

EE 3114 **Systems and Control**

9. Controllability and Observability

Semester B, 2017-18

LTI State-Space Systems

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

$\mathbf{x} \in \mathbb{R}^m$ is the state vector

$\mathbf{u} \in \mathbb{R}^n$ is the control vector

$\mathbf{y} \in \mathbb{R}^k$ is the output vector

$A_{m \times m}$ is the system matrix

$B_{m \times n}$ is the control matrix

$C_{k \times m}$ is the observation matrix

$D_{k \times n}$ is the input matrix (often, $D = 0$)

Multi-Input Multi-Output (MIMO): $n > 1, k > 1$

Single-Input Single-Output (SISO): $n = 1, k = 1$

Multi-Input Single-Output (MISO): $n > 1, k = 1$

Single-Input Multi-Output (SIMO): $n = 1, k > 1$

Reachability

Given an LTI system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$

Given a set of initial states: $I = \{\mathbf{x}(0)\}$

A state $\mathbf{x}^*$ is said to be **reachable** at time $t = T$ **if** there exist an initial state $\mathbf{x}_0 \in I$ and an input $\mathbf{u}$ to drive the state $\mathbf{x}$ trajectory from $\mathbf{x}_0$ at $t = 0$ to arrive $\mathbf{x}^*$ at $t = T$



Controllability

Given an LTI system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$

Given a set of initial states: $I = \{\mathbf{x}(0)\}$

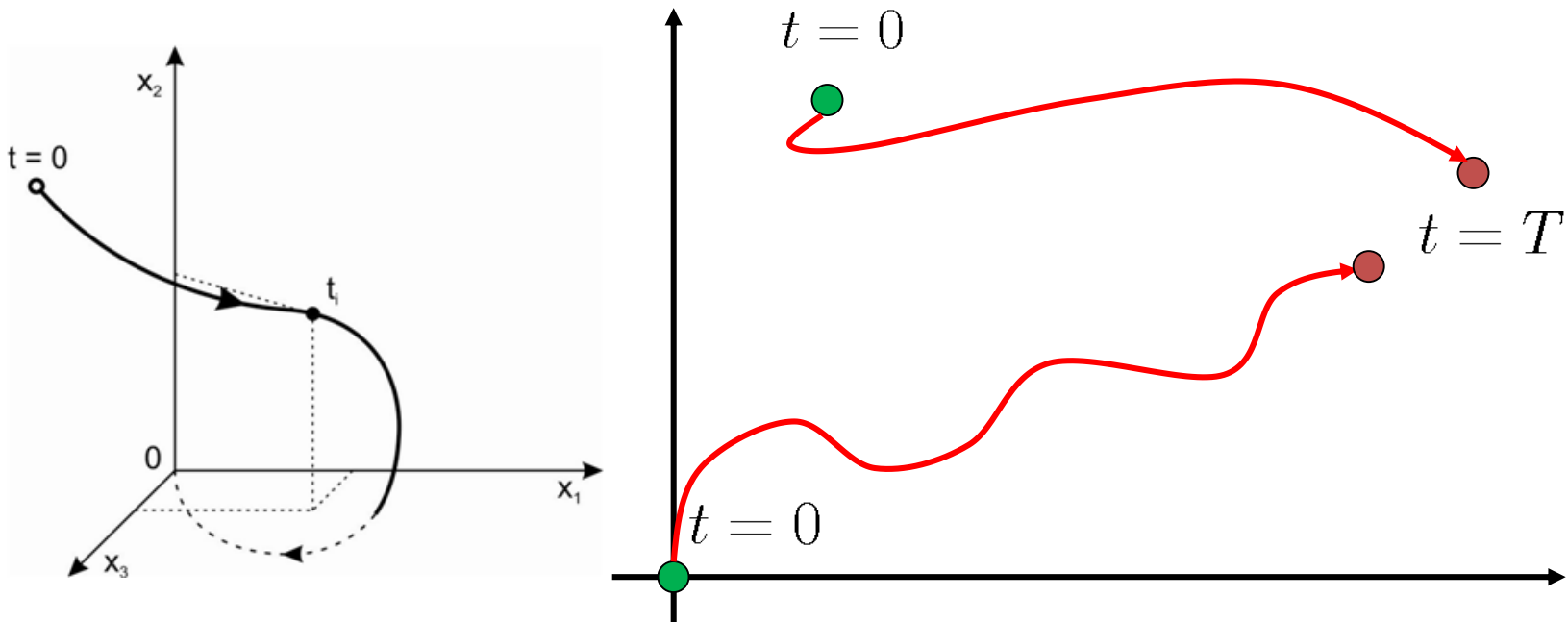
The LTI system is said to be (completely) **controllable** **if** its reachable initial set $I = \mathbb{R}^n$ (the whole state space) and the time $T < \infty$ (finite time)



The LTI system is said to be **controllable** **if** there **exists** an input $\mathbf{u}$ which can control the state $\mathbf{x}$ trajectory to reach **any** final state from **any** initial state in **finite** time

Controllability

The LTI system is said to be **controllable** if there **exists** an input u which can control the state x trajectory to reach **any** final state from **any** initial state in **finite** time



Controllability

Consider LTI system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, with $A_{n \times n}$ and $B_{n \times m}$

Theorem

LTI system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ is **controllable if and only if**

Matrix $[B \ AB \ A^2B \ A^3B \ \dots \ A^{n-1}B]$ has full row-rank

(Proof given below)

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow [B \ AB] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{ (Full row-rank)}$$

 This LTI system (A, B) is controllable

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow [B \ AB] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{ (Full row-rank)}$$

This LTI system (A, B) is controllable

Q: Can the state $\mathbf{x}(t)$ of this system go, say, from $(0,0)$ to $(5, -5)$?

A: Yes, even for constant control input $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$\mathbf{x}(t^*) = e^{At^*} \mathbf{x}(0) + \int_0^{t^*} e^{A(t^*-\tau)} B \mathbf{u} d\tau$$



$$\begin{bmatrix} 5 \\ -5 \end{bmatrix} = 0 + \int_0^{t^*} e^{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(t^*-\tau)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u} d\tau$$

$$= \int_0^{t^*} \begin{bmatrix} e^{t^*-\tau} & 0 \\ 0 & e^{t^*-\tau} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} d\tau \mathbf{u}$$

$$= \begin{bmatrix} 0 & \int_0^{t^*} e^{t^*-\tau} d\tau \\ \int_0^{t^*} e^{t^*-\tau} d\tau & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} (e^{t^*}-1)u_2 \\ (e^{t^*}-1)u_1 \end{bmatrix} \rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -5/(e^{t^*}-1) \\ 5/(e^{t^*}-1) \end{bmatrix}$$

This control input can drive $\mathbf{x}$ to move from $(0,0)$ at $t = 0$ to $(5, -5)$ at $t = t^*$

Theorem

LTI system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ is controllable if and only if

Matrix $[B \ AB \ A^2B \ A^3B \ \dots \ A^{n-1}B]$ has a full row-rank

Proof:

Starting, say, from $\mathbf{x}(0) = 0$ going to any $\mathbf{x}^*$ at $t = t^*$

$$\mathbf{x}(t^*) = e^{At^*} \mathbf{x}(0) + \int_0^{t^*} e^{A(t^*-\tau)} B\mathbf{u}(\tau) d\tau$$



$$\begin{aligned} \mathbf{x}^* &= 0 + \int_0^{t^*} e^{A(t^*-\tau)} B\mathbf{u}(\tau) d\tau \quad \downarrow \text{Cayley-Hamilton Theorem} \\ &= \int_0^{t^*} \left[I + \beta A(t^* - \tau) + \dots + \frac{\beta^{n-1}}{(n-1)!} A^{n-1}(t^* - \tau)^{n-1} \right] B\mathbf{u}(\tau) d\tau \\ &= [B \ AB \ \dots \ A^{n-1}B] [\gamma_1 \ \gamma_2 \ \dots \ \gamma_n]^T \quad (\gamma_i \text{ contains } \mathbf{u}, t^*, \beta) \end{aligned}$$

This has a solution $(\mathbf{u}, t^*)$ if and only if $[B \ AB \ \dots \ A^{n-1}B]$ has full row-rank

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[B \ AB \ A^2B] = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 3 & 5 & 9 \end{bmatrix} \quad (\text{Full row-rank})$$

➡ Controllable

➡ There exist a finite time t^* and a finite control input $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ such that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ can move from anywhere to anywhere else

Not all systems are controllable

Example:

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$[B \ AB \ A^2B] = \begin{bmatrix} 2 & -4 & 8 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{Not full row-rank})$$

 Uncontrollable

Controllability Canonical Form (1)

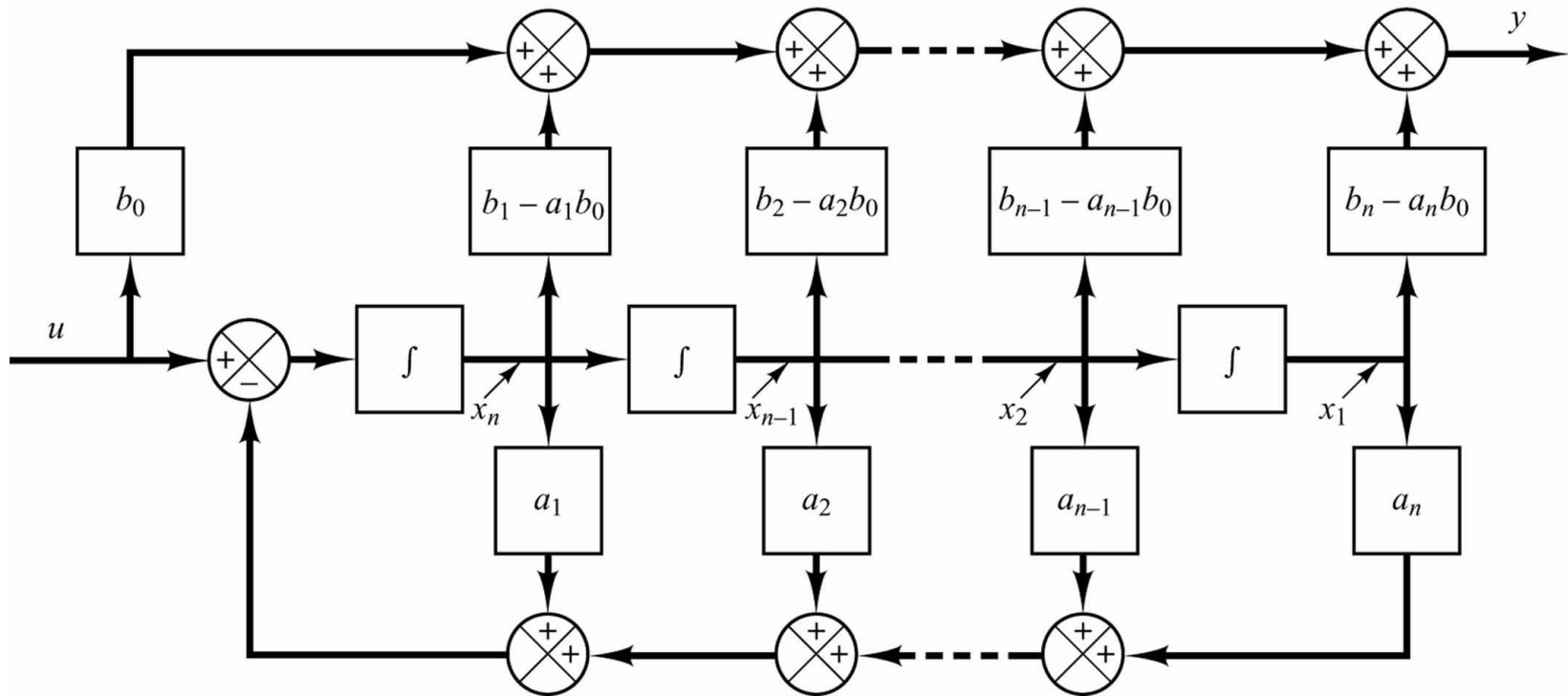
$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdot & \cdot & \cdot & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdot & \cdot & \cdot & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Block diagram representation of the system

-- Controllability canonical form (1)



Example:

$$\frac{Y(s)}{U(s)} = \frac{s + 3}{s^2 + 3s + 2}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Recall: _____

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

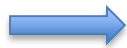
$$y = [b_n - a_n b_0 \quad b_{n-1} - a_{n-1} b_0 \quad \cdot \quad \cdot \quad \cdot \quad b_2 - a_2 b_0 \quad b_1 - a_1 b_0]$$

Controllability Canonical Form (2)

Diagonal Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s + p_1)(s + p_2) \cdots (s + p_n)}$$

$$= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \cdots + \frac{c_n}{s + p_n}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & 0 \\ & -p_2 & & \\ & & \ddots & \\ & & & \ddots & \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [c_1 \quad c_2 \quad \cdots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Example:

$$\frac{Y(s)}{U(s)} = \frac{(s+5)(s+4)}{(s+1)(s+2)(s+3)} = \frac{6}{s+1} - \frac{6}{s+2} + \frac{1}{s+3}$$

$$\Rightarrow \dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$
$$y = [6 \quad -6 \quad 1] \mathbf{x}$$

Recall Advantage:

$$e^{At} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

Controllability Canonical Form (3)

Jordan Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{(s + p_1)^3 (s + p_4)(s + p_5) \dots (s + p_n)}$$

$$\frac{Y(s)}{U(s)} = b_0 + \frac{c_1}{(s + p_1)^3} + \frac{c_2}{(s + p_1)^2} + \frac{c_3}{s + p_1} + \frac{c_4}{s + p_4} + \dots + \frac{c_n}{s + p_n}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -p_1 & 1 & \vdots & & \vdots \\ 0 & 0 & -p_1 & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & -p_4 & & 0 \\ \vdots & & \vdots & & \ddots & \\ \vdots & & \vdots & & & \ddots \\ \vdots & & \vdots & & & \ddots \\ 0 & \dots & 0 & 0 & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = [c_1 \quad c_2 \quad \dots \quad c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Example:

$$G(s) = \frac{s^2 + 6s + 8}{(s + 1)^2(s + 3)}$$
$$= \frac{1.5}{(s + 1)^2} + \frac{1.25}{s + 1} - \frac{0.25}{s + 3}$$



$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$
$$y = [1.5 \quad 1.25 \quad -0.25] \mathbf{x}$$

Gramian Controllability Condition

Recall:

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t - \tau)B\mathbf{u}(\tau)d\tau$$

For simplicity, let $\mathbf{x}(0) = 0$

$$\begin{aligned}\mathbf{x}(t) &= \int_0^t \Phi(t - \tau)B\mathbf{u}(\tau)d\tau \\ &= \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\end{aligned}$$

Gramian Matrix:

$$G_c(t^*) = \int_0^{t^*} e^{A(t^*-\tau)}BB^Te^{A^T(t^*-\tau)}d\tau \quad (n \times n)$$

Theorem:

LTI system is **controllable if and only if** there is a $t^* > 0$ such that $G_c(t^*)$ has full rank (i.e., is non-singular)

LTI State-Space Systems

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

Theorem:

LTI system is controllable if and only if the Gramian matrix

$$G_c(t^*) = \int_0^{t^*} e^{A\tau} B B^T e^{A^T \tau} d\tau \quad (n \times n)$$

has full rank for any $t^* > 0$ (e. g., $t^* = +\infty$)

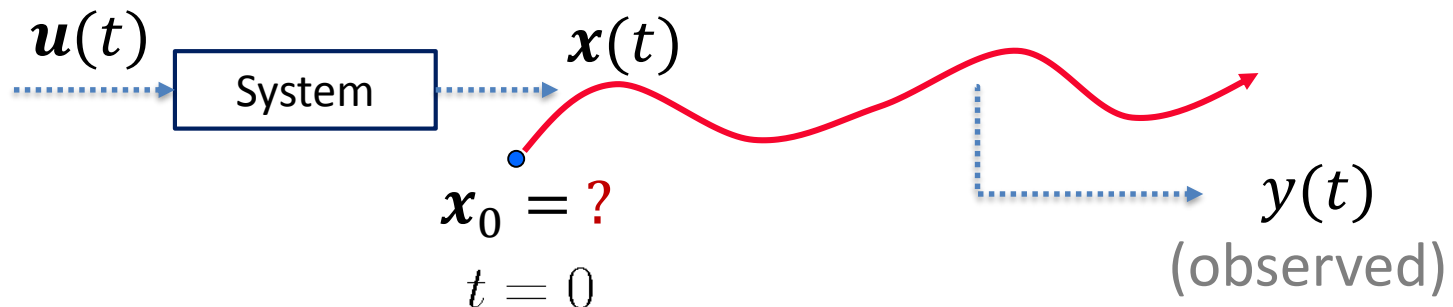
Observability

Consider LTI system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

This LTI system is said to be **observable** if given **any** input-output pair $(\mathbf{u}, \mathbf{y})$, the initial state $\mathbf{x}_0$ of the system can be **uniquely determined** (observed).



Observability

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)$$

Uniqueness means:

$$\mathbf{y}(t) = C\Phi(t)\mathbf{x}_1(0) + C \int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau + D\mathbf{u}(t)$$

$$\mathbf{y}(t) = C\Phi(t)\mathbf{x}_2(0) + C \int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau + D\mathbf{u}(t)$$

$$\rightarrow \mathbf{x}_1(0) = \mathbf{x}_2(0)$$

Condition:

There exists $t^* > 0$ such that $C\Phi(t^*)$ has full column-rank

(Proof on next page)

Proof:

Subtraction 

$$C\Phi(t)\mathbf{x}_1(0) - C\Phi(t)\mathbf{x}_2(0) = 0$$

$$\rightarrow C\Phi(t)[\mathbf{x}_1(0) - \mathbf{x}_2(0)] = 0$$

$$\rightarrow [C\Phi(t)]^T C\Phi(t)[\mathbf{x}_1(0) - \mathbf{x}_2(0)] = 0$$

$$\rightarrow [\Phi^T(t)C^T C\Phi(t)][\mathbf{x}_1(0) - \mathbf{x}_2(0)] = 0$$

If there exists $t^* > 0$ such that $C\Phi(t^*)$ has full column rank, then $[\Phi^T(t^*)C^T C\Phi(t^*)]$ has full rank (i.e., is non-singular)

$$\rightarrow [\mathbf{x}_1(0) - \mathbf{x}_2(0)] = 0$$

$$\rightarrow \mathbf{x}_1(0) = \mathbf{x}_2(0)$$

LTI State-Space Systems

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

$$\Phi(t - \tau) = e^{A(t-\tau)}$$

Theorem:

LTI system is observable **if and only if** the Gramian matrix

$$G_o(t^*) = \int_0^{t^*} e^{A^T(t^*-\tau)} C^T C e^{A(t^*-\tau)} d\tau \quad (n \times n)$$

or, simply,

$$G_o(t^*) = \int_0^{t^*} e^{A^T \tau} C^T C e^{A \tau} d\tau \quad (n \times n)$$

has full rank for any $t^* > 0$ (e. g., $t^* = +\infty$)

LTI State-Space Systems

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

Theorem

LTI system (A, B, C, D) is observable if and only if

Matrix $\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has a full column-rank

(Proof is similar to the controllability condition above)

Observability Canonical Form

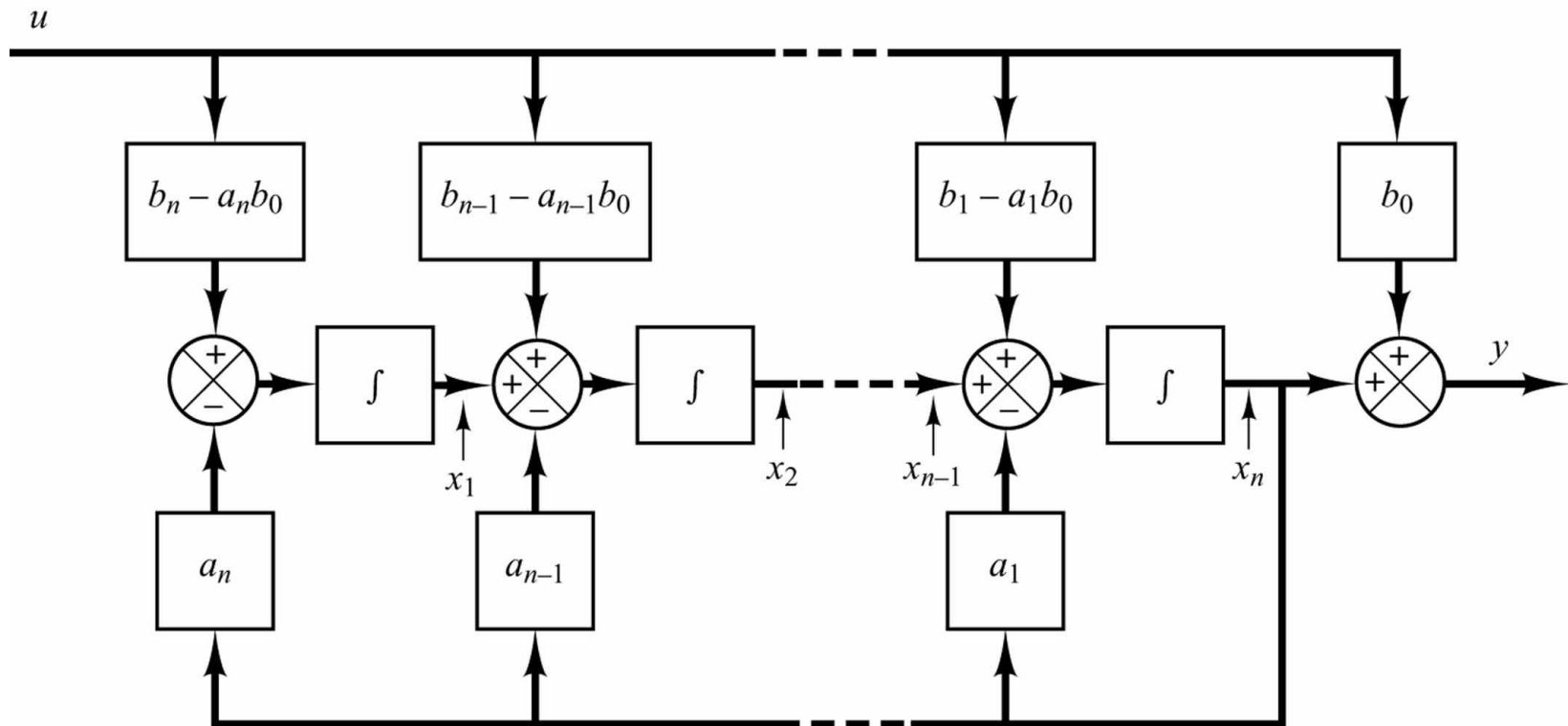
$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot & 0 & -a_n \\ 1 & 0 & 0 & \cdot & \cdot & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & 0 & -a_2 \\ 0 & 0 & 0 & \cdot & \cdot & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Note: Similarly to controllability canonical forms, there are other observability canonical forms

Block Diagram Representation (Observability Canonical Form)



Example:

$$\frac{Y(s)}{U(s)} = \frac{s + 3}{s^2 + 3s + 2}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \text{ (Full column-rank)} \rightarrow \text{Observable}$$

Recall:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

$$\begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$$

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[B \ AB \ A^2B] = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 4 \\ 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 1 & 9 & 1 \end{bmatrix} \quad (\text{Full row-rank})$$

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \quad (\text{Not full column-rank})$$

➡ Controllable but **not** Observable

Equivalence Transformation

Consider

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Equivalence Transformation: $\bar{x} = Px$ (P is nonsingular)



$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$

$$y = \bar{C}\bar{x} + \bar{D}u$$

where

$$\bar{A} = PAP^{-1}, \bar{B} = PB, \bar{C} = CP^{-1}, \bar{D} = D$$

Theorem:

The **controllability** and **observability** of an LTI system remain **unchanged** after an equivalence transformation

Proof: Because

$$[\bar{B} \quad \bar{A}\bar{B} \quad \dots \quad \bar{A}^{n-1}\bar{B}] = P[B \quad AB \quad \dots \quad A^{n-1}B]$$

$$\begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \bar{C}\bar{A}^2 \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} P^{-1}$$

Theorem:

The Transfer Matrix (Function) of an LTI system remains **unchanged** after an equivalence transformation

Recall:

$$\begin{array}{lll} \dot{x}(t) = Ax(t) + Bu(t) & \longleftrightarrow & \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\ y(t) = Cx(t) + Du(t) & x(t) = p\hat{x}(t) & y(t) = \hat{C}\hat{x}(t) + \hat{D}u(t) \end{array}$$

p is nonsingular

$$\hat{G}(s) = G(s)$$

These two LTI systems are **equivalent**

Pole-Zero Cancellation of Transfer Functions

Example:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C = [1 \quad 0], D = [0]$$

$$\text{Transfer Function: } G(s) = C(sI - A)^{-1}B = \frac{s-3}{(s-3)(s+2)} = \frac{1}{s+2}$$

- There is a pole-zero cancellation → reduced order
- Transfer function $G(s) = \frac{1}{s+3}$ is stable, but system (A, B, C, D) is not:

$$x_1(t) = x_1(0)e^{-2t} + \int_0^t e^{-2(t-\tau)} u(\tau) d\tau (= y(t))$$

$$x_2(t) = x_2(0)e^{3t} + 2 \int_0^t e^{3(t-\tau)} u(\tau) d\tau$$

Minimum Realization

Theorem:

If an LTI system is both controllable and observable, then its transfer function has no pole-zero cancelation.

- **Note:** The converse may not be true.
- **Recall:** Equivalence transformation does not change the Transfer matrix (Function) of an LTI system.
- An LTI system is called a **minimal realization** if its order cannot be reduced by any equivalence transformation.

Theorem:

LTI system (A, B, C, D) is a minimal realization **if and only if** it is both controllable and observable.

Duality

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

is controllable (observable)

if and only if

$$\dot{\mathbf{x}} = \mathbf{A}^T \mathbf{x} + \mathbf{C}^T \mathbf{u}$$

$$\mathbf{y} = \mathbf{B}^T \mathbf{x} + \mathbf{D}^T \mathbf{u}$$

is observable (controllable)

Proof:

The controllability (observability) matrix of the 1st system is the transpose of the observability (controllability) matrix of the 2nd system

State Space Decomposition

Diagonal Canonical Form

$$\begin{bmatrix} \dot{x}_{CO} \\ \dot{x}_{C\bar{O}} \\ \dot{x}_{\bar{C}O} \\ \dot{x}_{\bar{C}\bar{O}} \end{bmatrix} = \begin{bmatrix} A_{CO} & 0 & 0 & 0 \\ 0 & A_{C\bar{O}} & 0 & 0 \\ 0 & 0 & A_{\bar{C}O} & 0 \\ 0 & 0 & 0 & A_{\bar{C}\bar{O}} \end{bmatrix} \begin{bmatrix} x_{CO} \\ x_{C\bar{O}} \\ x_{\bar{C}O} \\ x_{\bar{C}\bar{O}} \end{bmatrix} + \begin{bmatrix} B_{CO} \\ B_{C\bar{O}} \\ 0 \\ 0 \end{bmatrix} u$$
$$y = [C_{CO} \quad 0 \quad C_{\bar{C}O} \quad 0]x$$

C - Controllable

$\bar{C}$ - Uncontrollable

O - Observable

$\bar{O}$ - Unobservable

State Space Decomposition

General Case

$$\begin{bmatrix} \dot{\hat{x}}_{CO} \\ \dot{\hat{x}}_{C\overline{O}} \\ \dot{\hat{x}}_{\overline{CO}} \\ \dot{\hat{x}}_{\overline{CO}} \end{bmatrix} = \begin{bmatrix} \hat{A}_{CO} & 0 & \hat{A}_{13} & 0 \\ \hat{A}_{21} & \hat{A}_{C\overline{O}} & \hat{A}_{23} & \hat{A}_{24} \\ 0 & 0 & \hat{A}_{\overline{CO}} & 0 \\ 0 & 0 & \hat{A}_{43} & \hat{A}_{\overline{CO}} \end{bmatrix} \begin{bmatrix} \hat{x}_{CO} \\ \hat{x}_{C\overline{O}} \\ \hat{x}_{\overline{CO}} \\ \hat{x}_{\overline{CO}} \end{bmatrix} + \begin{bmatrix} \hat{B}_{CO} \\ \hat{B}_{C\overline{O}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \hat{C}_{CO} & 0 & \hat{C}_{\overline{CO}} & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{CO} \\ \hat{x}_{C\overline{O}} \\ \hat{x}_{\overline{CO}} \\ \hat{x}_{\overline{CO}} \end{bmatrix} + Du$$

where

$$\left(\begin{bmatrix} \hat{A}_{CO} & 0 \\ \hat{A}_{21} & \hat{A}_{C\overline{O}} \end{bmatrix}, \begin{bmatrix} \hat{B}_{CO} \\ \hat{B}_{C\overline{O}} \end{bmatrix} \right) \text{ is controllable}$$

$$\left(\begin{bmatrix} \hat{C}_{CO} & \hat{C}_{\overline{CO}} \end{bmatrix}, \begin{bmatrix} \hat{A}_{CO} & \hat{A}_{13} \\ 0 & \hat{A}_{\overline{CO}} \end{bmatrix} \right) \text{ is observable.}$$



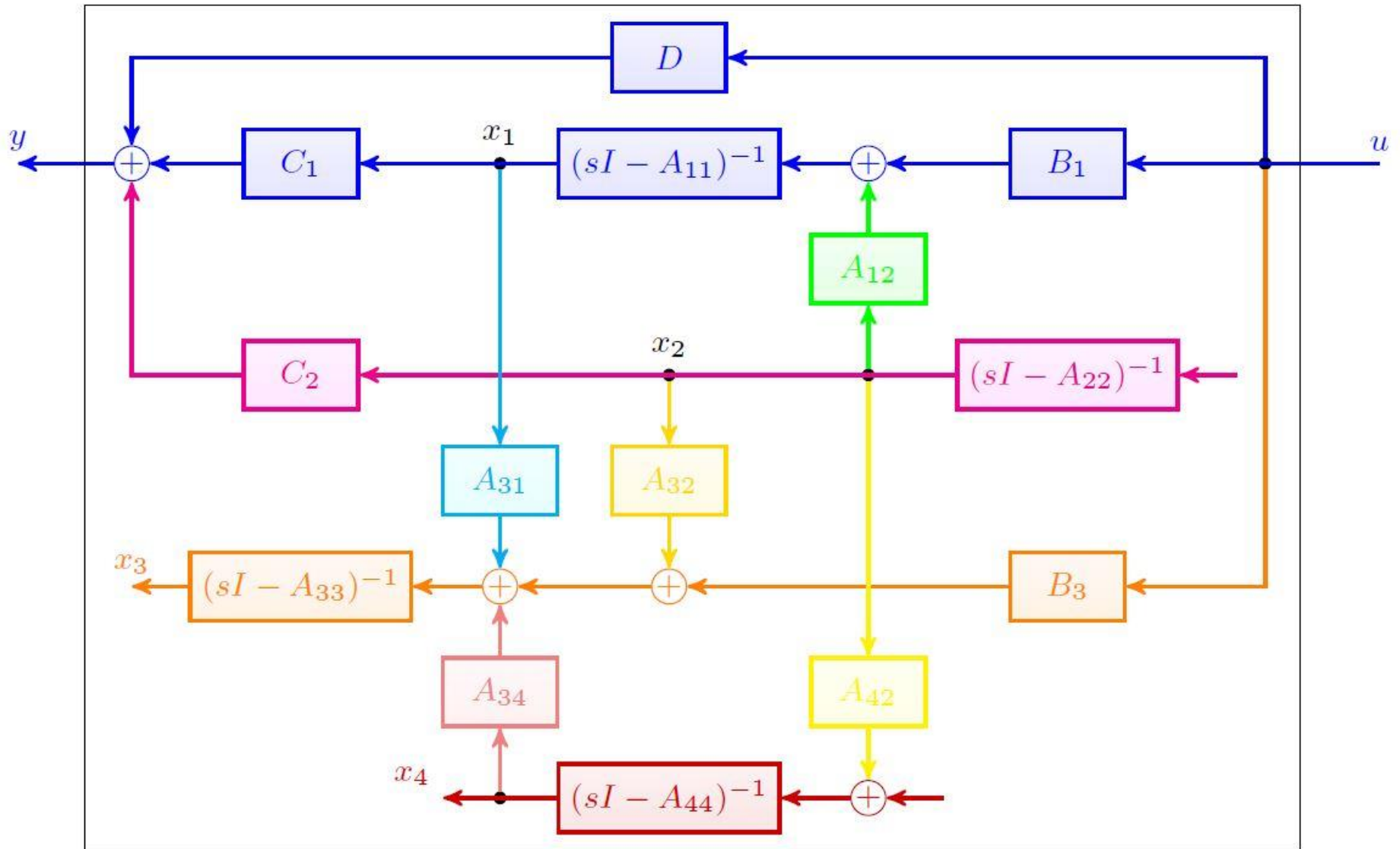
$$G(s) = C(sI - A)^{-1}B + D = \hat{C}_{CO}(sI - \hat{A}_{CO})^{-1}\hat{B}_{CO} + D$$

Transfer Matrix Decomposition

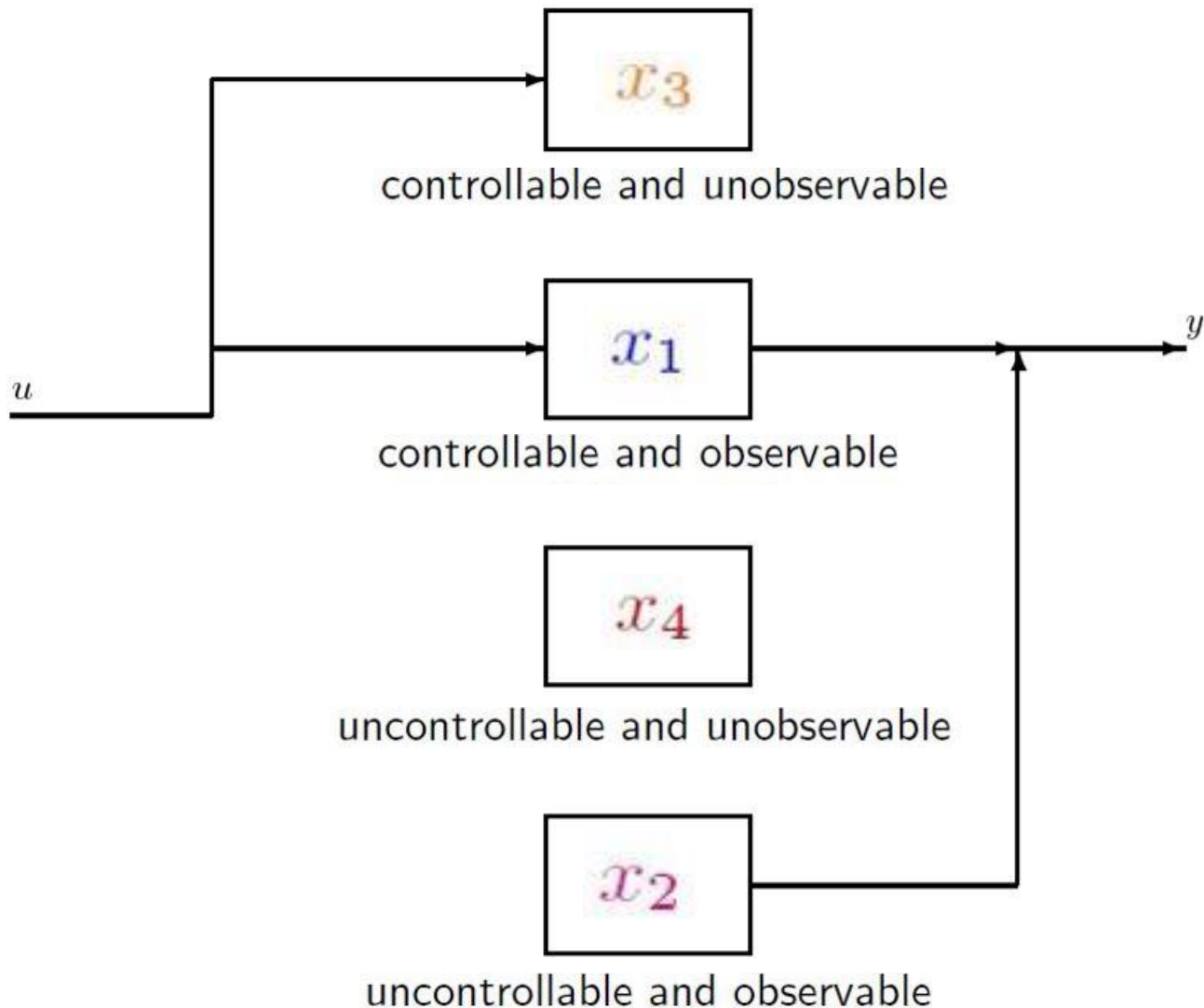
$$G(s) = \left[\begin{array}{cccc|c} A_{11} & A_{12} & 0 & 0 & B_1 \\ 0 & A_{22} & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & B_3 \\ 0 & A_{42} & 0 & A_{44} & 0 \\ \hline C_1 & C_2 & 0 & 0 & D \end{array} \right]$$

States	Decomposition
x_1	controllable and observable
x_2	uncontrollable and observable
x_3	controllable and unobservable
x_4	uncontrollable and unobservable

Block Diagram



State-Space Decomposition



LTV State-Space Systems

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\mathbf{u}$$

$$\mathbf{y} = C(t)\mathbf{x} + D(t)\mathbf{u}$$

$\mathbf{x} \in \mathbb{R}^m$ is the state vector
 $\mathbf{u} \in \mathbb{R}^n$ is the control vector
 $\mathbf{y} \in \mathbb{R}^k$ is the output vector

$A_{m \times m}$ is the system matrix
 $B_{m \times n}$ is the control matrix
 $C_{k \times m}$ is the observation matrix
 $D_{k \times n}$ is the input matrix (often, $D = 0$)

Multi-Input Multi-Output (MIMO): $n > 1, k > 1$

Single-Input Single-Output (SISO): $n = 1, k = 1$

Multi-Input Single-Output (MISO): $n > 1, k = 1$

Single-Input Multi-Output (SIMO): $n = 1, k > 1$

LTV State-Space Systems

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\mathbf{u}$$

$$\mathbf{y} = C(t)\mathbf{x} + D(t)\mathbf{u}$$

Theorem:

LTI system is **controllable if and only if** the Gramian matrix

$$G_c(t^*) = \int_0^{t^*} \Phi(t^* - \tau) B B^T \Phi^T(t^* - \tau) d\tau \quad (n \times n)$$

has full rank for any $t^* > 0$ (e.g., $t^* = +\infty$)

Note: For LTV systems, $\Phi(t - \tau) \neq e^{A(t-\tau)}$

LTV State-Space Systems

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\mathbf{u}$$

$$\mathbf{y} = C(t)\mathbf{x} + D(t)\mathbf{u}$$

Theorem:

LTV system is **observable if and only if** the Gramian matrix

$$G_o(t^*) = \int_0^{t^*} \Phi^T(t^* - \tau) C^T C \Phi(t^* - \tau) d\tau \quad (n \times n)$$

has full rank for any $t^* > 0$ (e. g., $t^* = +\infty$)

Note: For LTV systems, $\Phi(t - \tau) \neq e^{A(t-\tau)}$

END