

Homework 3 Solutions

Question 1: Consider the following function

$$f(x, y) = y^{\sqrt{x}}$$

(a) (**10 points**) Show that $\frac{\partial f}{\partial x} = \left(\frac{\ln y}{2\sqrt{x}}\right)y^{\sqrt{x}}$

$$(2 \text{ points}) \quad f(x, y) = \exp(\sqrt{x} \ln y)$$

$$(2 \text{ points}) \quad \frac{\partial f}{\partial x} = \exp(\sqrt{x} \ln y) \frac{\partial}{\partial x}(\sqrt{x} \ln y)$$

$$= y^{\sqrt{x}} \left(\frac{1}{2} x^{-1/2} \ln y \right)$$

$$(6 \text{ points}) \quad = \frac{\ln y}{2\sqrt{x}} y^{\sqrt{x}}$$

(b) (**10 points**) Show that $\frac{\partial f}{\partial y} = \sqrt{x} y^{\sqrt{x}-1}$

$$(10 \text{ points}) \quad \frac{\partial f}{\partial y} = \sqrt{x} y^{\sqrt{x}-1}$$

(Taking x as fixed, $\sqrt{x}$ is a constant and the derivative (w.r.t. y) of $y^{\sqrt{x}}$ immediately gives the answer. However, there is an alternative way of getting the same answer as follows

$$\frac{\partial f}{\partial y} = \exp(\sqrt{x} \ln y) \frac{\partial}{\partial y}(\sqrt{x} \ln y) = y^{\sqrt{x}} \left(\frac{\sqrt{x}}{y} \right) = \sqrt{x} y^{\sqrt{x}-1}$$

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Question 2: Consider the following equation

$$g(x, y) = e^{xy} + 2(x + y) = 5$$

(a) (**15 points**) Find $\frac{dy}{dx}$.

$$(15 \text{ points}) \quad \frac{dy}{dx} = -\frac{g_1(x, y)}{g_2(x, y)} = -\frac{ye^{xy} + 2}{xe^{xy} + 2}$$

(b) (**5 points**) Find the derivative of $y(x)$ when $x = 0$.

When $x = 0$,

$$e^0 + 2y = 5$$

$$(2 \text{ points}) \quad y = 2$$

$$(3 \text{ points}) \quad y'(0) = -\frac{4}{2} = -2$$

Question 3: (**20 points**) Find the degree of homogeneity of the following function

$$h(x, y) = x(x^2 + xy)^{1/5} + y^{3/5}x^{4/5}$$

$$(6 \text{ points}) \quad h(cx, cy) = (cx) \left((cx)^2 + (cx)(cy) \right)^{1/5} + (cy)^{3/5} (cx)^{4/5}$$

$$= c^{7/5} \left(x(x^2 + xy)^{1/5} + y^{3/5}x^{4/5} \right)$$

$$(10 \text{ points}) \quad = c^{7/5} h(x, y)$$

(4 points) So, the degree of homogeneity of the function is $7/5$.

Question 4: (**20 points**) Find and classify (i.e., determine the nature of) the critical points of the function

$$f(x, y) = x^2 - 2xy + 2y^2 - 2y + 2$$

The first-order conditions are

$$(2 \text{ points}) \quad f_1(x, y) = 2x - 2y = 0$$

$$(2 \text{ points}) \quad f_2(x, y) = -2x + 4y - 2 = 0$$

The above equations give

$$x = y = 1$$

(4 points) There is one critical point $(1, 1)$.

The second-derivatives are

$$(1 \text{ points}) \quad f_{11}(x, y) = 2$$

$$(1 \text{ points}) \quad f_{12}(x, y) = -2$$

$$(1 \text{ points}) \quad f_{22}(x, y) = 4$$

Then

$$(3 \text{ points}) \quad D = f_{11}f_{22} - f_{12}^2 = (2)(4) - (-2)^2 = 4 > 0$$

$$(2 \text{ points}) \quad f_{11}(1, 1) = 2 > 0$$

(4 points) Thus, the critical point is a minimum.

Question 5: (20 points) A monopoly manufactures two goods, X and Y , with demand functions

$$x = 12 - p^X \quad y = 18 - p^Y$$

The firm's cost function is $C(x, y) = x^2 + y^2 + 2xy$. Find the maximum profit achievable, and the quantities produced of each of X and Y in order to achieve this.

The profit function is

$$\begin{aligned} \Pi(x, y) &= xp^X + yp^Y - C(x, y) \\ &= x(12 - x) + y(18 - y) - (x^2 + y^2 + 2xy) \\ (4 \text{ points}) \quad &= 12x + 18y - 2x^2 - 2y^2 - 2xy \end{aligned}$$

The first-order conditions are

$$(2 \text{ points}) \quad \Pi_1(x, y) = 12 - 4x - 2y = 0$$

$$(2 \text{ points}) \quad \Pi_2(x, y) = -18 - 4y - 2x = 0$$

The above equations give

$$x = 1, y = 4$$

(4 points) There is one critical point $(1, 4)$.

The second-derivatives are

$$(1 \text{ points}) \quad \Pi_{11}(x, y) = -4$$

$$(1 \text{ points}) \quad \Pi_{12}(x, y) = -2$$

$$(1 \text{ points}) \quad \Pi_{22}(x, y) = -4$$

Then

$$(2 \text{ points}) \quad D = \Pi_{11}\Pi_{22} - \Pi_{12}^2 = (-4)(-4) - (-2)^2 = 12 > 0$$

$$(1 \text{ points}) \quad \Pi_{11}(1, 1) < 0$$

Thus, the critical point is a maximum.

(2 points) The maximum profit is $\Pi(1, 4) = 42$.