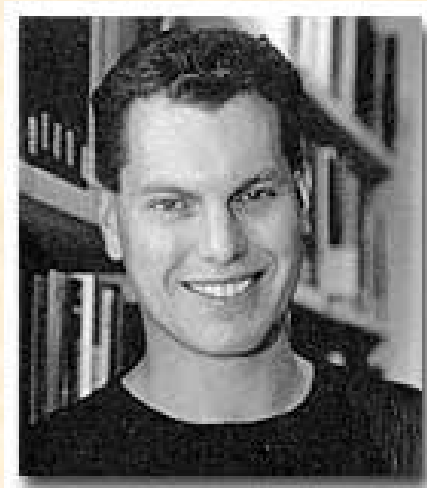
Illustration of Erdős-Rényi random-graph network model



Small-World Networks

“Collective dynamics of
'small-world' networks”

--- Nature, 393: 440-442, 1998



D. J. Watts



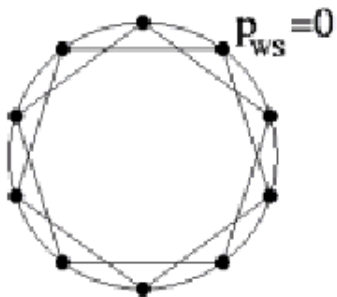
S. H. Strogatz

(Cornell University)

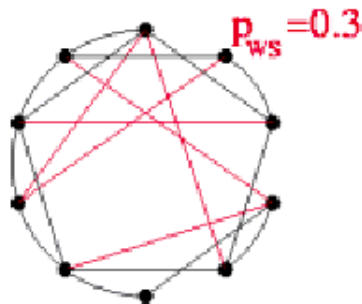
Small-World Networks

Watts-Strogatz

(Nature 393, 440 (1998))



N nodes forms a regular lattice. With probability p , each edge is rewired randomly



Features:

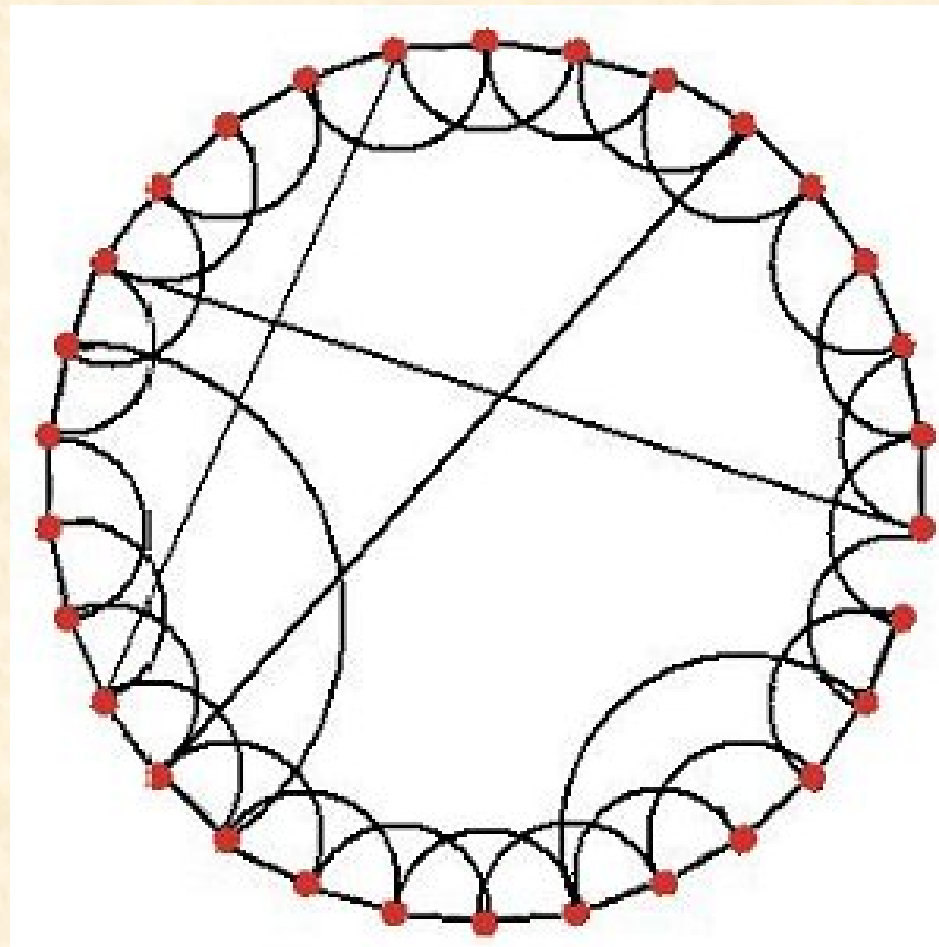
- **Connectivity** Poisson distribution
- **Homogeneous nature:** each node has roughly the same number of edges
- **Small average path length but large clustering coefficient**
- **Not growing**

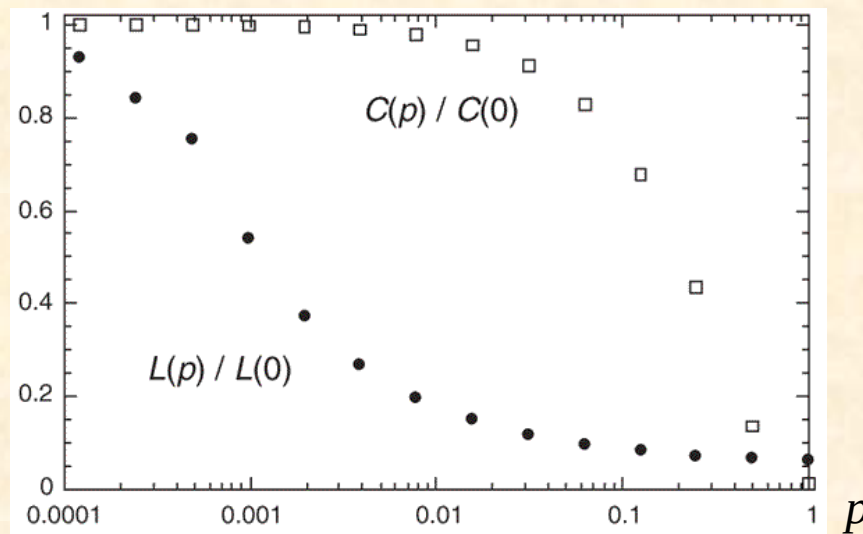
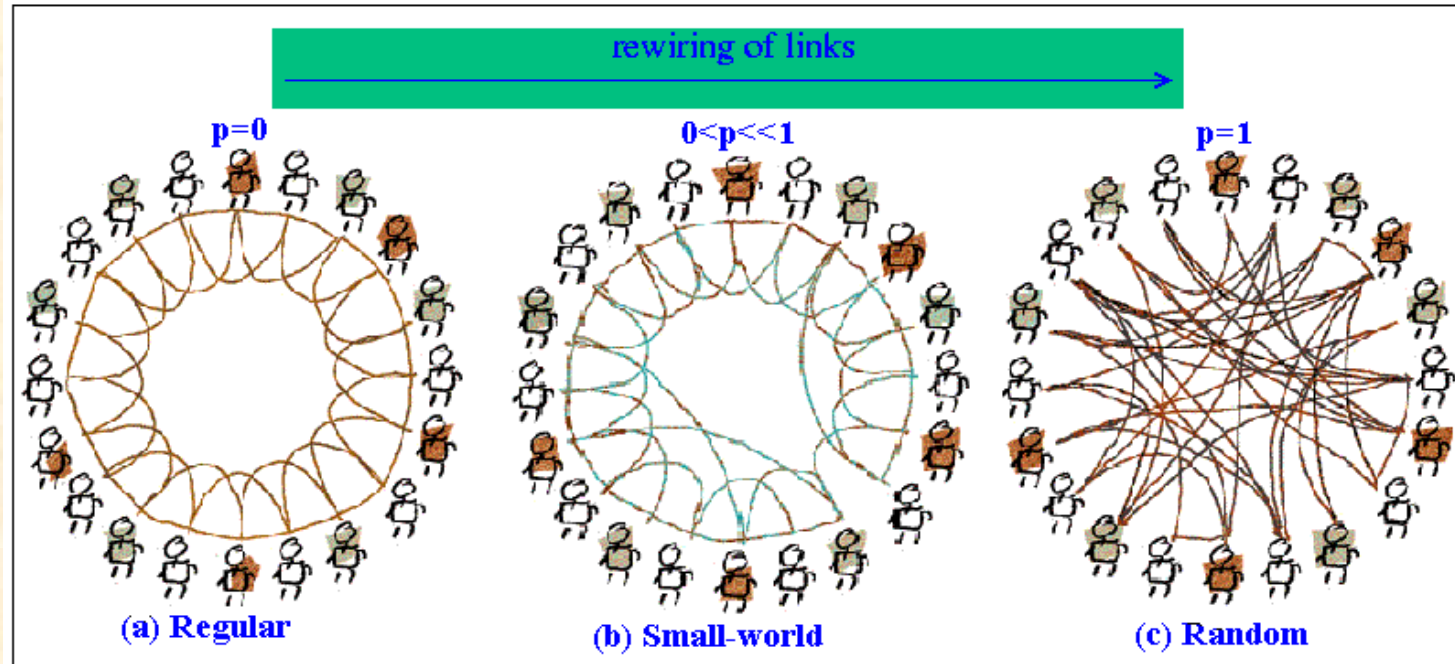
WS Small-World Network Model

Algorithm (rewiring):

- ❖ Start from a ring-shaped network with N nodes, in which each node is connected to its $2K$ neighbors, where K is a (usually small) positive integer.
- ❖ Go around the ring-shaped network clockwise (or counterclockwise). Pick a node and operate on its connections to the K neighbors one by one: an edge connecting this node to its neighbor will be kept unchanged; another end of the edge will be disconnected with probability p and then be re-connected to a randomly-picked node from the network. - ignore self-loops and multiple edges.

For example:





← Even a few long-distance edges are sufficient to drastically decrease L without significantly changing C

NW Small-World Network Model

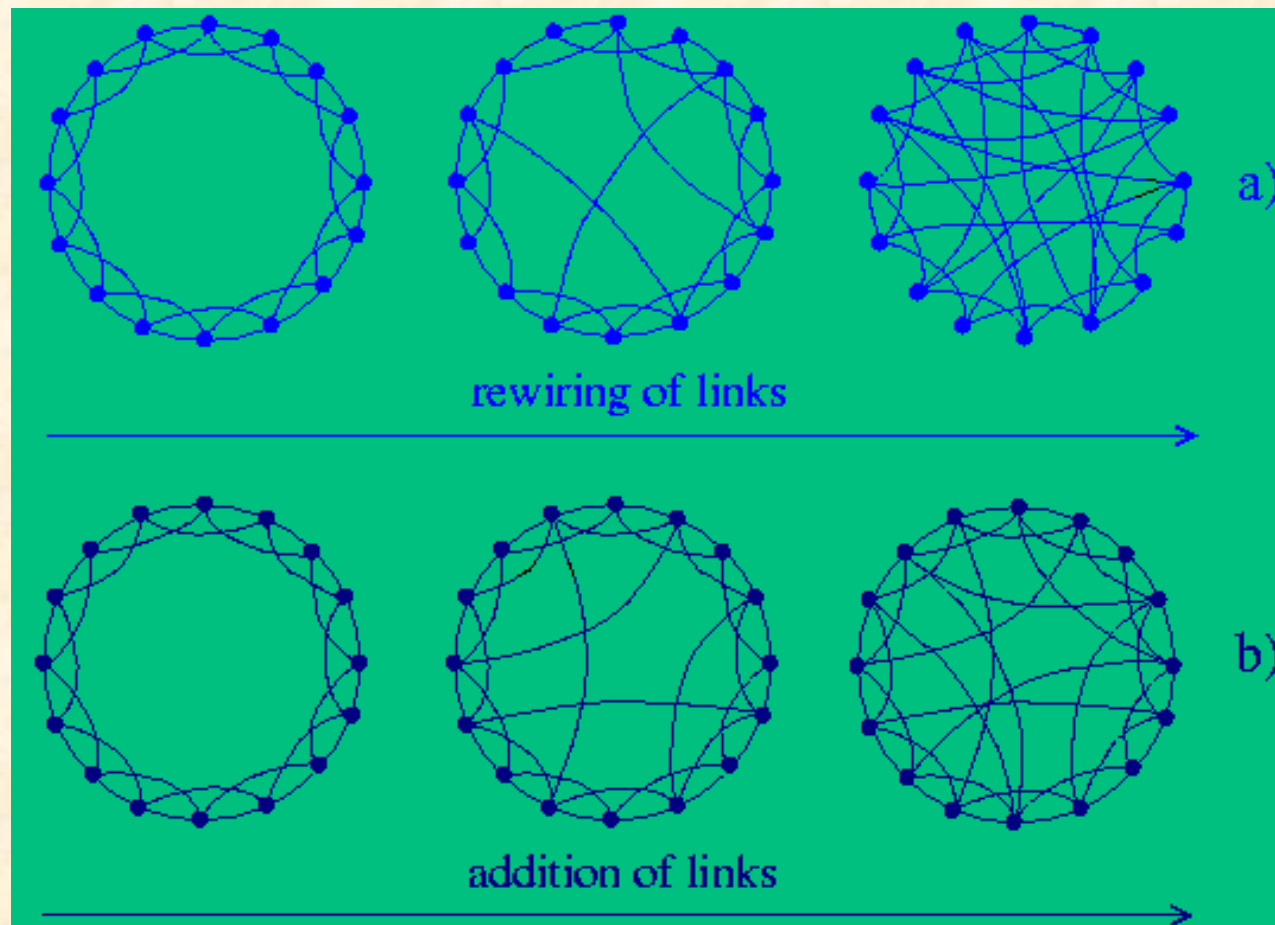
Algorithm (adding edges):

- ❖ Start from a ring-shaped network with N nodes, in which each node is connected to its $2K$ neighbors, where K is a (usually small) positive integer.
- ❖ With a (small) probability p , add an edge to each pair of nodes in the network.



Mark E J Newman

Newman-Watts Small-World Network Model



a) **WS** small-world network model b) **NW** small-world network model

Small-World Networks

Theorem: For large enough N ,

Clustering Coefficient of the WS small-world network model:

$$C(p) = \frac{3(K-2)}{4(K-1)}(1-p)^3$$

Clustering Coefficient of the NW small-world network model:

$$C(p) = \frac{3(K-2)}{4(K-1) + 4Kp(p+2)}$$

Small-World Networks

Theorem:

Average Path Length of **WS** small-world network model:

$$L(p) = \frac{2N}{K} f(NKp/2)$$
$$f(x) = \begin{cases} c & x \ll 1 \\ \ln x / x & x \gg 1 \end{cases} \quad (c \text{ — constant})$$

Average Path Length of **NW** small-world network model:

$$L(p) = \frac{2N}{K} f(NKp/2)$$
$$f(x) \approx \frac{1}{2\sqrt{x^2 + 2x}} \tanh^{-1} \sqrt{\frac{x}{x+2}}$$

Small-World Networks

Theorem:

Degree Distribution of WS small-world network model:

$$P(k) = \sum_{i=0}^{\min\{k-K/2, K/2\}} \binom{K/2}{i} (1-p)^i p^{(K/2)-i} \frac{(pK/2)^{k-i-K/2}}{(k-N-K/2)!} e^{-pK/2} \quad k \geq K/2$$

$$P(k) = 0 \quad k < K/2$$

Degree Distribution of NW small-world network model:

$$P(k) = \binom{N}{k-K} \left(\frac{Kp}{N} \right)^{k-K} \left(1 - \frac{Kp}{N} \right)^{N-k+K} \quad k \geq K$$

$$P(k) = 0 \quad k < K$$

Navigable Small-World Networks

International Congress of Mathematics (ICM)

22-28 August 2006, Madrid, Spain

Jon M Kleinberg (Cornell Univ.) received the
Nevanlinna Prize for Applied Mathematics

He gave a 45-minute talk -
“Complex Networks and
Decentralized Search Algorithms”

J M Kleinberg, “Navigation in a small world,”
Nature, 2000



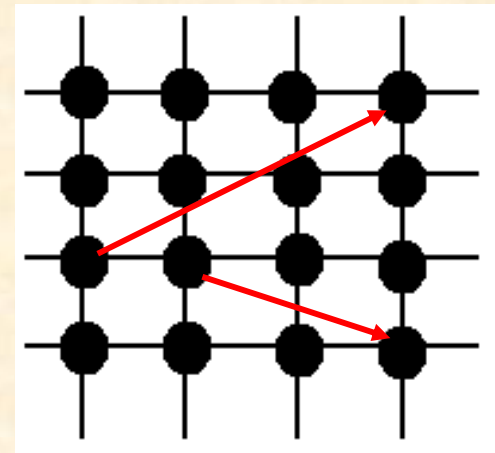
J M Kleinberg, The small-world phenomenon: An algorithmic perspective.
Proc. 32nd ACM Symposium on Theory of Computing, 2000: 163-170

Kleinberg's Navigable Networks

Problem -- To deliver a message from a source to a target on a lattice, in as few steps as possible

- (i) allowed to add m long-range edges anywhere in the lattice
- (ii) using only locally available information:
 - Location of the target on the lattice
 - Locations and long-range neighbors of all nodes that had previously processed the message

Suppose that every node does not know the long-range neighbors of the other nodes which had not previously processed the message (otherwise, the problem would be easy to solve)



Kleinberg's Navigable Networks

- Recall the NW small-world network model
- One may simply add m long-range edges to the given lattice to make it an NW small-world network
- But Kleinberg showed that a fast algorithm does not exist for NW model
- **Observations:**
 - In NW model, each node has an equal probability to receive new edge
 - Kleinberg argued that intuitively it would be easier for a new edge to connect to a nearby neighbor than to those far away (because usually everyone has some local information about its neighbors)
- So, he assumed that the connecting probability is reversely proportional to the distance between the two nodes

Kleinberg's Navigable Networks

Assumption: the probability of a new edge connecting a node u to any other node v in the given lattice:

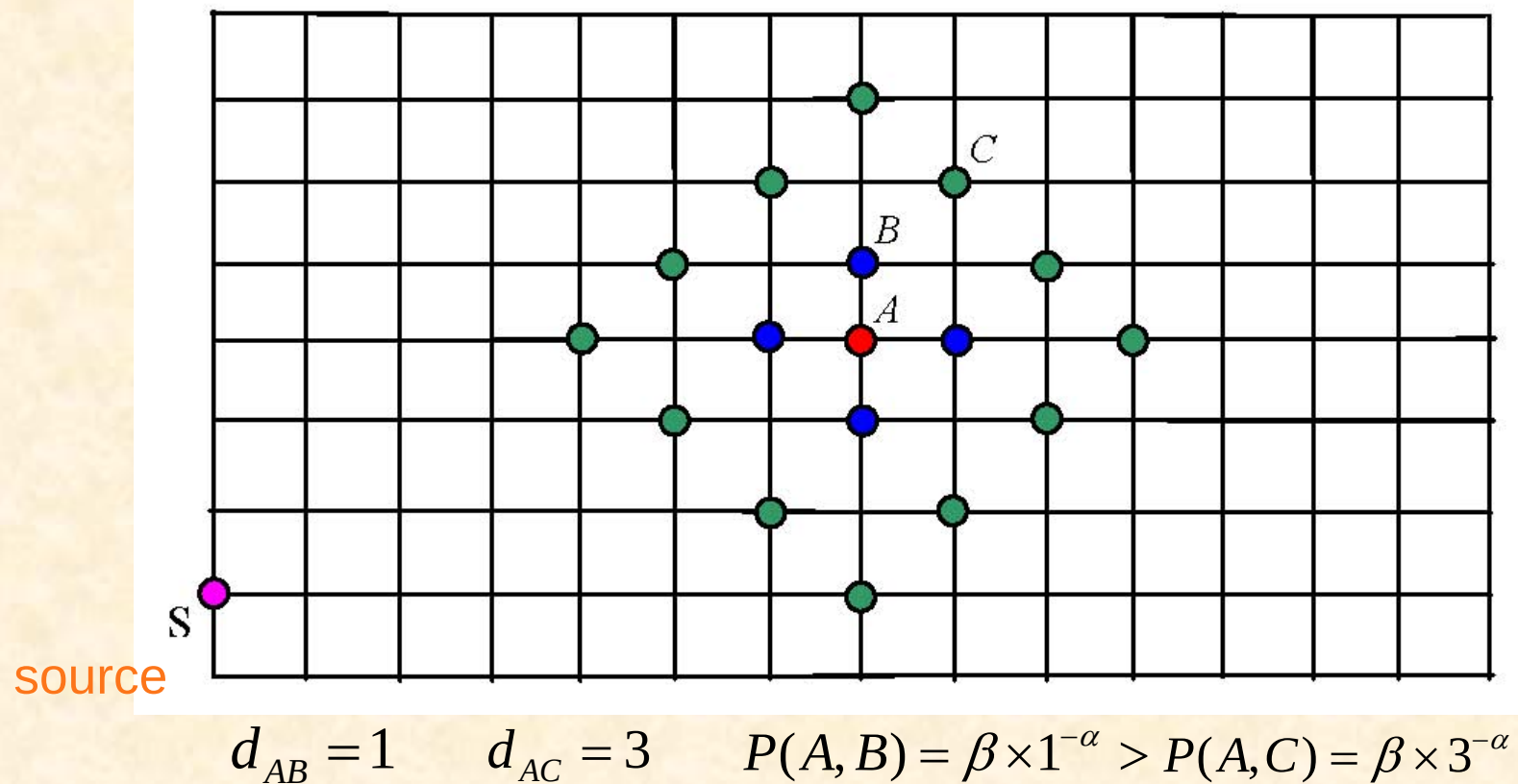
$$P(u, v) = \beta d_{uv}^{-\alpha}$$

where d_{uv} is the distance between u and v with parameters $\alpha \geq 0, \beta > 0$ satisfying $\beta \sum_{(u,v)} d_{u,v}^{-\alpha} = 1$ as a probability measure.

When $\alpha = 0$, the new edge-addition probability is a constant, so every node has the same probability to receive the new edge; in this case, the model reduces to the original NW small-world network model.

Kleinberg's Navigable Networks

Graphical illustration of the new edge-distribution mechanism:



Kleinberg's Navigable Networks

- ❑ Noticing that other algorithms may lead the message to go through some detours in approaching the target
- ❑ Kleinberg's idea was to find a way that guarantees the message to move at least one step closer to the target each time
- ❑ He applied a greedy algorithm, where at every step the current message holder will pass the message to a neighbor who is as close to the target as possible

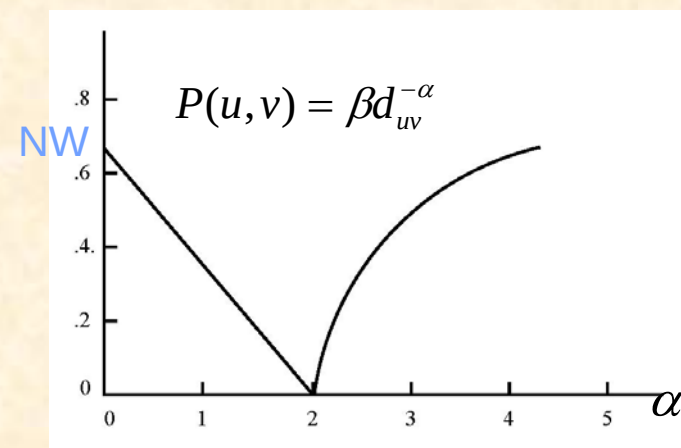
Kleinberg's Navigable Networks

When $\alpha = 0$ the edge-addition probability is a constant, so the model is the original NW small-world network model

As α increases, a decentralized algorithm can take more advantage of the geographic structure with long-range connections (everyone has some long-range information)

As α further increases, however, the number of long-range edges become very small therefore less useful

So, there is a best value of α in between, which is found to be $\alpha = 2$



Kleinberg's Navigable Networks

A network is *navigable* if it requires at most $\text{poly}(\log N)$ time-steps to deliver a message to any target from any source on the network.

Kleinberg's Theorem.

For $\alpha = 2$, the lattice is navigable in $O(\log N)^2$ time-steps at a rate depending on m , but for all $\alpha < 0$ it is not navigable.

For $0 \leq \alpha < 2$, it takes at least $O(N^{(2-\alpha)/3})$ time-steps to deliver a message from any source to any target.

For $\alpha > 2$, it will take at least $O(N^{(\alpha-2)/(\alpha-1)})$ time-steps.

For the Kleinberg problem formulated in the original NW ring-shaped network setting, see:

M E J Newman: *Networks: An Introduction*. Oxford, UK, 2010: 713-718

BREAK

10 minutes

Another Breakthrough: Scale Free
“Emergence of scaling in random networks”
Science 286: 509 (1999)



A.-L. Barabási



R. Albert

(Norte Dame University)

BA Scale-Free Network Model

Start with a **connected** network having $m_0 \geq 1$ nodes

(i) Add new nodes:

Add 1 **new node** into the network:

This node is connected to m ($m \leq m_0$) existing nodes simultaneously

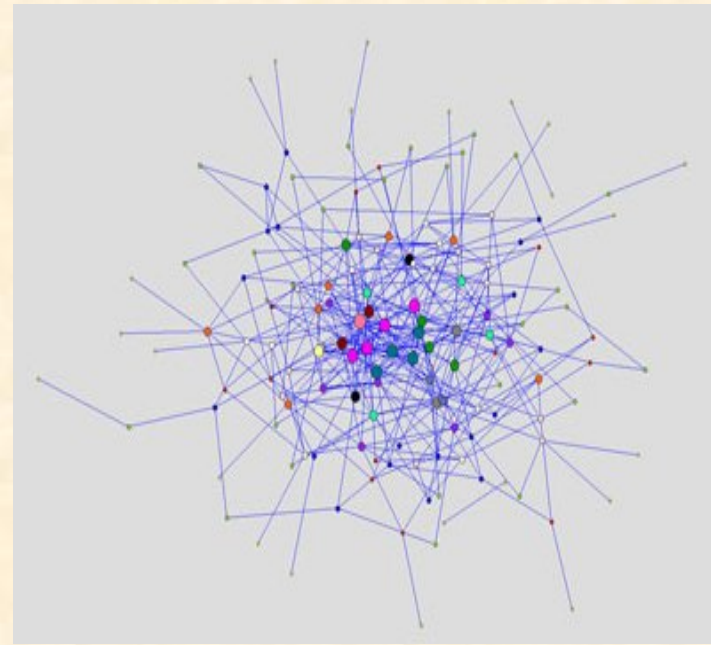
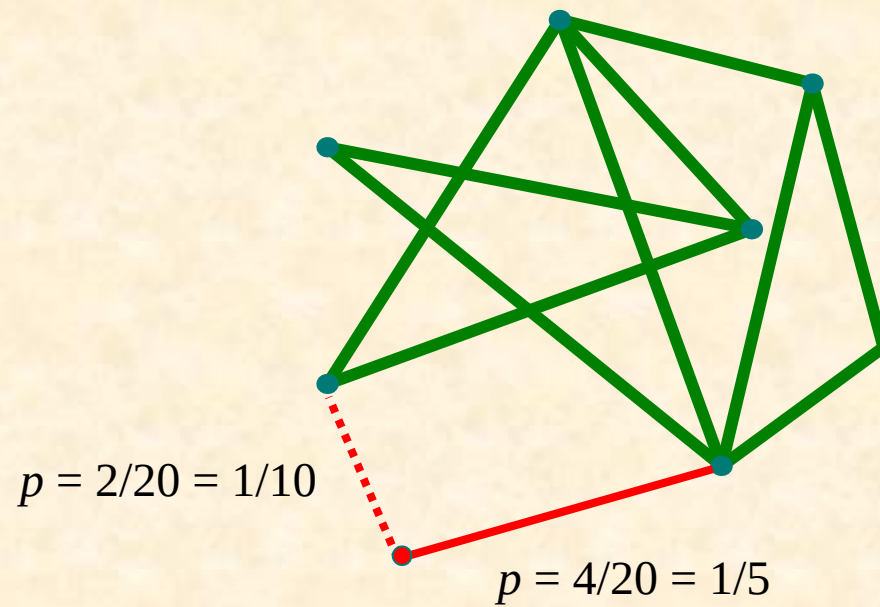
(ii) Add new edges:

The way to add the m **new edges** into the network:

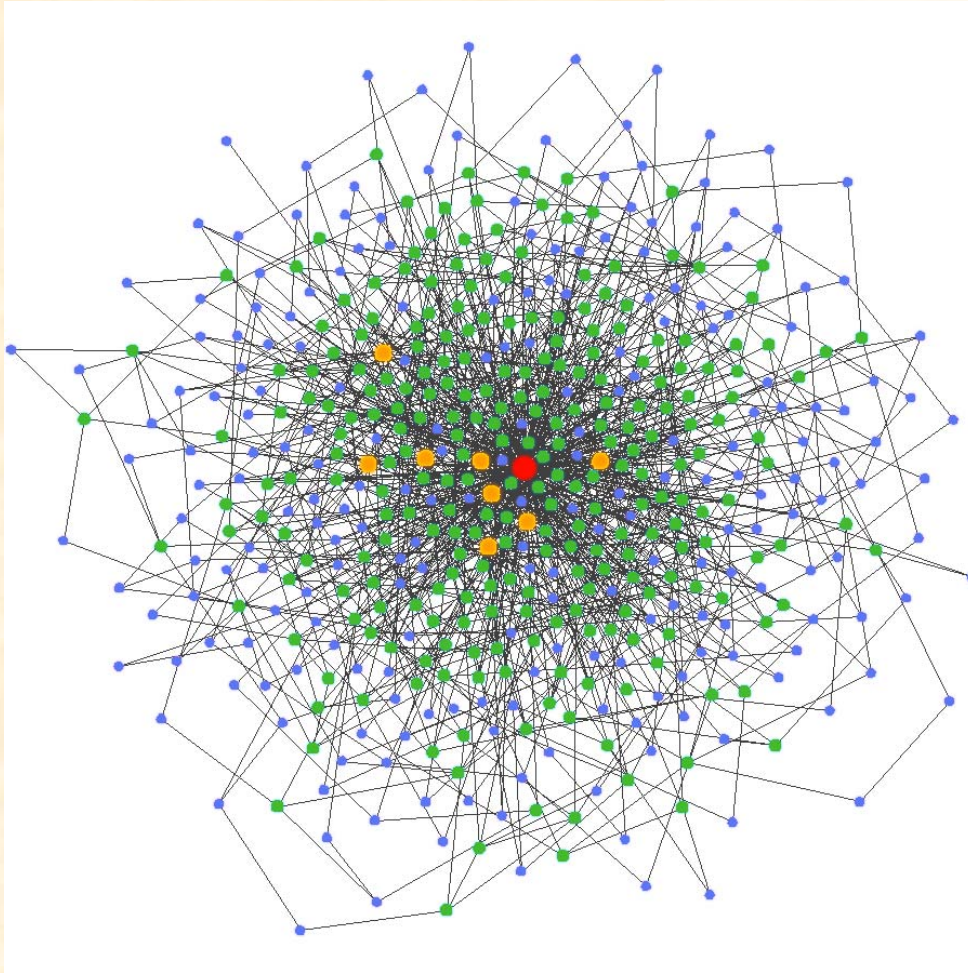
Every existing node is to be chosen with probability

$$\Pi(k_i) = \frac{k_i}{\sum_l k_l}$$

Scale-Free Network Modeling



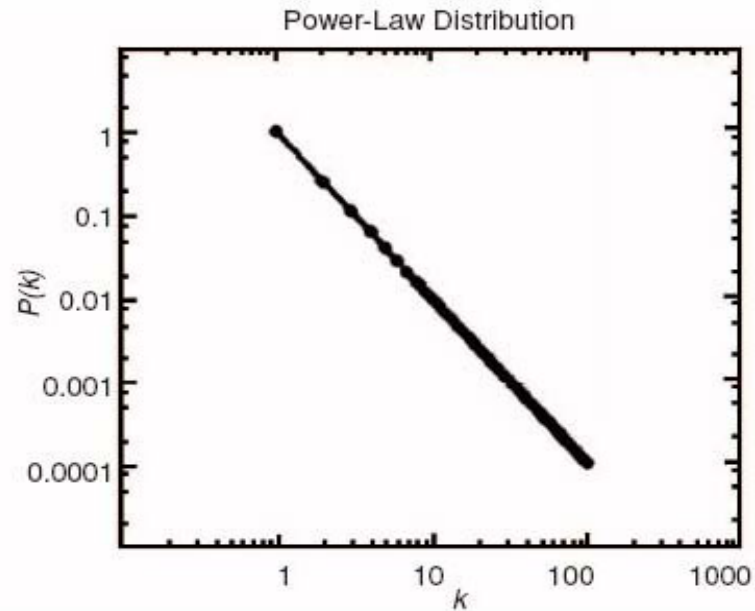
Scale-Free Networks



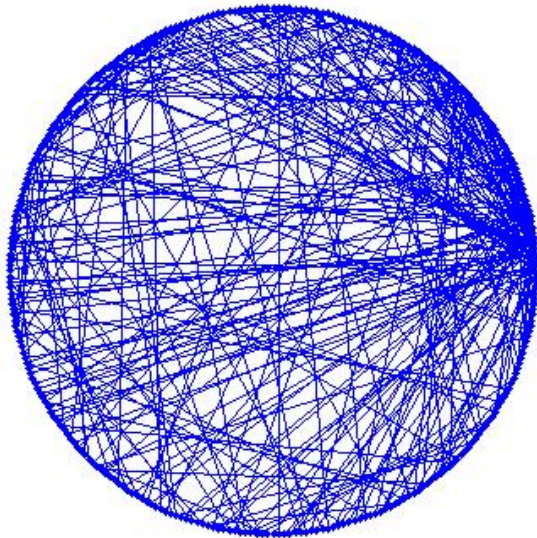
Features:

- ❖ **Connectivity:**
in power-law form
- ❖ **Non-homogeneous nature:**
very few nodes have many links but most nodes have very few links
- ❖ **Growing**

(Hawoong Jeong)



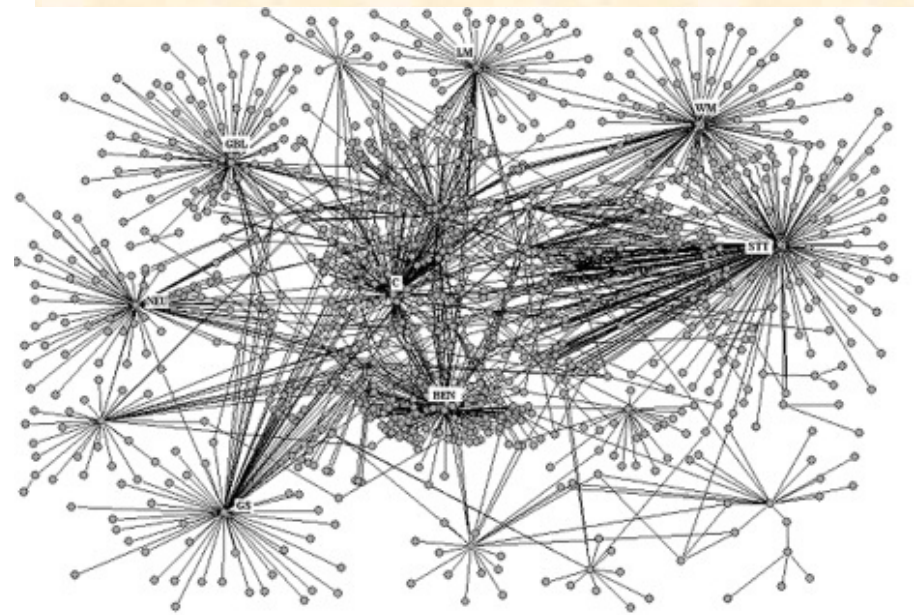
(b) Power-law distribution (log-log)



Power-Law Degree Distribution

$$P(k) = 2m^2 k^{-3}$$

Two ways to displace



How to derive the power law ? – An illustration

Let i be a node, which was added to the network at time t_i . Suppose this node has degree k when it is being picked up at time $t \geq t_i$.

Imaging that k is a continuous variable, so that probability $\Pi(k_i) = k_i / \sum_j k_j$ can be viewed as a continuous rate of change of k_i therefore

$$\frac{\partial k_i}{\partial t} = a\Pi(k_i) = ak_i / \sum_j k_j$$

Since the new node brings m edges in, the change of connectivity at each step is m , one has $a = m$. Also, $\sum_j k_j = 2mt$ so

$$\frac{\partial k_i}{\partial t} = \frac{mk_i}{2mt} = \frac{k_i}{2t}$$

Solving this equation, with the initial condition that node i was added to the network at t_i with connectivity $k_i(t_i) = m$ yields $k_i(t) = m\sqrt{\frac{t}{t_i}}$ namely $t_i = \frac{m^2 t}{k_i^2}$

On the other hand, it follows from $t_i = \frac{m^2 t}{k_i^2}$ that

$$P(k_i(t) < k) = P\left(t_i > \frac{m^2 t}{k^2}\right)$$

Assuming the time intervals $\{t_i\}$ are equally distributed, one has $P(t_i) = \frac{1}{t}$ for all t_i so

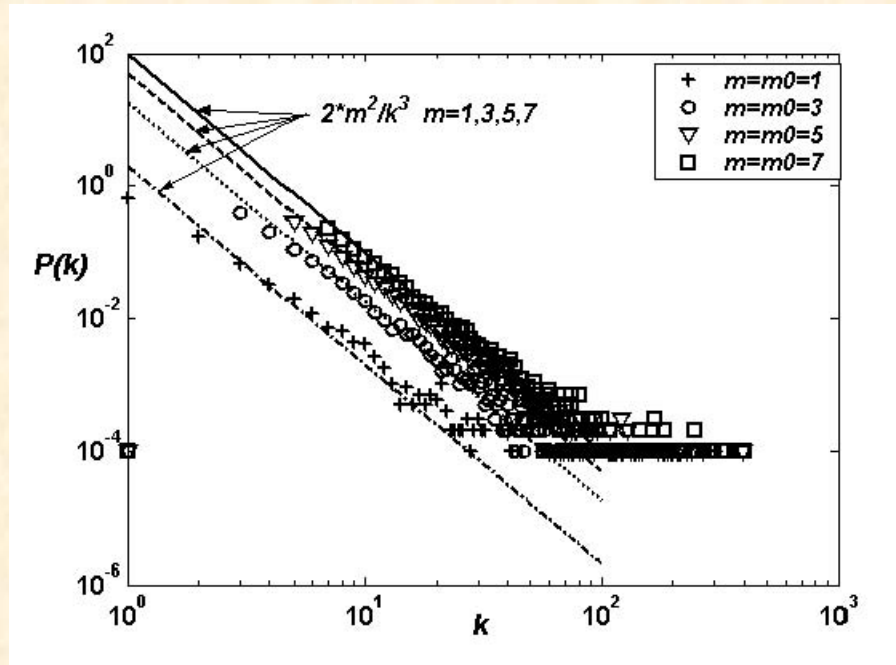
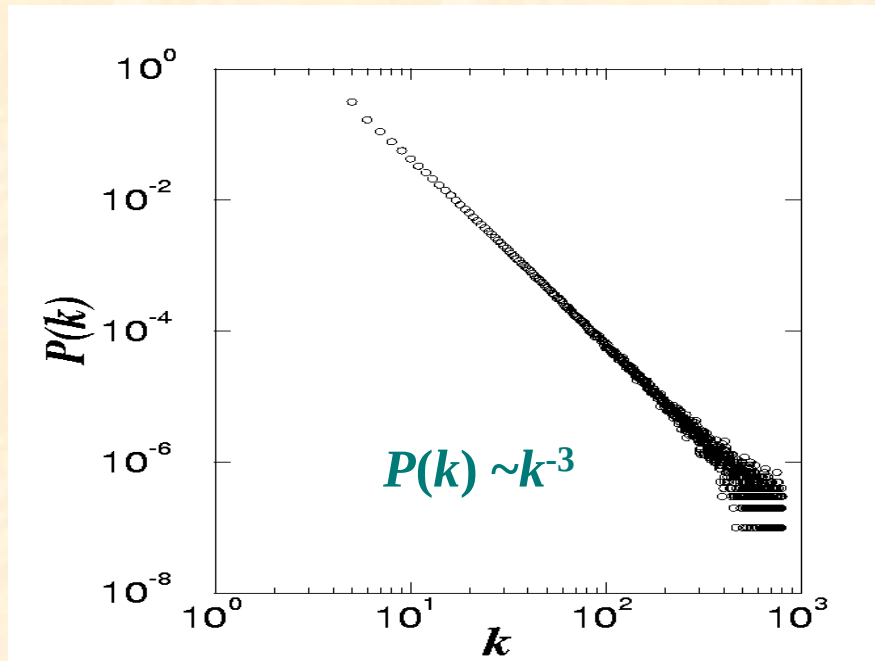
$$P\left(t_i > \frac{m^2 t}{k^2}\right) = 1 - P\left(t_i \leq \frac{m^2 t}{k^2}\right) = 1 - \frac{m^2 t}{k^2} \cdot P(t_i) = 1 - \frac{m^2 t}{k^2} \frac{1}{t} = 1 - \frac{m^2}{k^2}$$

Consequently

$$P(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = 2m^2 k^{-3}$$

This is a power law of the form $c \cdot k^{-\gamma}$ with $\gamma = 3$

BA Model



A.-L.Barabási, R. Albert, *Science* 286, 509 (1999)

Power-Law Distributions: Scale-Free

Consider a probability distribution function $f(x)$.
If, for any given constant a , there is a constant b
such that the following “scale-free” property holds:

$$f(ax) = b f(x)$$

then, with the assumption that $f(1) f'(1) \neq 0$, the
function $f(x)$ is uniquely determined by a power-law:

$$f(x) = f(1) x^{-\gamma} \quad \gamma = -f'(1) / f(1)$$

BA Model: $P(k) = 2m^2 k^{-3}$

Power-Law Distributions: Scale-Free

Proof. In $f(ax) = b f(x)$, let $x = 1$. Then, $f(a) = b f(1)$, so $b = f(a) / f(1)$; so,

$$f(ax) = \frac{f(a) f(x)}{f(1)}$$

Since this equality holds for arbitrary a and x , one may also consider a as a variable and take a derivative w.r.t. a :

$$\frac{df(ax)}{d(ax)} \frac{d(ax)}{da} = \frac{f(x)}{f(1)} \frac{df(a)}{da}$$

Letting $a = 1$ gives

$$x \frac{df(x)}{d(x)} = \frac{f'(1)}{f(1)} f(x)$$

which has a unique solution

$$f(x) = f(1) x^{-\gamma} \quad \gamma = -f'(1) / f(1)$$

BA Scale-Free Network Model

Theorem:

Average Path Length: $L \sim \frac{\ln N}{\ln \ln N}$

Clustering Coefficient: $C = \frac{m^2(m+1)^2}{4(m-1)} \left(\ln \left(\frac{m+1}{m} \right) - \frac{1}{m+1} \right) \frac{(\ln t)^2}{t}$

Degree Distribution: $P(k) = 2m^2 k^{-3}$

Extended BA (EBA) Model

The BA Model can describe scale-free networks having a power law with $r = 3$, but many real-world networks do not satisfy $r = 3$

EBA model (Albert and Barabasi, 2000)

Start with a fully-connected network with $m_0 \geq 0$ nodes.

(i) Add new nodes (incremental growth):

With probability p , one new node is added into the network

→ **(ii) Re-wiring:**

With probability q , m ($m \leq m_0$) edges are rewired

(iii) Add new edges (preferential attachment):

m ($m \leq m_0$) new edges are added into the network with probability

$$\Pi(k_i) = \frac{k_i}{\sum_l k_l}$$

EBA Model

In this model, $0 < p \leq 1$ and $0 \leq q < 1 - p$

If $q < \min(1 - p, (1 - p + m)/(1 + 2m))$, then the degree distribution of nodes will be in a **power-law** form:

$$P(k) \sim (k + A(p, q, m) + 1)^{-\gamma}$$

where $\gamma = 1 + B$ (typically, $2 < \gamma < 3$), and

$$A(p, q, m) = (p - q) \left(\frac{2m(1 - q)}{1 - p - q} + 1 \right)$$

$$B(p, q, m) = \frac{2m(1 - q) + 1 - p - q}{m}$$

Fitness Model

Recall: During the growth of the BA model, the degree of a node is changing by

$$k_i(t) = \sqrt{\frac{t}{t_i}}$$

where $k_i(t)$ is the degree of node i at time t and t_i is the instant at which node i is being added into the network.

This implies that the older a node, the higher its degree

In real life, however, this may not be true. For example:

- A teenager can have more social connections than an elder man
- A new website in WWW can receive more links than an old one
-

Fitness Model

1. Growth: Start from a small network of size $m_0 \geq 1$ and introduce one new node to the existing network each time, and with probability $\rho(\eta)$ this node is given a fitness value, $0 < \eta_i < 1$

2. Preferential Attachment: The new node is connected to $1 \leq m \leq m_0$ existing nodes, each is according to the probability

$$\Pi_i = \frac{\eta_i k_i}{\sum_{j=1}^n \eta_j k_j}$$

where $n = m_0 + t - 1$ is the total number of existing nodes at the $(t-1)$ st step of the process.

Basic Properties: similar to the BA model

Main difference: Preferential attachment is proportional to both
(1) node degree; (2) fitness value

A Simple Model

with power-law degree distribution

1. Start with nothing: no nodes, no links
2. At each time, with probability p , one new node is added to the existing network.
3. For every pair of unconnected nodes, with probability $q = 1 - p$, one edge is added to connect them.

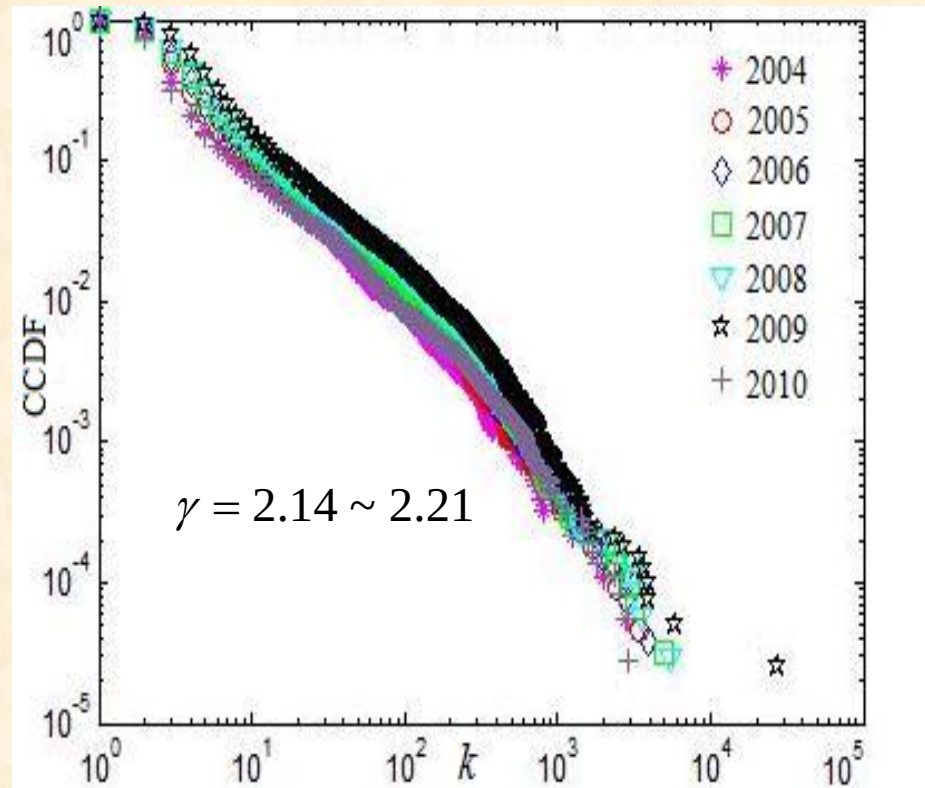
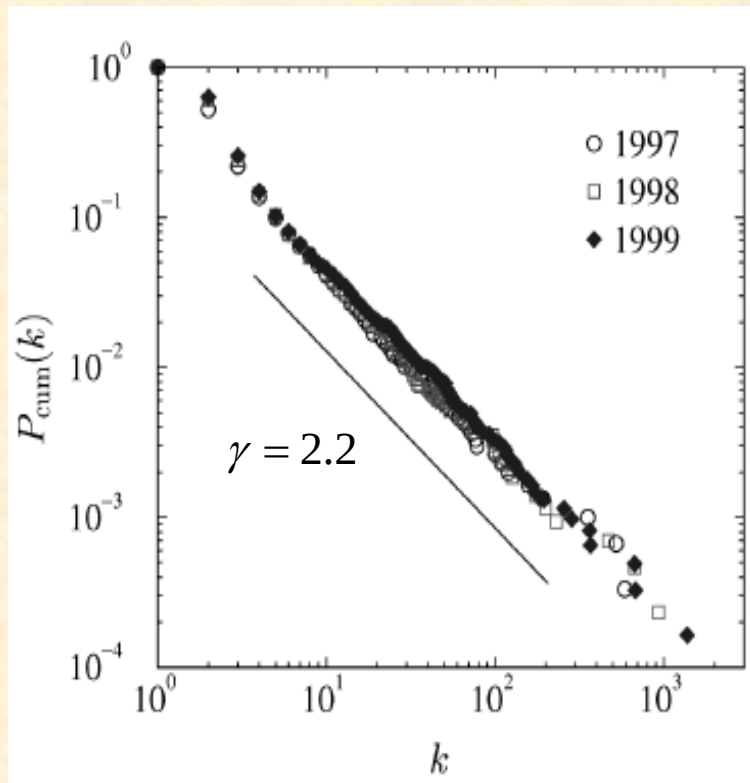
Theorem: The degree distribution of the above network model has a power-law form, $P(k) \sim k^{-\gamma}$, where $\gamma = 1 + q^{-1}$

Corollary: If $1/2 < q < 1$ then $2 < \gamma < 3$

Aiello, Chung and Lu (2002)

Scale-Free Internet

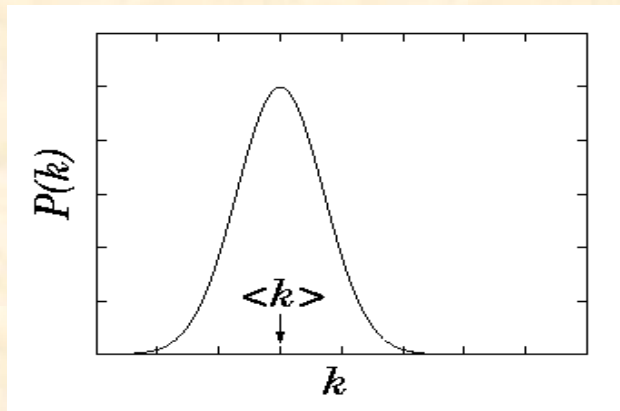
$$P(k) \sim k^{-\gamma}$$



(Autonomous Systems level)

Road Map vs Airline Routing Map

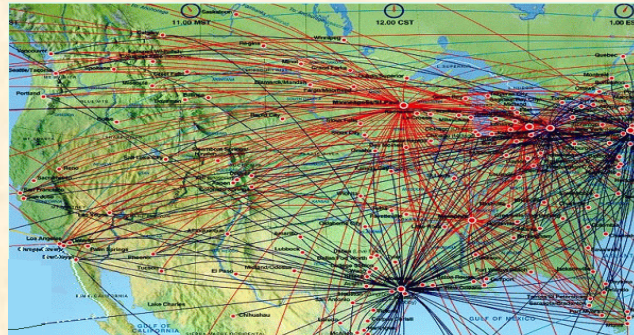
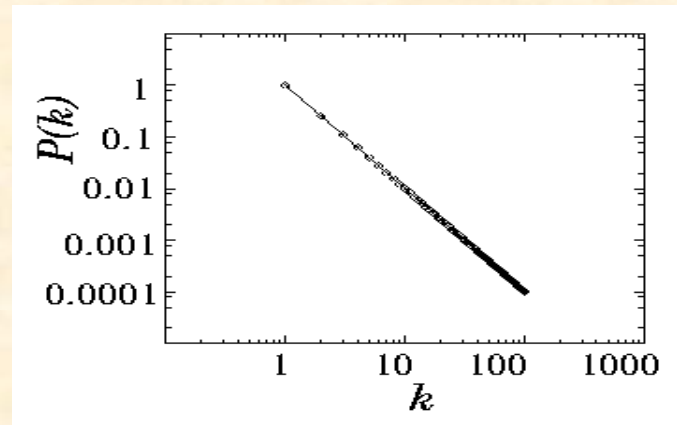
Poisson distribution



Small-world Network

(nodes: cities edges: highways)

Power-law distribution $p(k) \sim k^{-\gamma}$



Scale-free Network

(nodes: airports edges: flights)

Robustness and Fragility of Scale-Free Networks

“Achilles’ heel”

R. Albert, H. Jeong, A. L. Barabasi, *Nature*, 406, 387-482 (2000)

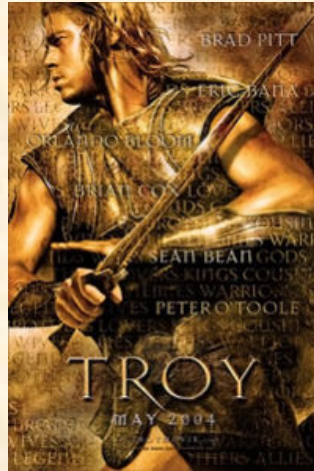


In Greek mythology, when Achilles was a baby, his mother Thetis took him to the River Styx, which was supposed to offer powers of invulnerability, and dipped his body into the water. But as mother held Achilles by the heel, his heel was not washed over by the water of the magical river. Achilles grew up to be a man of war who survived many great battles. On one day, a poisonous arrow shot at him was lodged in his heel, killing him shortly after.

“Achilles' heel”



Corfu, Greece



2004 film: TROY

特洛伊



Trojan Hours



Robustness versus Fragility

Illustration:

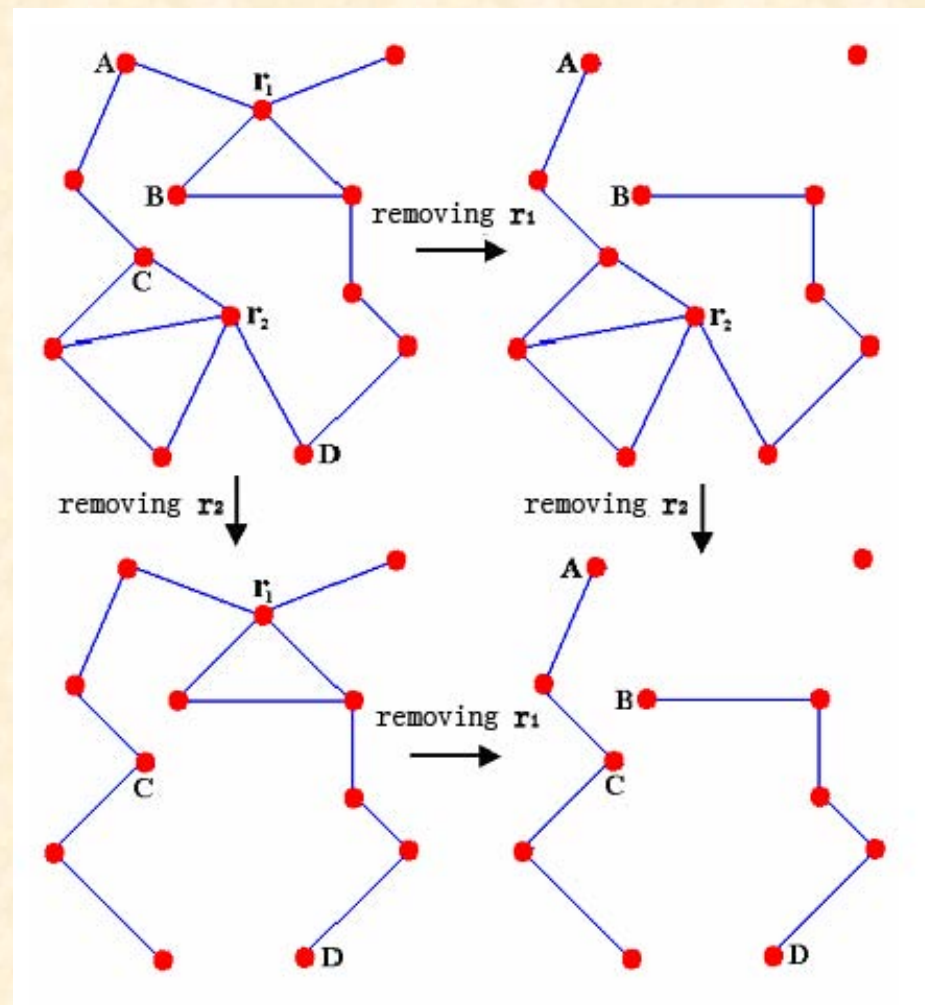
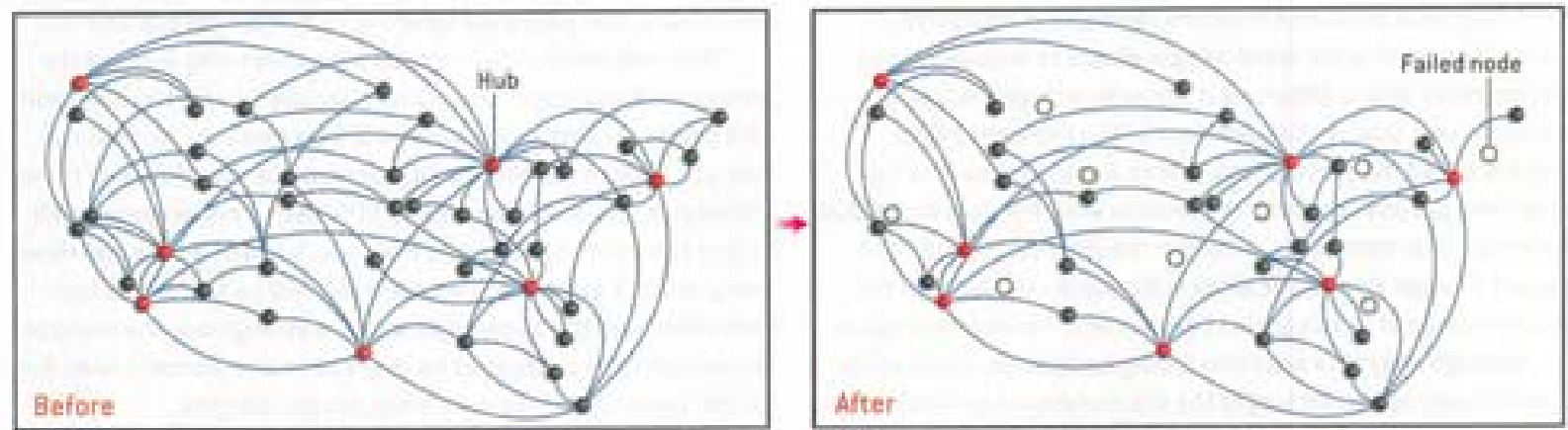
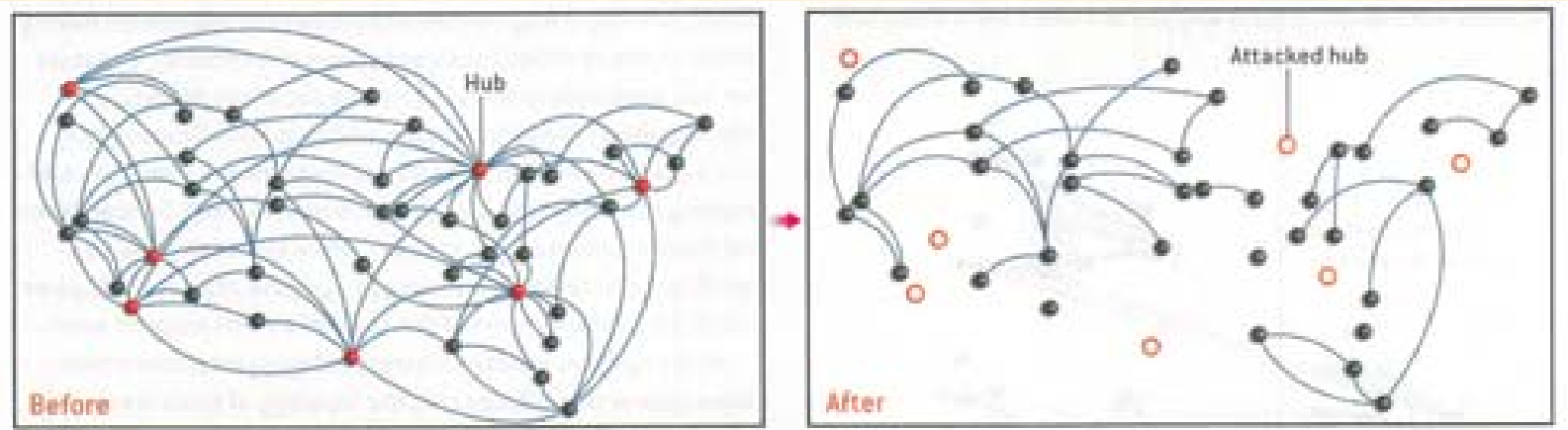


Illustration Example:

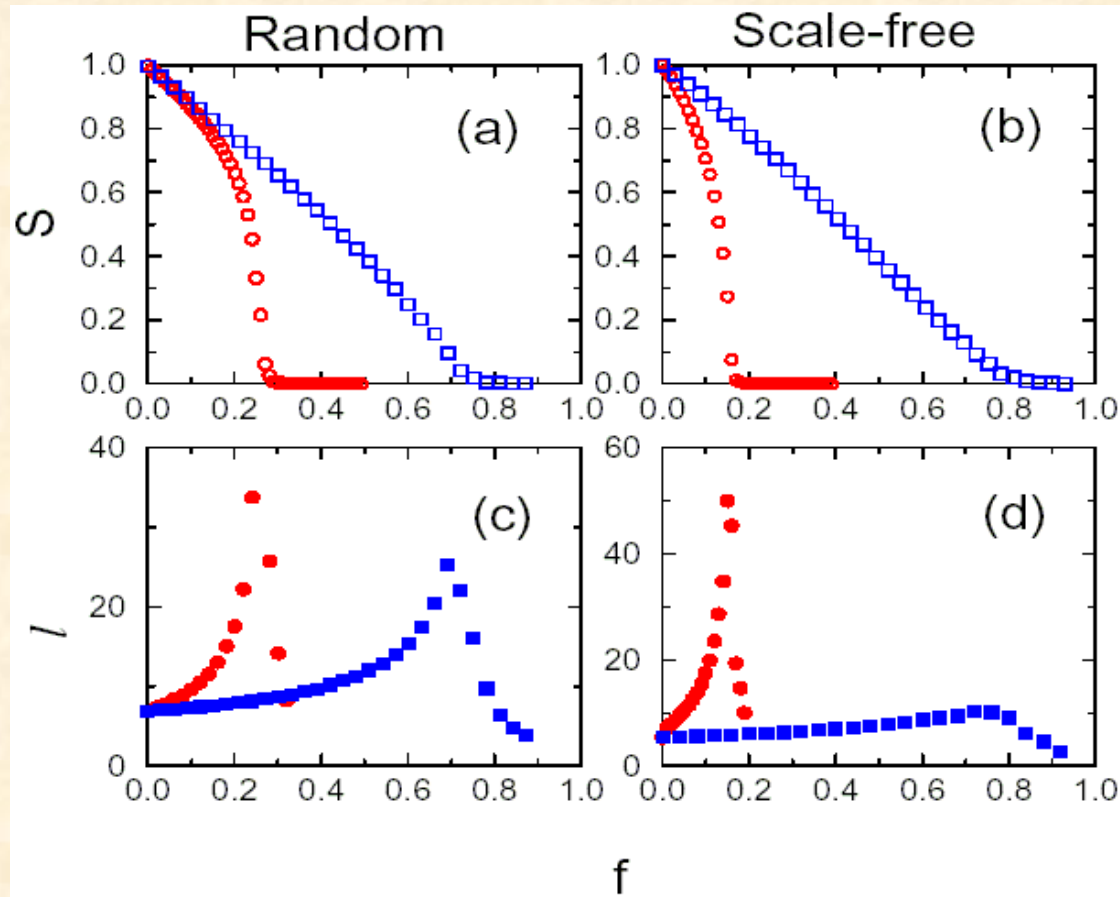


(a) Random failures



(b) Intentional attacks

Simulation Example:



S – size of largest subgraph

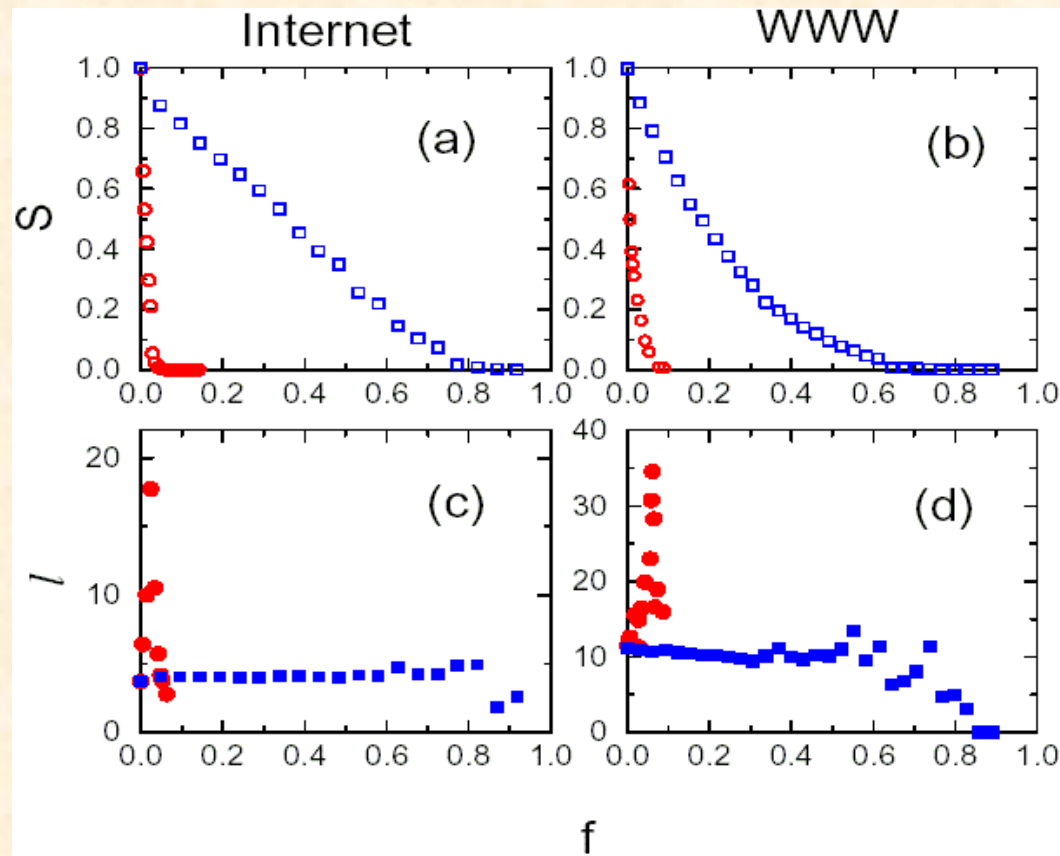
l – average path length

f – fraction of attacks

R. Albert, H. Jeong, A. L. Barabasi,
Nature, 406, 387-482 (2000)

Robustness and fragility of ER random-graph and BA scale-free networks:
 (a) and (c): ER random-graph networks; (b) and (d): BA scale-free networks;
 squares—random removal of nodes; circles—intentional removal of nodes

Internet and WWW: Simulation Results



S – size of largest subgraph

l – average path length

f – fraction of attacks

R. Albert, H. Jeong, A. L. Barabasi,
Nature, 406, 387-482 (2000)

Robustness and fragility of the Internet and WWW against intentional attacks:

(a) and (c): Internet; (b) and (d): WWW

squares—random failures, circles—intentional attacks

End

