

EF2452: Mathematics for Economics and Finance

Lecture 9

Part II of II

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Main Topics

- Determinant of a square matrix
- Cramer's rule
- Inverse of a 3×3 matrix via determinants
- Eigenvalues and eigenvectors

Determinants

- Determinants can be used to solve a system of linear equations and to find the inverse of a matrix.
- We have already discussed the (matrix) method of solving a system of linear equations and finding the inverse using elementary row operations. Here, we provide alternative ways of doing so using determinants.
- The **determinant** of a square matrix A is a particular number associated with A , written $\det A$ or $|A|$.
- When A is a 2×2 matrix, the determinant is given by the formula

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- Determinants are only defined for square matrices.

Determinants: 3×3 matrix

- The determinant of a 3×3 matrix is given as follows:

$$\begin{aligned} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

Determinants: 3×3 matrix

- In this way, the computation of a determinant of order 3 can be reduced to calculating three determinants of order 2.
- Note that a_{11} is multiplied by the second-order determinant obtained by deleting row 1 and column 1 of $|A|$.
- Likewise a_{12} , with a minus sign attached, is multiplied by the determinant obtained by deleting row 1 and column 2 of $|A|$.
- Finally, a_{13} is multiplied by the determinant obtained by deleting row 1 and column 3 of $|A|$.

The Determinant as a Test for Invertibility

- A square matrix A is invertible if and only if $\det A \neq 0$.
- A system of n linear equations in n unknowns $A\mathbf{x} = \mathbf{b}$ has a **unique solution** if and only if $\det A \neq 0$.

Solving a system of linear equations

- We illustrate Cramer's rule for the system of two equations in two unknowns, but it could be used for any $n \times n$ matrix A where A is invertible.
- Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

- We have $A\mathbf{x} = \mathbf{b}$, where the coefficient matrix A is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Solving a system of linear equations

- Then x_1 and x_2 are given by (with $|A| \neq 0$)

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

- Thus, $x_i = \Delta_i/|A|$, where $|A|$ is the determinant of A and Δ_i is the determinant of the matrix obtained by replacing the i th column of A with $\mathbf{b}$.

An Example in Macroeconomics

- IS-LM analysis is part of elementary macroeconomics that studies the equilibrium values of the national income Y and the interest rate r in terms of certain policy parameters and behavioral parameters.
- IS stands for 'investment savings' and LM for 'liquidity money'.
- First, we consider equilibrium in the 'goods sector'. This can be represented by the equation

$$Y = C + I + G$$

i.e., the national income Y is equal to the national expenditure. Expenditure can be split into consumption C , investment I , and government spending G .

- G is considered as a policy parameter.

An Example in Macroeconomics

- Behavioral parameters enter when we consider how consumption is related to the national income and how investment is affected by the interest rate:

$$C = C_0 + bY \qquad I = I_0 - ar$$

a, b, C_0, I_0 are positive constants.

- As income increases, consumption increases too: intuitive.
- As interest rates increase, borrowing becomes costlier, and hence investment decreases.
- Thus, the equilibrium condition in the goods sector becomes

$$Y = (C_0 + bY) + (I_0 - ar) + G$$

An Example in Macroeconomics

- Equilibrium in the 'money sector' is represented by the statement that the supply of money is equal to the demand for money: $M_s = M_d$.
- The money supply (M_s) is assumed to be given; it is another policy parameter.
- The money demand M_d is assumed to depend on Y and r :

$$M_d = M_0 + fY - gr$$

f, g, M_0 are positive constants. A higher interest rate means savings are more attractive, and therefore, the demand for money goes down.

- The equilibrium condition in the money sector is

$$M_s = M_0 + fY - gr$$

An Example in Macroeconomics

- The two equilibrium conditions can be written as a system of two linear equations in the two unknowns Y and r :

$$Y = (C_0 + bY) + (I_0 - ar) + G \quad \implies (1-b)Y + ar = C_0 + I_0 + G$$

$$M_s = M_0 + fY - gr \quad \implies fY - gr = M_s - M_0$$

- Therefore, we have

$$\begin{pmatrix} 1-b & a \\ f & -g \end{pmatrix} \begin{pmatrix} Y \\ r \end{pmatrix} = \begin{pmatrix} C_0 + I_0 + G \\ M_s - M_0 \end{pmatrix}$$

- Use Cramer's rule to solve for Y and r (see ICA 5).

Inverse of a Matrix using Determinants

- We can find the inverse of a matrix using another method that involves determinants.
- We will illustrate it using a 3×3 matrix. Suppose we have the following matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- A is invertible when $|A| \neq 0$.
- For each i and j between 1 and 3, let A_{ij} be the determinant of the 2×2 matrix obtained from A by deleting row i and column j . See the next page.

Inverse of a Matrix using Determinants

$$\begin{aligned}
 A_{11} &= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} &
 A_{12} &= \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} &
 A_{13} &= \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
 A_{21} &= \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} &
 A_{22} &= \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} &
 A_{23} &= \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\
 A_{31} &= \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} &
 A_{32} &= \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} &
 A_{33} &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}
 \end{aligned}$$

Co-factor Matrix

- Then write down the co-factor matrix

$$C = \begin{pmatrix} +A_{11} & -A_{12} & +A_{13} \\ -A_{21} & +A_{22} & -A_{23} \\ +A_{31} & -A_{32} & +A_{33} \end{pmatrix}$$

Notice the “+” sign whenever $i + j$ is even and “−” sign when $i + j$ is odd.

Adjugate of a Matrix

- Next, take the transpose of the co-factor matrix to obtain what is known as the adjugate or adjoint of matrix A :

$$\text{adj}(A) = C^T = \begin{pmatrix} +A_{11} & -A_{21} & +A_{31} \\ -A_{12} & +A_{22} & -A_{32} \\ +A_{13} & -A_{23} & +A_{33} \end{pmatrix}$$

Inverse of a Matrix

- Finally, the inverse of the matrix A is

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \begin{pmatrix} +A_{11} & -A_{21} & +A_{31} \\ -A_{12} & +A_{22} & -A_{32} \\ +A_{13} & -A_{23} & +A_{33} \end{pmatrix}$$

Homogeneous System of Equations

- Consider the special case of $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \mathbf{0}$ so that, we have

$$A\mathbf{x} = \mathbf{0}$$

- Then the system of n equations and n unknowns is called **homogeneous**.
- A homogeneous system always has the so-called *trivial solution* $x_1 = x_2 = \cdots = x_n = 0$.
- The homogeneous system has *nontrivial* solutions if and only if $|A| = 0$.

Eigenvalues and Eigenvectors

- A nonzero vector $\mathbf{x}$ that solves

$$A\mathbf{x} = \lambda\mathbf{x}$$

is called an eigenvector of A , and the associated λ is called an eigenvalue.

- Note that λ is a number or a scalar; it is not a vector.

Eigenvalues and Eigenvectors

- Consider the following system when A is a 2×2 matrix:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- The system is

$$a_{11}x_1 + a_{12}x_2 = \lambda x_1$$

$$a_{21}x_1 + a_{22}x_2 = \lambda x_2$$

- Rewriting the above system, we have

$$(a_{11} - \lambda)x_1 + a_{12}x_2 = 0$$

$$a_{21}x_1 + (a_{22} - \lambda)x_2 = 0$$

Eigenvalues and Eigenvectors

- Writing the above in matrix form, we have

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

where I is the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- The above homogeneous system has *nontrivial* solution $\mathbf{x} \neq \mathbf{0}$ if and only if $|A - \lambda I| = 0$.

Eigenvalues and Eigenvectors

- $|A - \lambda I| = 0$ implies

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

$$\implies \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

- So the eigenvalues λ are the solutions of this quadratic equation, and the eigenvectors are the non-zero vectors $\mathbf{x}$ that satisfy system $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

References

Following chapters of the textbook:

20 Determinants