

Problem Set 4: Systems of Linear differential Equations

1. By putting $x_1 = x$, $x_2 = \frac{dx}{dt} = x'$, $x_3 = \frac{d^2x}{dt^2} = x''$, show that the third order differential equation $x''' + x'' - x' - x = 0$ can be written in the form $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Hence find the general solution.

2. Solve the initial-value problem $\begin{cases} x_1' = x_1 \\ x_2' = x_2 - x_3 \\ x_3' = x_2 + x_3 \end{cases}$ with initial condition $x_1(0) = 1, x_2(0) = 1, x_3(0) = 1$.

3. Find all solutions of the differential system $\frac{d\vec{u}}{dt} = A\vec{u}$, where $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{pmatrix}$ and

$$\vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

4. (a) Convert the system $\begin{cases} \frac{dx}{dt} - 3x - 6y = 0 \\ \frac{dx}{dt} + \frac{dy}{dt} - 3y = 0 \end{cases}$ into an equivalent system of the form $\frac{d\vec{u}}{dt} = A\vec{u}$, where $A = \begin{pmatrix} 3 & 6 \\ -3 & -3 \end{pmatrix}$ and $\vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

- (b) It is given that $A = \begin{pmatrix} 3 & 6 \\ -3 & -3 \end{pmatrix}$ has eigenvalues $3i$ and $-3i$. For the eigenvalue $3i$, a corresponding eigenvector is $\begin{pmatrix} 1+i \\ -1 \end{pmatrix}$. Hence, or otherwise, solve the system

$$\begin{cases} \frac{dx}{dt} - 3x - 6y = 0 \\ \frac{dx}{dt} + \frac{dy}{dt} - 3y = 0 \end{cases} \quad \text{subject to the initial conditions } x(0) = 0 \text{ and } y(0) = 1.$$

5. Solve the differential system $\begin{cases} x_1' = 3x_1 + 3x_2 + 8 \\ x_2' = x_1 + 5x_2 + 4e^{3t} \end{cases}$. That is, $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}$, with $A = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{f} = \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}$.
6. Solve the differential system $\vec{u}' = A\vec{u} + \vec{f}$, with $A = \begin{pmatrix} 1 & -10 \\ -1 & 4 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{f} = \begin{pmatrix} e^t \\ \sin t \end{pmatrix}$ by using variation of parameters.
7. Use variation of parameters to solve the initial value problem $\frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}$, $\vec{u}(0) = \vec{u}_0$, in which $A = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}$, $\vec{f} = \begin{pmatrix} 2e^t \\ 2e^t \end{pmatrix}$, $\vec{u}_0 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.
8. (a) Convert the system $\begin{cases} \frac{d^2x}{dt^2} = \frac{dx}{dt} + y \\ \frac{dy}{dt} = \frac{dx}{dt} + y \end{cases}$ into an equivalent system of three first order ordinary differential equations $\frac{d}{dt}[\vec{u}(t)] = A\vec{u}(t)$ where $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $\vec{u}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$.
- (b) Solve the system $\frac{d}{dt}[\vec{u}(t)] = A\vec{u}(t)$ to obtain the general solutions of $x = x(t)$ and $y = y(t)$ to $\begin{cases} \frac{d^2x}{dt^2} = \frac{dx}{dt} + y \\ \frac{dy}{dt} = \frac{dx}{dt} + y \end{cases}$.
9. (a) Convert the system $\begin{cases} \frac{d^2x}{dt^2} = y + e^{-t} \\ \frac{dy}{dt} = -2x - 5\frac{dx}{dt} - 4y + e^{-t} \end{cases}$ into an equivalent system of three first order ordinary differential equations $\frac{d}{dt}[\vec{u}(t)] = A\vec{u}(t) + \vec{f}(t)$, where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{pmatrix}$, $\vec{u}(t) = \begin{bmatrix} x(t) \\ z(t) \\ y(t) \end{bmatrix}$ and $\vec{f}(t) = \begin{pmatrix} 0 \\ e^{-t} \\ e^{-t} \end{pmatrix}$.

(b) It is given that $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{pmatrix}$ has eigenvalue -2 with a corresponding eigenvector

$$\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \text{ and eigenvalue } -1 \text{ of multiplicity 2 with a corresponding eigenvector } \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Find the general solution in matrix form to the system $\frac{d}{dt}[\vec{u}(t)] = A\vec{u}(t)$.

(c) Find a particular solution in matrix form to the system $\frac{d}{dt}[\vec{u}(t)] = A\vec{u}(t) + \vec{f}(t)$.

(d) Find the general solution in matrix form to the system
$$\begin{cases} \frac{d^2x}{dt^2} = y + e^{-t} \\ \frac{dy}{dt} = -2x - 5\frac{dx}{dt} - 4y + e^{-t} \end{cases}.$$

-End-