

EE2104 Introduction to Electromagnetics
Assignment 4 Solution

Deadline (for uploading): 11:59 pm, 2 Dec 2022
Please put down your name and university ID.

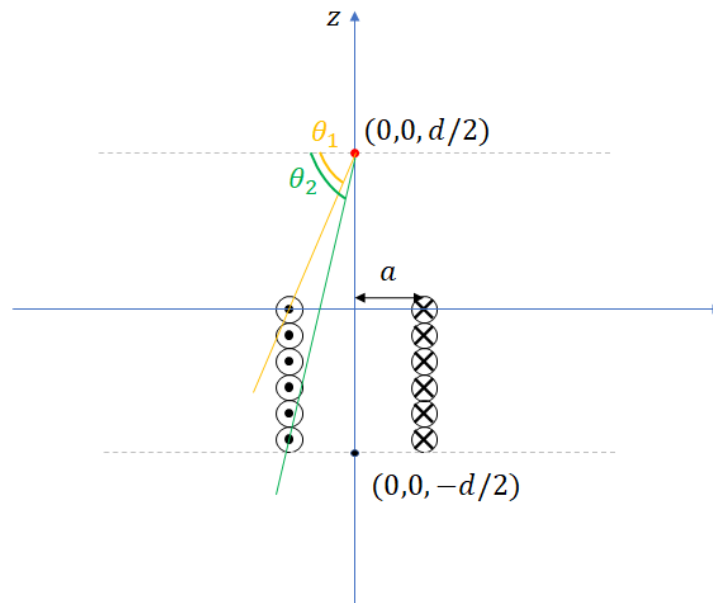
- Given the expression of the B -field at distance z directly above the center of a circular coil of radius a carrying current I is

$$B = \frac{\mu_0}{2} \frac{Ia^2}{(z^2 + a^2)^{3/2}}$$

A stack of the coils are connected in series to form a solenoid of turn density n . The solenoid is centered along the z -axis. It has a finite length extending from $z = -d/2$ to 0 . By superposition, find B at the $(x, y, z) = (0, 0, +d/2)$.

Solution:

Consider the segment from z to $z + dz$:



The number of turns included is ndz .
So the differential B -field is:

$$d\mathbf{B} = \hat{\mathbf{a}}_z \frac{\mu_0}{2} \frac{nIa^2}{(z^2 + a^2)^{3/2}} dz$$

$$\mathbf{B} = \hat{\mathbf{a}}_z \int_{-d/2}^0 \frac{\mu_0}{2} \frac{nIa^2}{(z^2 + a^2)^{3/2}} dz = \hat{\mathbf{a}}_z \frac{\mu_0 n I a^2}{2} \int_{-d/2}^0 \frac{1}{(z^2 + a^2)^{3/2}} dz$$

But

$$z = a \tan \theta \Rightarrow dz = a \sec^2 \theta d\theta$$
$$z^2 + a^2 = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta$$

So we have

$$\mathbf{B} = \hat{\mathbf{a}}_z \frac{\mu_0 n I a^2}{2} \int_{\theta_2}^{\theta_1} \frac{1}{a^3 \sec^3 \theta} a \sec^2 \theta d\theta = \hat{\mathbf{a}}_z \frac{\mu_0 n I}{2} \int_{\theta_2}^{\theta_1} \cos \theta d\theta = \hat{\mathbf{a}}_z \frac{\mu_0 n I}{2} \sin \theta \Big|_{\theta_2}^{\theta_1}$$

where at the extremes:

$$\sin \theta_1 = \frac{-d}{\sqrt{4a^2 + d^2}}$$

$$\sin \theta_2 = \frac{-d}{\sqrt{a^2 + d^2}}$$

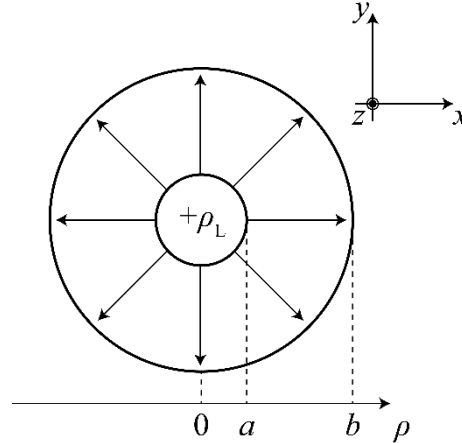
Therefore,

$$\mathbf{B} \left(0, 0, \frac{d}{2} \right) = \frac{\mu_0 n I d}{2} \left(\frac{1}{\sqrt{a^2 + d^2}} - \frac{1}{\sqrt{4a^2 + d^2}} \right) \hat{\mathbf{a}}_z$$

2. Consider a long coaxial cable of inner and outer radii a and b , respectively. The gap is filled with air. The length of the cable is L . Find
- the capacitance C using Gauss's law,
 - the inductance L using Ampere's law, and
 - calculate a value called the characteristic impedance $Z = (L/C)^{1/2}$ if $b = 5$ mm, and $a = 1$ mm.

Solution:

The cross-section of the coaxial cable is shown below:



- (a) Assume the inner conductor is charged with a charge density of $+\rho_L$ (C/m).

Gauss' law is applied for a cylindrical surface S of radius ρ and length L , we have the E-field derived as follows (based on the approach in the lecture notes 4-25):

- By symmetry, $\mathbf{D} = D(\rho)\hat{\mathbf{a}}_\rho$
- Then,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \rho_L L$$

$$\int_{S_{top}} (D\hat{\mathbf{a}}_\rho) \cdot (dS\hat{\mathbf{a}}_z) + \int_{S_{bottom}} (D\hat{\mathbf{a}}_\rho) \cdot (dS(-\hat{\mathbf{a}}_z)) + \int_{S_{curved}} (D\hat{\mathbf{a}}_\rho) \cdot (dS\hat{\mathbf{a}}_\rho) = \rho_L L$$

$$\int_{S_{curved}} D(\rho) dS = \rho_L L$$

$$D(\rho) \int_{S_{curved}} dS = \rho_L L$$

- Therefore,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \hat{\mathbf{a}}_\rho$$

- The voltage can be calculated as:
-

$$V = - \int_{\rho=b}^a \mathbf{E} \cdot d\mathbf{L} = - \int_{\rho=b}^a \frac{\rho_L}{2\pi\epsilon_0\rho} d\rho = \frac{\rho_L}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

We can obtain the capacitance (as the inner conductor carries a total charge of $\rho_L L$)

$$C = \frac{\rho_L L}{V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

- (b) Assume a current $I_0 \hat{\mathbf{a}}_z$ is flowing along the positive z direction via the inner conductor and a current $-I_0 \hat{\mathbf{a}}_z$ along the negative z -direction via the outer conductor.

Ampere's law is applied over a closed integration loop C of radius ρ on a plane perpendicular to the z -axis. We have the B -field derived as follows (based on the approach in the lecture notes 9-14):

- i) By symmetry, $\mathbf{H} = H(\rho) \hat{\mathbf{a}}_\phi$
 ii) Using the Ampere's Law, we have in between the two conductors:

$$\oint_C \mathbf{H} \cdot d\mathbf{L} = I_0$$

$$\oint_C H dL = I_0$$

$$H(\rho) \oint_C dL = I_0$$

$$H(\rho) 2\pi\rho = I_0$$

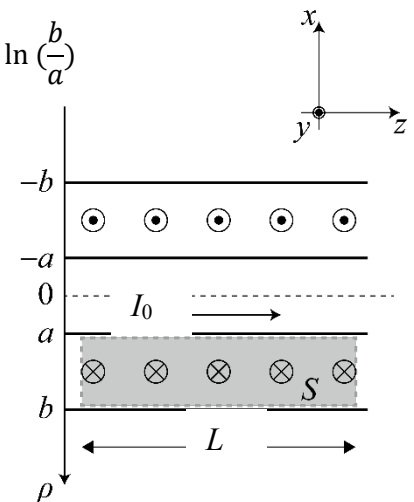
$$\therefore \mathbf{B} = \frac{\mu_0 I_0}{2\pi\rho} \hat{\mathbf{a}}_\phi$$

- iii) The magnetic flux through a surface S in the figure below can be written as:

$$\Phi = \int_S \frac{\mu_0 I_0}{2\pi\rho} dS = \int_{z=0}^L \int_{\rho=a}^b \frac{\mu_0 I}{2\pi\rho} d\rho dz = L \frac{\mu_0 I_0}{2\pi} \ln\left(\frac{b}{a}\right)$$

- iv) Finally, the inductance is given by:

$$L = \frac{\Phi}{I} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) L$$



- (c) Combining the above results, we have for the impedance:

$$Z = \sqrt{L/C} = \sqrt{\left[\frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) L \right] / \left[\frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)} \right]} = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \ln\left(\frac{b}{a}\right)$$

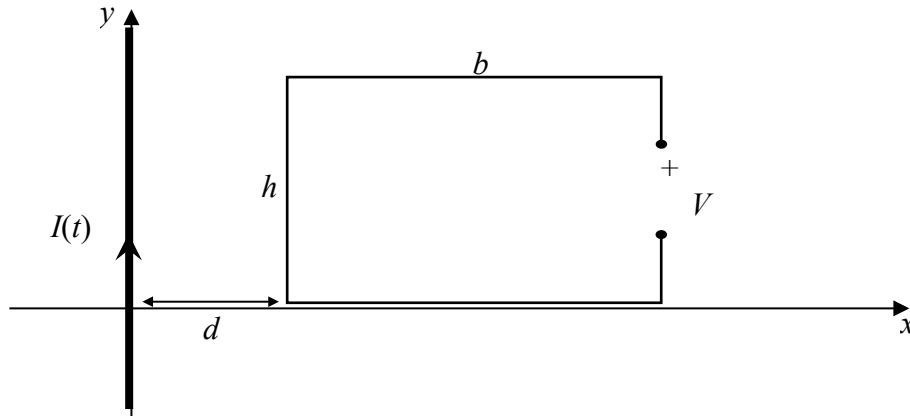
When $b = 5$ mm and $a = 1$ mm,

$$Z = \sqrt{L/C} = \sqrt{\frac{\mu_0}{\epsilon_0}} \cdot \ln\left(\frac{5}{1}\right) \cdot \frac{1}{2\pi} = 96.6(\Omega)$$

3. An infinitely long wire of current I flowing towards the $+y$ -direction is put on the y -axis. The current is:

$$I = I_0 \exp(-\gamma_0 t)$$

where I_0 is the amplitude and γ_0 is a rate. A rectangular open wire loop of height h and width b is put at distance d away, as the following figure shows. Find the induced voltage V across the loop.



Solution:

The magnetic field on the xy -plane is

$$\mathbf{B} = -\frac{\mu_0 I}{2\pi x} \hat{\mathbf{a}}_z = -\frac{\mu_0 I_0}{2\pi x} \exp(-\gamma_0 t) \hat{\mathbf{a}}_z$$

The flux contained by the coil is

$$\Phi = \int_{\text{coil area}} \mathbf{B} \cdot d\mathbf{S} = \int_{x=d}^{x=d+b} \frac{\mu_0 I_0}{2\pi x} \exp(-\gamma_0 t) (-\hat{\mathbf{a}}_z \cdot -\hat{\mathbf{a}}_z) h dx = \frac{h\mu_0 I_0}{2\pi} \exp(-\gamma_0 t) \ln \left(1 + \frac{b}{d}\right)$$

According to the Faraday's law, the induced voltage is

$$V = -\frac{d\Phi}{dt} = \frac{\mu_0 I_0}{2\pi} \ln \left(1 + \frac{b}{d}\right) \gamma_0 h \exp(-\gamma_0 t)$$

4. The following considers electromagnetic wave propagating in air (without charges or currents).

(a) State the Maxwell's equations in terms of $\mathbf{E}$ and $\mathbf{H}$ for the free-space.

(b) Show that the four equations are satisfied by:

$$\mathbf{E} = E_0 \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{x}}$$

$$\mathbf{H} = \frac{E_0}{c\mu_0} \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{y}}$$

for some constants E_0 and ω .

Solution:

(a) Because of the absence of charges, the charge density ρ_v and current density $\mathbf{J}$ are both zero. The Maxwell's equations become:

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{D} &= 0 \\ \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

Because of the absence of materials, there is no polarization $\mathbf{P}$ nor magnetization $\mathbf{M}$. So we have

$$\begin{aligned}\mathbf{D} &= \epsilon_0 \mathbf{E} \\ \mathbf{B} &= \mu_0 \mathbf{H}\end{aligned}$$

The Maxwell's equations further reduce to:

$$\begin{aligned}\nabla \times \mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \\ \nabla \times \mathbf{H} &= \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{H} &= 0\end{aligned}$$

(b) For the first equation,

$$\begin{aligned}\text{LHS: } \nabla \times \left[E_0 \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{x}} \right] &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0 \cos\left(\omega\left(\frac{z}{c} - t\right)\right) & 0 & 0 \end{vmatrix} \\ &= -\hat{\mathbf{y}} \left(0 - \frac{\partial \left[E_0 \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \right]}{\partial z} \right) = -\hat{\mathbf{y}} \frac{\omega E_0}{c} \sin\left(\omega\left(\frac{z}{c} - t\right)\right)\end{aligned}$$

$$\text{RHS: } -\mu_0 \frac{\partial}{\partial t} \left[\frac{E_0}{c\mu_0} \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{y}} \right] = -\hat{\mathbf{y}} \frac{\omega E_0}{c} \sin\left(\omega\left(\frac{z}{c} - t\right)\right)$$

which equals LHS.

For the second equation,

$$\begin{aligned}
 \text{LHS: } \nabla \times \left[\frac{E_0}{c\mu_0} \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{y}} \right] &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{E_0}{c\mu_0} \cos\left(\omega\left(\frac{z}{c} - t\right)\right) & 0 \end{vmatrix} \\
 &= \hat{\mathbf{x}} \left(0 - \frac{\partial \left[\frac{E_0}{c\mu_0} \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \right]}{\partial z} \right) = \hat{\mathbf{x}} \frac{\omega}{c} \frac{E_0}{\mu_0} \sin\left(\omega\left(\frac{z}{c} - t\right)\right) \\
 &= \hat{\mathbf{x}} \epsilon_0 \omega E_0 \sin\left(\omega\left(\frac{z}{c} - t\right)\right)
 \end{aligned}$$

$$\text{RHS: } \epsilon_0 \frac{\partial}{\partial t} \left[E_0 \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{x}} \right] = \hat{\mathbf{x}} \epsilon_0 \omega E_0 \sin\left(\omega\left(\frac{z}{c} - t\right)\right)$$

which equals LHS.

For the third equation,

$$\begin{aligned}
 \text{LHS: } \nabla \cdot \left(E_0 \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{x}} \right) \\
 = \left(\frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right) \cdot \left(E_0 \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{x}} \right) = 0
 \end{aligned}$$

which is the RHS.

For the fourth equation,

$$\text{LHS: } \nabla \cdot \left(\frac{E_0}{c\mu_0} \cos\left(\omega\left(\frac{z}{c} - t\right)\right) \hat{\mathbf{y}} \right) = 0$$

which equals RHS.

Hence, the four equations are satisfied.