

# EF2452: Mathematics for Economics and Finance

## Lecture 3

### Part II of II

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# Main Topics

- Composite Function Rule OR Chain Rule
- Derivative of an Inverse Function
- Logarithmic Differentiation

# Composite Function

- Suppose that  $y$  is a function of  $x$ , and that  $z$  is a function of  $y$ . In this case, we say that  $z$  is a composite function of  $x$ .
- An example:

$$y = 5x^2 + 3x + 7 \quad z = y^2$$

- We can then write  $z$  as a (composite) function of  $x$ :

$$z = (5x^2 + 3x + 7)^2$$

# Chain Rule

- Now suppose that  $x$  changes. This gives rise to two-stage “chain-reaction”: first,  $y$  reacts directly to the change in  $x$ ; second,  $z$  reacts to this induced change in  $y$ .
- Chain rule says

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

- In the example, we have

$$\frac{dz}{dy} = \frac{d}{dy}y^2 = 2y = 2(5x^2 + 3x + 7)$$

$$\frac{dy}{dx} = \frac{d}{dx}(5x^2 + 3x + 7) = 10x + 3$$

$$\text{Hence, } \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = 2(5x^2 + 3x + 7)(10x + 3)$$

# Inverse Function

- If  $y = f(x)$ , then the inverse function for  $f$  (if it exists) is defined as follows

$$x = f^{-1}(y)$$

- Letting  $g = f^{-1}$ , we have

$$g(f(x)) = x$$

- An example: In  $y = f(x) = 2x + 7$ ,  $x$  is the input and  $y$  is the output.
- To obtain a formula for the inverse function, we have to find out  $x$  in terms of  $y$ :

$$2x = y - 7 \implies x = \frac{y - 7}{2} = g(y)$$

# Derivative of an Inverse Function

- If  $g = f^{-1}$ , then we have

$$g(f(x)) = x$$

- That is, we have  $y$  is a function of  $x$ ,  $y = f(x)$ , and  $g$  is a function of  $y$ ,  $g(y)$ .
- Differentiating both sides w.r.t.  $x$ , we have

$$\frac{d}{dx}g(y) = 1$$

- By chain rule, the L.H.S. is

$$\frac{dg}{dx} = \frac{dg}{dy} \cdot \frac{dy}{dx} = g'(y) \cdot f'(x)$$

# Derivative of an Inverse Function

$$g'(y) \cdot f'(x) = 1$$

- Then, we have the following

$$g'(y) = \frac{1}{f'(x)}$$

# Properties of Natural Exponential Function

- Natural exponential function:  $f(x) = \exp(x) = e^x$ , where  $e \approx 2.71828$ .

$$f'(x) = f(x) = e^x$$

$$e^a e^b = e^{a+b}$$

$$\frac{e^a}{e^b} = e^{a-b}$$

$$(e^a)^b = e^{ab}$$

# Properties of Natural Logarithmic Function

- By definition,  $e^{\ln x} = x$  for all  $x > 0$ , and  $\ln e^x = x$  for all  $x$ .
- So,  $\ln e = 1$
- For  $x > 0, y > 0$ ,

$$\ln xy = \ln x + \ln y$$

$$\ln \left( \frac{x}{y} \right) = \ln x - \ln y$$

$$\ln x^p = p \ln x$$

# Logarithmic Differentiation

I.  $f(x) = \ln x$ , then  $f'(x) = \frac{1}{x}$

II.  $f(x) = a^x$ , then  $f'(x) = a^x(\ln a)$ , where  $a$  is a constant

III.  $f(x) = x^x$ , then  $f'(x) = x^x(1 + \ln x)$

Usually, statement I above is taken as a rule, and the other two statements are derived using rule I and other rules of differentiation (see the next few slides).

# Logarithmic Differentiation II

- If  $y = a^x$ , find  $\frac{dy}{dx}$ .

$$y = a^x$$

- Taking the natural logarithm of both sides, we have

$$\ln y = x \ln a$$

- Differentiating both sides w.r.t.  $x$ , we get

$$\frac{d}{dx} \ln y = \ln a$$

# Logarithmic Differentiation II

- If we write  $z = \ln y$ , then by chain rule the L.H.S. is:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \frac{d}{dy}(\ln y) \cdot \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx}$$

- Then, we have

$$\frac{1}{y} \frac{dy}{dx} = \ln a$$

$$\implies \frac{dy}{dx} = y \cdot \ln a = a^x \ln a$$

# Logarithmic Differentiation III

- If  $y = x^x$ , find  $\frac{dy}{dx}$ .

$$y = x^x$$

- Taking the natural logarithm of both sides, we have

$$\ln y = x \ln x$$

- Differentiating both sides w.r.t.  $x$ , we get

$$\frac{d}{dx} \ln y = \frac{d}{dx} (x \ln x)$$

## Logarithmic Differentiation III

- If we write  $z = \ln y$ , then by chain rule the L.H.S. is:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \frac{d}{dy}(\ln y) \cdot \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx}$$

- The R.H.S. becomes (using product rule):

$$\frac{d}{dx}(x \ln x) = x \frac{d}{dx}(\ln x) + (\ln x) \frac{d}{dx}(x) = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$$

- Therefore,

$$\frac{1}{y} \frac{dy}{dx} = 1 + \ln x$$

$$\implies \frac{dy}{dx} = y \cdot (1 + \ln x) = x^x (1 + \ln x)$$

# References

Following chapters of the textbook:

- 6 Introduction to Calculus
- 7 Some Special Functions