

Midterm Test 2 Solutions

Question 1 [25 marks]

Consider the system with state equation:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

(a) **[10 marks]** Find state transition matrix e^{At} of this system.

(b) **[15 marks]** Calculate the response of unit step input at time $t = 1$ with initial state $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution:

(a) The state transition matrix can be calculated by $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$, where

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ -2 & s-1 \end{bmatrix}^{-1} = \frac{1}{(s-2)(s+1)} \begin{bmatrix} s-1 & 1 \\ 2 & s \end{bmatrix} = \frac{1}{3} \begin{bmatrix} \frac{1}{s-2} + \frac{2}{s+1} & \frac{1}{s-2} - \frac{1}{s+1} \\ \frac{2}{s-2} - \frac{2}{s+1} & \frac{s-2}{s-2} + \frac{s+1}{s+1} \end{bmatrix}.$$

Therefore,

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = \frac{1}{3} \begin{bmatrix} e^{2t} + 2e^{-t} & e^{2t} - e^{-t} \\ 2e^{2t} - 2e^{-t} & 2e^{2t} + e^{-t} \end{bmatrix}.$$

(b) The state transition equation of the system is given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

Plugging the e^{At} obtained in (a), $t = 1$, $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $u(t) = 1$ into the above equation, we have

$$\begin{aligned} x(1) &= \frac{1}{3} \begin{bmatrix} e^2 + 2e^{-1} \\ 2e^2 - 2e^{-1} \end{bmatrix} + \frac{1}{3} \int_0^1 \begin{bmatrix} e^{2\tau} - e^{-\tau} \\ 2e^{2\tau} + e^{-\tau} \end{bmatrix} d\tau \\ &= \frac{1}{3} \begin{bmatrix} e^2 + 2e^{-1} \\ 2e^2 - 2e^{-1} \end{bmatrix} + \frac{1}{3} \begin{bmatrix} \frac{1}{2}e^2 + e^{-1} \\ e^2 - e^{-1} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^2 + e^{-1} - \frac{1}{2} \\ e^2 - e^{-1} \end{bmatrix}. \end{aligned}$$

Question 2 [20 marks]

Consider the following state-space representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$$

- (a) [10 marks] Given the transformation $x = Pz$ where $P = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$, calculate the transformed state-space representation in terms of z_1 and z_2 .
- (b) [10 marks] Is there an invertible transformation matrix such that the resulting representation is controllable? Show your reason.

Solution:

- (a) (10 marks) The inverse of the given P is

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{bmatrix}.$$

Using the transformation $x = Pz$, the resulting representation is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \bar{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \bar{B}u,$$

$$y = \bar{C} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \bar{D}u,$$

where

$$\bar{A} = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\bar{B} = P^{-1}B = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix},$$

$$\bar{C} = CP = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix},$$

$$\bar{D} = D = 1.$$

- (b) (10 marks) We know that

$$\begin{bmatrix} \bar{B} & \bar{A}\bar{B} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix},$$

whose rank is $1 < 2$. Moreover, since invertible linear transformation cannot change the controllability of the system, there does not exist a desired linear transformation.

Question 3 [30 marks]

Consider the system with state-space representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 1 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

- (a) [10 marks] Determine its transfer function $\frac{Y(s)}{U(s)}$ with unknown parameter k .
- (b) [10 marks] Find the necessary and sufficient condition on k such that the system is a minimal realization.
- (c) [10 marks] For k such that the system is controllable, give its controllability canonical form.

Solution:

- (a) Since the state-space representation is in diagonal form, its transfer function can be directly written by

$$\frac{Y(s)}{U(s)} = \frac{1}{s-1} + \frac{1}{s+k} + \frac{k}{s+2} = \frac{(k+2)s^2 + (k^2+3)s - k^2 + 2k - 2}{s^3 + (k+1)s^2 + (k-2)s - 2k}.$$

- (b) First, we check the controllability of the given system. Calculate the controllability matrix as

$$W_c = [B \quad AB \quad A^2B] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -k & k^2 \\ 1 & -2 & 4 \end{bmatrix},$$

for which it holds that

$$W_c \xrightarrow{\substack{-R_1+R_2 \rightarrow R_2 \\ -R_1+R_3 \rightarrow R_3}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -k-1 & k^2-1 \\ 0 & -3 & 3 \end{bmatrix} \xrightarrow{C_3+C_2 \rightarrow C_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & k^2-k-2 & k^2-1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Therefore, $\text{rank}(W_c) = 3$ if and only if $k^2 - k - 2 \neq 0$. That is, $k \neq 2$ and $k \neq -1$.

As for the observability, the observability matrix is computed by

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & k \\ 1 & -k & -2k \\ 1 & k^2 & 4k \end{bmatrix},$$

for which one has that

$$W_o \xrightarrow{\substack{-C_1+C_2 \rightarrow C_2 \\ -kC_1+C_3 \rightarrow C_3}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -k-1 & -3k \\ 1 & k^2-1 & 3k \end{bmatrix} \xrightarrow{R_3+R_2 \rightarrow R_2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & k^2-k-2 & 0 \\ 1 & k^2-1 & 3k \end{bmatrix}.$$

Thus, $\text{rank}(W_o) = 3$ if and only if $k \neq 0$ and $k^2 - k - 2 \neq 0$, which means $k \neq 0$, $k \neq 2$, and $k \neq -1$.

In summary, the system is a minimum realization if and only if $k \neq 0$, $k \neq 2$, and $k \neq -1$.

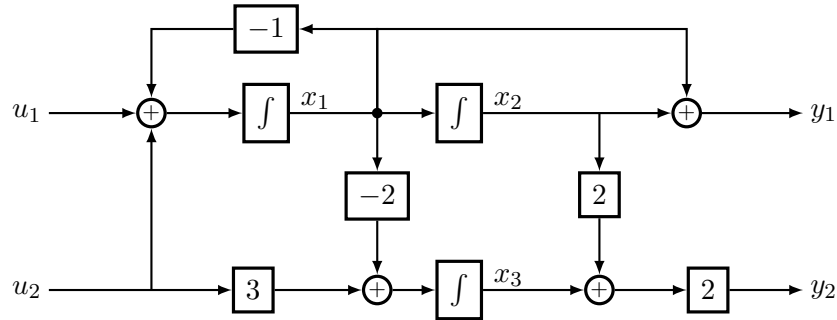
- (c) Its controllable canonical form can be directly obtained from the transfer function obtained in (a)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2k & 2-k & -k-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} -k^2 + 2k - 2 & k^2 + 3 & k + 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Question 4 [25 marks]

The block diagram shown below is an LTI system with two inputs $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and two outputs $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.



- (a) [15 marks] Find the state-space representation for this system with x_1 , x_2 , and x_3 as states.
- (b) [10 marks] Determine if the state-space representation is controllable and observable.

Solution:

- (a) From the block diagram, it is obvious that

$$\begin{cases} \dot{x}_1 = -x_1 + u_1 + u_2, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = -2x_1 + 3u_2, \\ y_1 = x_1 + x_2, \\ y_2 = 2(2x_2 + x_3), \end{cases} \quad \Rightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) Since the row rank of $[B \ AB] = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 3 & -2 & -2 \end{bmatrix}$ is already 3, the row rank of the controllability matrix $[B \ AB \ A^2B]$ is also 3, which means that the system is controllable. Construct the observability matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whose column rank is $2 < 3$, so the system is not observable.