

Problem 1

- (a) The position vectors of A , B and C are given by

$$\overrightarrow{OA} = \vec{i} + \vec{j}, \quad \overrightarrow{OB} = 2\vec{j} + 3\vec{k} \quad \text{and} \quad \overrightarrow{OC} = 2\vec{i} - \vec{j}$$

(b) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\vec{j} + 3\vec{k}) - (\vec{i} + \vec{j}) = -\vec{i} + \vec{j} + 3\vec{k}.$

$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = (\vec{i} + \vec{j}) - (2\vec{i} - \vec{j}) = -\vec{i} + 2\vec{j}.$$

- (c) Note that

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (2\vec{i} - \vec{j}) - (2\vec{j} + 3\vec{k}) = 2\vec{i} - 3\vec{j} - 3\vec{k}.$$

Since the $\vec{i}$ -component of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are not the same, thus $\overrightarrow{AB} \neq \overrightarrow{BC}$.

(d) $\widehat{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{-\vec{i} + \vec{j} + 3\vec{k}}{\sqrt{(-1)^2 + 1^2 + 3^2}} = -\frac{1}{\sqrt{11}}\vec{i} + \frac{1}{\sqrt{11}}\vec{j} + \frac{3}{\sqrt{11}}\vec{k}$

$$\widehat{BC} = \frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} = \frac{2\vec{i} - 3\vec{j} - 3\vec{k}}{\sqrt{(2)^2 + (-3)^2 + (-3)^2}} = \frac{2}{\sqrt{22}}\vec{i} - \frac{3}{\sqrt{22}}\vec{j} - \frac{3}{\sqrt{22}}\vec{k}$$

(e) (i) $\vec{a} = 3 \times (-\widehat{AB}) = \frac{3}{\sqrt{11}}\vec{i} - \frac{3}{\sqrt{11}}\vec{j} - \frac{9}{\sqrt{11}}\vec{k}$

(ii) $\vec{b} = 5 \times (\widehat{BC}) = \frac{10}{\sqrt{22}}\vec{i} - \frac{15}{\sqrt{22}}\vec{j} - \frac{15}{\sqrt{22}}\vec{k}.$

Problem 2

(a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (\vec{i} + 2\vec{j}) - (\vec{j} - \vec{k}) = \vec{i} + \vec{j} + \vec{k}.$

$$\overrightarrow{AX} = \underbrace{\frac{2}{3}|\overrightarrow{AB}|}_{\text{magnitude}} \times \underbrace{(\widehat{AB})}_{\text{direction}} = \frac{2}{3}|\overrightarrow{AB}| \times \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{2}{3}(\vec{i} + \vec{j} + \vec{k}) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}.$$

- (b) Using the fact that $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$, we have

$$\overrightarrow{OX} = \overrightarrow{AX} + \overrightarrow{OA} = \left(\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right) + (\vec{j} - \vec{k}) = \frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} + \frac{5}{3}\vec{k}.$$

Problem 3

(a) $|\vec{a}| = \sqrt{2^2 + (-3)^2 + 5^2} = \sqrt{38}$
 $|\vec{a} - 2\vec{b}| = |-9\vec{j} + 5\vec{k}| = \sqrt{(-9)^2 + 5^2} = \sqrt{106}.$

(b) $\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{i} + 3\vec{j}}{\sqrt{1^2 + 3^2}} = \frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j}.$

- (c) Note that

$$|2\vec{a} + \vec{b}| = |5\vec{i} - 3\vec{j} + 10\vec{k}| = \sqrt{5^2 + (-3)^2 + 10^2} = \sqrt{134}.$$

Thus the vector $\vec{c}$ is given by

$$\vec{c} = \underbrace{|2\vec{a} + \vec{b}|}_{\text{magnitude}} \times \underbrace{\hat{b}}_{\text{direction}} = \sqrt{134} \left(\frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j} \right) = \frac{\sqrt{134}}{\sqrt{10}}\vec{i} + \frac{3\sqrt{134}}{\sqrt{10}}\vec{j}.$$

Problem 4

(a) $\vec{a} \cdot \vec{b} = 1(-2) + 3(1) - 2(3) = -5.$

(b) Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$. Then we have

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{-5}{\sqrt{1^2 + 3^2 + (-2)^2} \sqrt{(-2)^2 + 1^2 + 3^2}} = \frac{-5}{\sqrt{14}\sqrt{14}} \Rightarrow \theta \approx 110.92^\circ.$$

(c) $\vec{c}$ and $\vec{b}$ are perpendicular, then $\vec{c} \cdot \vec{b} = |\vec{c}||\vec{b}| \cos 90^\circ = 0$

$$\Rightarrow (3\vec{i} + x\vec{j} - 2\vec{k}) \cdot (-2\vec{i} + \vec{j} + 3\vec{k}) = 0$$

$$\Rightarrow 3(-2) + x(1) + (-2)(3) = 0 \Rightarrow x = 12.$$

(d) $\vec{d} = y\vec{a} + 3\vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular, then $\vec{d} \cdot (\vec{a} - \vec{b}) = |\vec{d}||\vec{a} - \vec{b}| \cos 90^\circ = 0$

$$\Rightarrow (y\vec{a} + 3\vec{b}) \cdot (\vec{a} - \vec{b}) = 0$$

$$\Rightarrow y(\vec{a} \cdot \vec{a}) + (3 - y)(\vec{a} \cdot \vec{b}) - 3(\vec{b} \cdot \vec{b}) = 0$$

$$\Rightarrow y|\vec{a}|^2 + (3 - y)(\vec{a} \cdot \vec{b}) - 3|\vec{b}|^2 = 0$$

$$\Rightarrow y(\sqrt{14})^2 + (3 - y)(-5) - 3(\sqrt{14})^2 = 0$$

$$\Rightarrow y = 3.$$

Problem 5

(a) Let $\theta = \angle ABC$, then θ satisfies

$$\cos \theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} \dots \dots (*)$$

Since $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \vec{i} - 2\vec{k}$, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 2\vec{i} - 2\vec{k}$ and $\overrightarrow{BA} \cdot \overrightarrow{BC} = (1)(2) + (0)(0) + (-2)(-2) = 6$, then we can deduce from (*) that

$$\cos \theta = \frac{6}{\sqrt{(1)^2 + (-2)^2} \sqrt{(2)^2 + (-2)^2}} = \frac{6}{\sqrt{5}\sqrt{8}} \Rightarrow \theta \approx 18.43^\circ.$$

(b) Given that DE is perpendicular to EF , we then have $\overrightarrow{ED} \cdot \overrightarrow{EF} = |\overrightarrow{ED}||\overrightarrow{EF}| \cos 90^\circ = 0.$

Since $\begin{cases} \overrightarrow{ED} = \overrightarrow{OD} - \overrightarrow{OE} = (x-1)\vec{i} - \vec{j} \\ \overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = 3\vec{i} - 6\vec{j} - 2\vec{k} \end{cases}$ then we have

$$\overrightarrow{ED} \cdot \overrightarrow{EF} = 0 \Rightarrow (x-1)(3) + (-1)(-6) + (0)(-2) = 0 \Rightarrow x = -1.$$

Problem 6

(a) Let $\theta = \angle ABC$, θ is found to be

$$\cos \theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} \dots \dots (*)$$

Note that $\begin{cases} \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = 3\vec{i} - \vec{j} \\ \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{i} + 2\vec{j} \end{cases}$ then we obtain from (*) that

$$\cos \theta = \frac{3(1) + (-1)(2)}{\sqrt{3^2 + (-1)^2} \sqrt{1^2 + 2^2}} = \frac{1}{\sqrt{10}\sqrt{5}} \Rightarrow \theta \approx 81.87^\circ.$$

(b) We need to check whether $\angle ABD = \frac{1}{2}\angle ABC.$

Let $\phi = \angle ABD$, then we have

$$\cos \phi = \frac{\overrightarrow{BA} \cdot \overrightarrow{BD}}{|\overrightarrow{BA}| |\overrightarrow{BD}|} \dots \dots (**)$$

Since $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = 2\vec{i} + 2\vec{j}$, then we can obtain from (**) that

$$\cos \phi = \frac{3(2) + (-1)(2)}{\sqrt{10}\sqrt{8}} = \frac{4}{\sqrt{80}} \Rightarrow \phi \approx 63.43^\circ.$$

It is obvious that $\angle ABD \neq \frac{1}{2}\angle ABC$. Therefore BD is not an angle bisector of BC .

Problem 7

(Method 1)

Let $\theta = \angle BAC$, then

$$\cos \theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \frac{2}{(4)(4)} = \frac{1}{8}.$$

Using cosine law, we have

$$BC^2 = AB^2 + AC^2 - 2(AB)(AC) \cos \theta = 4^2 + 4^2 - 2(4)(4) \left(\frac{1}{8}\right) = 28 \Rightarrow BC = \sqrt{28}.$$

(Method 2)

Using the fact that $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{AC} - \overrightarrow{AB}$ and $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$, we get

$$\begin{aligned} BC &= |\overrightarrow{BC}| = \sqrt{\overrightarrow{BC} \cdot \overrightarrow{BC}} = \sqrt{(\overrightarrow{AC} - \overrightarrow{AB}) \cdot (\overrightarrow{AC} - \overrightarrow{AB})} \\ &= \sqrt{(\overrightarrow{AC} \cdot \overrightarrow{AC}) - 2(\overrightarrow{AC} \cdot \overrightarrow{AB}) + (\overrightarrow{AB} \cdot \overrightarrow{AB})} = \sqrt{|\overrightarrow{AC}|^2 - 2(\overrightarrow{AB} \cdot \overrightarrow{AC}) + |\overrightarrow{AB}|^2} = \sqrt{4^2 - 2(2) + 4^2} = \sqrt{28}. \end{aligned}$$

Problem 8

(a) Let θ be the angle between $\vec{a}$ and $\vec{b}$, then we have

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{1}{1(2)} = \frac{1}{2} \Rightarrow \theta = 30^\circ.$$

(b) Using properties of scalar product, we get

$$\begin{aligned} (3\vec{a} - 2\vec{b}) \cdot (\vec{a} + 3\vec{b}) &= 3(\vec{a} \cdot \vec{a}) + 7(\vec{a} \cdot \vec{b}) - 6(\vec{b} \cdot \vec{b}) = 3|\vec{a}|^2 + 7(\vec{a} \cdot \vec{b}) - 6|\vec{b}|^2 \\ &= 3(1)^2 + 7(1) - 6(2)^2 = -14. \\ |\vec{a} - 2\vec{b}| &= \sqrt{(\vec{a} - 2\vec{b}) \cdot (\vec{a} - 2\vec{b})} = \sqrt{(\vec{a} \cdot \vec{a}) - 4(\vec{a} \cdot \vec{b}) + 4(\vec{b} \cdot \vec{b})} = \sqrt{|\vec{a}|^2 - 4(\vec{a} \cdot \vec{b}) + 4|\vec{b}|^2} \\ &= \sqrt{1^2 - 4(1) + 4(2)^2} = \sqrt{13}. \end{aligned}$$

(c) Let ϕ be the angle between the vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$. We have

$$\cos \phi = \frac{(\vec{a} - 2\vec{b}) \cdot (2\vec{a} + 3\vec{b})}{|\vec{a} - 2\vec{b}| |2\vec{a} + 3\vec{b}|} \dots \dots (*)$$

Since

$$\begin{aligned} (\vec{a} - 2\vec{b}) \cdot (2\vec{a} + 3\vec{b}) &= 2(\vec{a} \cdot \vec{a}) - (\vec{a} \cdot \vec{b}) - 6(\vec{b} \cdot \vec{b}) = 2|\vec{a}|^2 - (\vec{a} \cdot \vec{b}) - 6|\vec{b}|^2 \\ &= 2(1)^2 - (1) - 6(2)^2 = -23. \end{aligned}$$

and

$$\begin{aligned} |2\vec{a} + 3\vec{b}| &= \sqrt{(2\vec{a} + 3\vec{b}) \cdot (2\vec{a} + 3\vec{b})} = \sqrt{4(\vec{a} \cdot \vec{a}) + 12(\vec{a} \cdot \vec{b}) + 9(\vec{b} \cdot \vec{b})} \\ &= \sqrt{4|\vec{a}|^2 + 12(\vec{a} \cdot \vec{b}) + 9|\vec{b}|^2} = \sqrt{4(1)^2 + 12(1) + 9(2)^2} = \sqrt{52}, \end{aligned}$$

then we can conclude from (*) that

$$\cos \phi = \frac{-23}{\sqrt{13}\sqrt{52}} = -\frac{23}{26} \Rightarrow \phi \approx 152.2^\circ.$$

Problem 9

(a) Let θ be the angle between $\vec{a} - 2\vec{b}$ and $-9\vec{a} + 2\vec{b}$, then we have

$$\cos \theta = \frac{(\vec{a} - 2\vec{b}) \cdot (-9\vec{a} + 2\vec{b})}{|\vec{a} - 2\vec{b}| |-9\vec{a} + 2\vec{b}|} \dots (*)$$

We first note that

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \left(\cos^{-1} \frac{3}{5} \right) = 2(3) \left(\frac{3}{5} \right) = \frac{18}{5}.$$

Then we have

$$\begin{aligned} (\vec{a} - 2\vec{b}) \cdot (-9\vec{a} + 2\vec{b}) &= -9(\vec{a} \cdot \vec{a}) + 20(\vec{a} \cdot \vec{b}) - 4(\vec{b} \cdot \vec{b}) \\ &= -9|\vec{a}|^2 + 20(\vec{a} \cdot \vec{b}) - 4|\vec{b}|^2 = -9(2)^2 + 20\left(\frac{18}{5}\right) - 4(3)^2 = 0. \end{aligned}$$

Then we obtain from (*) that

$$\cos \theta = \frac{0}{|\vec{a} - 2\vec{b}| |-9\vec{a} + 2\vec{b}|} = 0 \Rightarrow \theta = 90^\circ.$$

This shows that the two vectors are perpendicular to each other.

(b) Let ϕ be the angle between $\vec{a}$ and $\vec{a} + k\vec{b}$, then we have

$$\cos \phi = \frac{\vec{a} \cdot (\vec{a} + k\vec{b})}{|\vec{a}| |\vec{a} + k\vec{b}|} \Rightarrow \frac{\vec{a} \cdot (\vec{a} + k\vec{b})}{|\vec{a}| |\vec{a} + k\vec{b}|} = \frac{1}{2} \dots (**).$$

Note that

$$\vec{a} \cdot (\vec{a} + k\vec{b}) = |\vec{a}|^2 + k(\vec{a} \cdot \vec{b}) = 4 + \frac{18}{5}k,$$

$$|\vec{a} + k\vec{b}| = \sqrt{(\vec{a} + k\vec{b}) \cdot (\vec{a} + k\vec{b})} = \sqrt{|\vec{a}|^2 + 2k(\vec{a} \cdot \vec{b}) + k^2|\vec{b}|^2} = \sqrt{4 + \frac{36}{5}k + 9k^2},$$

then we obtain from (**) that

$$\begin{aligned} \frac{4 + \frac{18}{5}k}{2\sqrt{4 + \frac{36}{5}k + 9k^2}} &= \frac{1}{2} \Rightarrow \dots \Rightarrow 99k^2 + 540k + 300 = 0 \\ \Rightarrow k &= -\frac{40\sqrt{3}}{33} - \frac{30}{11} \text{ or } k = \frac{40\sqrt{3}}{33} - \frac{30}{11} \end{aligned}$$

Problem 10

In the following, θ refers to the angle between the vectors $\vec{a}$ and $\vec{b}$

(a) Note that

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{3(1) + (-4)(-18)}{5\sqrt{325}} = \frac{3}{\sqrt{13}} \Rightarrow \theta < 90^\circ.$$

Then the required projection vector is given by

$$proj_{\vec{b}} \vec{a} = \underbrace{(|\vec{a}| \cos \theta)}_{\text{magnitude}} \times \underbrace{(\hat{b})}_{\text{direction}} \stackrel{\hat{b} = \frac{\vec{b}}{|\vec{b}|}}{=} 5 \left(\frac{3}{\sqrt{13}} \right) \times \frac{\vec{i} - 18\vec{j}}{\sqrt{325}} = \frac{3}{13}\vec{i} - \frac{54}{13}\vec{j}.$$

(b) Note that

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{2(6) + (-3)(-2) + (-6)(11)}{\sqrt{49}\sqrt{161}} = \frac{-48}{7\sqrt{161}} \Rightarrow \theta > 90^\circ.$$

Then the required projection vector is given by

$$\begin{aligned} \text{proj}_{\vec{b}} \vec{a} &= \underbrace{(|\vec{a}| \cos(180^\circ - \theta))}_{\text{magnitude}} \times \underbrace{(-\hat{b})}_{\text{direction}} \stackrel{\hat{b} = \frac{\vec{b}}{|\vec{b}|}}{\cong} -|\vec{a}| \cos \theta \times \left(-\frac{6\vec{i} - 2\vec{j} + 11\vec{k}}{\sqrt{161}} \right) \\ &= -7 \left(\frac{-48}{7\sqrt{161}} \right) \times \frac{-6\vec{i} + 2\vec{j} - 11\vec{k}}{\sqrt{161}} = -\frac{288}{161}\vec{i} + \frac{96}{161}\vec{j} - \frac{528}{161}\vec{k}. \end{aligned}$$

(c) Note that

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{(-1)(1) + (2)(7) + (2)(7)}{3\sqrt{99}} = \frac{9}{\sqrt{99}} \Rightarrow \theta < 90^\circ.$$

Then the required projection vector is given by

$$\text{proj}_{\vec{b}} \vec{a} = \underbrace{(|\vec{a}| \cos \theta)}_{\text{magnitude}} \times \underbrace{(\hat{b})}_{\text{direction}} \stackrel{\hat{b} = \frac{\vec{b}}{|\vec{b}|}}{\cong} 3 \left(\frac{9}{\sqrt{99}} \right) \times \frac{\vec{i} + 7\vec{j} + 7\vec{k}}{\sqrt{99}} = \frac{3}{11}\vec{i} + \frac{21}{11}\vec{j} + \frac{21}{11}\vec{k}.$$

Problem 11

(a) Using the same method as in Example 10, we need to find $|\overrightarrow{AC}|$ and $|\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC}|$.

First, we note that

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -\vec{i}, \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{i} + \vec{j} + 3\vec{k}$$

It remains to find the magnitude of $\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC}$.

Let θ be the angle between $\overrightarrow{AC}$ and $\overrightarrow{AB}$

$$\cos \theta = \frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{|\overrightarrow{AC}||\overrightarrow{AB}|} = \frac{(-1)(-2) + 0(1) + 0(3)}{(1)\sqrt{14}} = \frac{2}{\sqrt{14}} \Rightarrow \theta < 90^\circ.$$

Thus the magnitude of $\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC}$ is given by

$$|\overrightarrow{AC}| \cos \theta = \sqrt{1} \left(\frac{2}{\sqrt{14}} \right) = \frac{2}{\sqrt{14}}.$$

Therefore the required distance d is given by

$$d = \sqrt{|\overrightarrow{AC}|^2 - |\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC}|^2} = \sqrt{(1)^2 - \left(\frac{2}{\sqrt{14}} \right)^2} = \sqrt{\frac{10}{14}}.$$

(b) Using the same method as in Example 10, we need to find $|\overrightarrow{DF}|$ and $|\text{proj}_{\overrightarrow{DE}} \overrightarrow{DF}|$.

First, we note that

$$\overrightarrow{DF} = \overrightarrow{OF} - \overrightarrow{OD} = -\vec{i} + 4\vec{j} - 2\vec{k}, \quad \overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = -2\vec{i} + \vec{j}$$

It remains to find the magnitude of $\text{proj}_{\overrightarrow{DE}} \overrightarrow{DF}$.

Let ϕ be the angle between $\overrightarrow{DF}$ and $\overrightarrow{DE}$

$$\cos \phi = \frac{\overrightarrow{DF} \cdot \overrightarrow{DE}}{|\overrightarrow{DF}||\overrightarrow{DE}|} = \frac{(-1)(-2) + 4(1) + (-2)(0)}{\sqrt{21}\sqrt{5}} = \frac{6}{\sqrt{105}} \Rightarrow \phi < 90^\circ.$$

Thus the magnitude of $\text{proj}_{\overrightarrow{DE}} \overrightarrow{DF}$ is given by

$$|\overrightarrow{DF}| \cos \phi = \sqrt{21} \left(\frac{6}{\sqrt{105}} \right) = \frac{6}{\sqrt{5}}.$$

Therefore the required distance d is given by

$$d = \sqrt{|\overrightarrow{DF}|^2 - |\text{proj}_{\overrightarrow{DE}} \overrightarrow{DF}|^2} = \sqrt{(\sqrt{21})^2 - \left(\frac{6}{\sqrt{5}}\right)^2} = \sqrt{\frac{69}{5}}.$$

Problem 12

- (a) $\vec{a} \times \vec{b} = (\vec{i} + 3\vec{j}) \times (-2\vec{j} + 5\vec{k}) = -2(\vec{i} \times \vec{j}) + 5(\vec{i} \times \vec{k}) - 6(\vec{j} \times \vec{j}) + 15(\vec{j} \times \vec{k})$
 $= -2\vec{k} + 5(-\vec{j}) - 6(\vec{0}) + 15\vec{i} = 15\vec{i} - 5\vec{j} - 2\vec{k}.$
- (b) $\vec{a} \times \vec{b} = (\vec{i} + \vec{j} - 2\vec{k}) \times (-3\vec{i} + 2\vec{j} + 5\vec{k})$
 $= -3(\vec{i} \times \vec{i}) + 2(\vec{i} \times \vec{j}) + 5(\vec{i} \times \vec{k}) - 3(\vec{j} \times \vec{i}) + 2(\vec{j} \times \vec{j}) + 5(\vec{j} \times \vec{k}) + 6(\vec{k} \times \vec{i}) - 4(\vec{k} \times \vec{j})$
 $- 10(\vec{k} \times \vec{k})$
 $= -3(\vec{0}) + 2\vec{k} + 5(-\vec{j}) - 3(-\vec{k}) + 2(\vec{0}) + 5\vec{i} + 6\vec{j} - 4(-\vec{i}) - 10(\vec{0}) = 9\vec{i} + \vec{j} + 5\vec{k}.$
- (c) $\vec{a} \times \vec{b} = (-3\vec{i} + \vec{j} + 3\vec{k}) \times (6\vec{j} + \vec{k})$
 $= -18(\vec{i} \times \vec{j}) - 3(\vec{i} \times \vec{k}) + 6(\vec{j} \times \vec{j}) + (\vec{j} \times \vec{k}) + 18(\vec{k} \times \vec{j}) + 3(\vec{k} \times \vec{k})$
 $= -18\vec{k} - 3(-\vec{j}) + 6(\vec{0}) + \vec{i} + 18(-\vec{i}) + 3(\vec{0}) = -17\vec{i} + 3\vec{j} - 18\vec{k}.$
- (d) $\vec{a} \times \vec{b} = (\vec{j} + \vec{k}) \times (3\vec{i} - \vec{j} + 2\vec{k})$
 $= 3(\vec{j} \times \vec{i}) - (\vec{j} \times \vec{j}) + 2(\vec{j} \times \vec{k}) + 3(\vec{k} \times \vec{i}) - (\vec{k} \times \vec{j}) + 2(\vec{k} \times \vec{k})$
 $= 3(-\vec{k}) - (\vec{0}) + 2\vec{i} + 3\vec{j} - (-\vec{i}) + 2(\vec{0}) = 3\vec{i} + 3\vec{j} + 3\vec{k}.$

Problem 13

According to the definition of vector product, the vector $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Then we conclude that

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = |\vec{a}| |\vec{a} \times \vec{b}| \cos 90^\circ = 0.$$

Problem 14

- (a) We first note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\vec{i} - 3\vec{j} - 2\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -3\vec{i} - 2\vec{j} + \vec{k} \end{cases}$. According to the definition of vector product, the vector $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Thus the required vector is found to be
- $$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= (2\vec{i} - 3\vec{j} - 2\vec{k}) \times (-3\vec{i} - 2\vec{j} + \vec{k}) \\ &= -6(\vec{i} \times \vec{i}) - 4(\vec{i} \times \vec{j}) + 2(\vec{i} \times \vec{k}) + 9(\vec{j} \times \vec{i}) + 6(\vec{j} \times \vec{j}) - 3(\vec{j} \times \vec{k}) + 6(\vec{k} \times \vec{i}) + 4(\vec{k} \times \vec{j}) \\ &\quad - 2(\vec{k} \times \vec{k}) \\ &= -6(\vec{0}) - 4(\vec{k}) + 2(-\vec{j}) + 9(-\vec{k}) + 6(\vec{0}) - 3(\vec{i}) + 6(\vec{j}) + 4(-\vec{i}) - 2(\vec{0}) \\ &= -7\vec{i} + 4\vec{j} - 13\vec{k}. \end{aligned}$$
- (b) Note that $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -5\vec{i} + \vec{j} + 3\vec{k}$, the required vector is given by
- $$\begin{aligned} \vec{a} &= |\overrightarrow{BC}| \times (\widehat{AB \times AC}) = \sqrt{35} \times \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|} = \sqrt{35} \times \frac{-7\vec{i} + 4\vec{j} - 13\vec{k}}{\sqrt{234}} \\ &= -\frac{7\sqrt{35}}{\sqrt{234}}\vec{i} + \frac{4\sqrt{35}}{\sqrt{234}}\vec{j} - \frac{13\sqrt{35}}{\sqrt{234}}\vec{k}. \end{aligned}$$

- (c) For any point $P = (x, y, z)$ in the plane, the vector $\overrightarrow{AP}$ lies on the same plane and its perpendicular to the vector $\overrightarrow{AB} \times \overrightarrow{AC}$. Note that $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x-1)\vec{i} + (y-2)\vec{j} + z\vec{k}$, then we have

$$\begin{aligned}\overrightarrow{AP} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) &= 0 \\ \Rightarrow -7(x-1) + 4(y-2) - 13(z) &= 0 \\ \Rightarrow 7x - 4y + 13z &= -1.\end{aligned}$$

Thus the equation of plane is $7x - 4y + 13z = -1$.

Problem 15

- (a) According to the definition, the vector $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. The required vector is found to be

$$\begin{aligned}\vec{c} &= \vec{a} \times \vec{b} = (2\vec{i} - \vec{j} + 2\vec{k}) \times (4\vec{i} - 4\vec{j} + 3\vec{k}) \\ &= 8(\vec{i} \times \vec{i}) - 8(\vec{i} \times \vec{j}) + 6(\vec{i} \times \vec{k}) - 4(\vec{j} \times \vec{i}) + 4(\vec{j} \times \vec{j}) - 3(\vec{j} \times \vec{k}) + 8(\vec{k} \times \vec{i}) - 8(\vec{k} \times \vec{j}) \\ &\quad + 6(\vec{k} \times \vec{k}) \\ &= 8(\vec{0}) - 8(\vec{k}) + 6(-\vec{j}) - 4(-\vec{k}) + 4(\vec{0}) - 3(\vec{i}) + 8(\vec{j}) - 8(-\vec{i}) + 6(\vec{0}) \\ &= 5\vec{i} + 2\vec{j} - 4\vec{k}.\end{aligned}$$

- (b) The area of the triangle is

$$= \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} \sqrt{5^2 + 2^2 + (-4)^2} = \frac{\sqrt{45}}{2}.$$

- (c) For any point $P = (x, y, z)$ in the plane, the vector $\overrightarrow{AP}$ lies on the same plane and its perpendicular to the vector $\vec{c} = \vec{a} \times \vec{b}$. Note that $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x-1)\vec{i} + (y-1)\vec{j} + (z-1)\vec{k}$, then we have

$$\begin{aligned}\overrightarrow{AP} \cdot \vec{c} &= 0 \\ \Rightarrow 5(x-1) + 2(y-1) - 4(z-1) &= 0 \\ \Rightarrow 5x + 2y - 4z &= 3.\end{aligned}$$

Thus the equation of plane is $5x + 2y - 4z = 3$.

- (d) The volume of parallelepiped is given by

$$V = |\vec{a} \cdot (\vec{a} \times \vec{b})| = |(\vec{i} + 2\vec{k}) \cdot (5\vec{i} + 2\vec{j} - 4\vec{k})| = |(1)(5) + (0)(2) + 2(-4)| = |-3| = 3.$$

Since the volume is non-zero, thus the vectors $\vec{a}$, $\vec{b}$ and $\vec{d}$ are not coplanar.

Problem 16

Note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -3\vec{i} + 8\vec{j} - 5\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -12\vec{i} + 4\vec{j} - 6\vec{k} \end{cases}$ and

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (-3\vec{i} + 8\vec{j} - 5\vec{k}) \times (-12\vec{i} + 4\vec{j} - 6\vec{k}) \\ &= 36(\vec{i} \times \vec{i}) - 12(\vec{i} \times \vec{j}) + 18(\vec{i} \times \vec{k}) - 96(\vec{j} \times \vec{i}) + 32(\vec{j} \times \vec{j}) - 48(\vec{j} \times \vec{k}) + 60(\vec{k} \times \vec{i}) - 20(\vec{k} \times \vec{j}) \\ &\quad + 30(\vec{k} \times \vec{k}) \\ &= 36(\vec{0}) - 12(\vec{k}) + 18(-\vec{j}) - 96(-\vec{k}) + 32(\vec{0}) - 48(\vec{i}) + 60(\vec{j}) - 20(-\vec{i}) - 30(\vec{0}) \\ &= -28\vec{i} + 42\vec{j} + 84\vec{k}.\end{aligned}$$

Then

- the area of $\Delta ABC = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \sqrt{(-28)^2 + 42^2 + 84^2} = 49$.

- the area of parallelogram $= |\overrightarrow{AB} \times \overrightarrow{AC}| = 98$.

Problem 17

Recall that the points A , B and C are coplanar if and only if $|\overrightarrow{AB} \times \overrightarrow{AC}| = 0$

- (a) We note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\vec{i} + 4\vec{j} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\vec{i} + \vec{j} - \vec{k} \end{cases}$ and $\overrightarrow{AB} \times \overrightarrow{AC} = \dots = -4\vec{i} + 3\vec{j} - 5\vec{k}$. Since $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-4)^2 + 3^2 + (-5)^2} = \sqrt{50} \neq 0$, so A , B and C are not coplanar.
- (b) We note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{i} - \vec{j} + 3\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\vec{i} + \vec{j} - 3\vec{k} \end{cases}$ and $\overrightarrow{AB} \times \overrightarrow{AC} = \dots = 0\vec{i} + 0\vec{j} + 0\vec{k}$. Since $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{0^2 + 0^2 + 0^2} = 0$, so A , B and C are coplanar.

Problem 18

Recall that the points A , B , C and D are coplanar if and only if the volume of parallelepiped with A , B , C and D as adjacent vertices is 0, i.e.

$$|\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})| = 0 \Leftrightarrow \overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = 0.$$

- (a) Note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{i} + 2\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -4\vec{i} - 2\vec{j} + 6\vec{k} \\ \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 2\vec{j} - 2\vec{k} \end{cases}$, then $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \dots = \overrightarrow{AB} \cdot (-8\vec{i} - 8\vec{j} - 8\vec{k}) = (-2\vec{i} + 2\vec{k}) \cdot (-8\vec{i} - 8\vec{j} - 8\vec{k}) = (-2)(-8) + 0(-8) + 2(-8) = 0$. Therefore, the four points are coplanar.
- (b) Note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{j} + 2\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -2\vec{i} - \vec{j} + \vec{k} \\ \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \vec{i} - 2\vec{j} + \vec{k} \end{cases}$, then $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \dots = \overrightarrow{AB} \cdot (\vec{i} + 3\vec{j} + 5\vec{k}) = (-2\vec{j} + 2\vec{k}) \cdot (\vec{i} + 3\vec{j} + 5\vec{k}) = (0)(1) + (-2)(3) + 2(5) = 4$. Therefore, the four points are not coplanar.

Problem 19

- (a) Following the procedure as in Example 15, we need to obtain the vectors $\overrightarrow{AD}$ and $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$, one can find that

- $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -2\vec{i} + 2\vec{j} - \vec{k}$,
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{i} + 2\vec{j} + 3\vec{k}$ and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -2\vec{i} + \vec{j} + \vec{k}$
 $\Rightarrow \vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \dots = -\vec{i} - 5\vec{j} + 3\vec{k}$.

Let θ be the angle between $\overrightarrow{AD}$ and $\vec{n}$, then we have

$$\cos \theta = \frac{\overrightarrow{AD} \cdot \vec{n}}{|\overrightarrow{AD}| |\vec{n}|} = \frac{(-2)(-1) + (2)(-5) + (-1)(3)}{3\sqrt{35}} = \frac{-11}{3\sqrt{35}}.$$

Then the required distance is simply the magnitude of the projection vector $proj_{\vec{n}} \overrightarrow{AD}$ and is given by

$$||\overrightarrow{AD}| \cos \theta| = |3 \left(\frac{-11}{3\sqrt{35}} \right)| = \frac{11}{\sqrt{35}}.$$

(b) We first note that $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \vec{i} + 2\vec{j} - 7\vec{k}$.

Let θ be the angle between $\overrightarrow{BA}$ and $\vec{n}$, then we have

$$\cos \theta = \frac{\overrightarrow{BA} \cdot \vec{n}}{|\overrightarrow{BA}| |\vec{n}|} = \frac{(1)(-1) + 2(-1) + (-7)(-1)}{\sqrt{54}\sqrt{3}} = \frac{4}{\sqrt{162}}$$

The required distance is the magnitude of the projection vector $proj_{\vec{n}} \overrightarrow{BA}$ and is given by

$$|\overrightarrow{BA}| \cos \theta = \sqrt{54} \left(\frac{4}{\sqrt{162}} \right) = \frac{4}{\sqrt{3}}.$$

Problem 20

(a) We let $A = (5, 0, -1)$ and $B = (6, 2, -2)$ be two points on the line L_1 . We also let $C = (2, 4, 0)$ and $D = (3, 3, 1)$ be two points on the line L_2 .

Following the procedure as in Example 16, we need to obtain the vectors $\overrightarrow{AD}$ and $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.

- $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -2\vec{i} + 3\vec{j} + 2\vec{k}$.
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{i} + 2\vec{j} - \vec{k}$ and $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \vec{i} - \vec{j} + \vec{k}$.

The normal vector $\vec{n}$ is found to be

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD} = (\vec{i} + 2\vec{j} - \vec{k}) \times (\vec{i} - \vec{j} + \vec{k}) = \vec{i} - 2\vec{j} - 3\vec{k}.$$

Let θ be the angle between $\overrightarrow{AD}$ and $\vec{n}$, then we have

$$\cos \theta = \frac{\overrightarrow{AD} \cdot \vec{n}}{|\overrightarrow{AD}| |\vec{n}|} = \frac{(-2)(1) + (3)(-2) + (2)(-3)}{\sqrt{17}\sqrt{14}} = \frac{-14}{\sqrt{17}\sqrt{14}}.$$

Then the required distance is simply the magnitude of the projection vector $proj_{\vec{n}} \overrightarrow{DA}$ and is given by

$$|\overrightarrow{DA}| \cos \theta = |\sqrt{17} \left(\frac{-14}{\sqrt{17}\sqrt{14}} \right)| = \sqrt{14}.$$

(b) We let $A = (1, 1, 1)$ and $B = (2, 1, 2)$ be two points on the line L_1 . We also let $C = (2, 1, 0)$ and $D = (3, 2, 0)$ be two points on the line L_2 .

Following the procedure as in Example 16, we need to obtain the vectors $\overrightarrow{AD}$ and $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.

- $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 2\vec{i} + \vec{j} - \vec{k}$.
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{i} + \vec{k}$ and $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \vec{i} + \vec{j}$.

The normal vector $\vec{n}$ is found to be

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD} = (\vec{i} + \vec{k}) \times (\vec{i} + \vec{j}) = -\vec{i} + \vec{j} + \vec{k}.$$

Let θ be the angle between $\overrightarrow{AD}$ and $\vec{n}$, then we have

$$\cos \theta = \frac{\overrightarrow{AD} \cdot \vec{n}}{|\overrightarrow{AD}| |\vec{n}|} = \frac{(2)(-1) + (1)(1) + (-1)(1)}{\sqrt{6}\sqrt{3}} = \frac{-2}{\sqrt{18}}.$$

Then the required distance is simply the magnitude of the projection vector $proj_{\vec{n}} \overrightarrow{DA}$ and is given by

$$|\overrightarrow{DA}| \cos \theta = |\sqrt{6} \left(\frac{-2}{\sqrt{18}} \right)| = \frac{2}{\sqrt{3}}.$$

Problem 21

(a) Since

$$|\vec{a} \times \vec{b}| = |(\vec{i} - 2\vec{j}) \times (2\vec{i} + \vec{j})| = |5\vec{k}| = 5 \neq 0,$$

thus $\vec{a}$ and $\vec{b}$ are not collinear and these two vectors are linearly independent.

(b) Note that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (-17\vec{i} + 9\vec{j} - 11\vec{k}) = (\vec{i} - 2\vec{j} + 3\vec{k}) \cdot (-17\vec{i} + 9\vec{j} - 11\vec{k}) \\ = (1)(-17) - 2(9) + (3)(-11) = -68 \neq 0.$$

The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are not coplanar, these three vectors are linearly independent.

(c) Note that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (-4\vec{i} - 8\vec{j} - 4\vec{k}) = (\vec{i} + 2\vec{j} - 5\vec{k}) \cdot (-4\vec{i} - 8\vec{j} - 4\vec{k}) \\ = (1)(-4) + 2(-8) + (-5)(-4) = 0.$$

The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, these three vectors are linearly dependent.

Problem 22

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent, then we may have

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0 \Rightarrow \vec{a} \cdot (-5m\vec{i} - 10\vec{j} - 10m\vec{k}) = 0 \\ \Rightarrow ((1-m)\vec{i} + 6\vec{j} + 5\vec{k}) \cdot (-5m\vec{i} - 10\vec{j} - 10m\vec{k}) = 0 \\ \Rightarrow -5m(1-m) + 6(-10) + 5(-10m) = 0 \\ \Rightarrow m^2 - 11m - 12 = 0 \Rightarrow (m-12)(m+1) = 0 \\ \Rightarrow m = 12 \text{ or } m = -1.$$

Problem 23

$$(a) \quad (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = \underbrace{(\vec{a} \times \vec{a})}_{\vec{0}} + (\vec{a} \times \vec{b}) - (\vec{b} \times \vec{a}) - \underbrace{(\vec{b} \times \vec{b})}_{\vec{0}} = (\vec{a} \times \vec{b}) - (-(\vec{a} \times \vec{b})) \\ = 2(\vec{a} \times \vec{b}).$$

(b) Given that $\vec{a}$ and $\vec{b}$ are perpendicular, we have $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos 90^\circ = 0$.

$$|\vec{a} + \vec{b}| = \sqrt{(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})} = \sqrt{(\vec{a} \cdot \vec{a}) + 2(\vec{a} \cdot \vec{b}) + (\vec{b} \cdot \vec{b})} \\ \stackrel{\vec{a} \cdot \vec{b} = 0}{\cong} \sqrt{(\vec{a} \cdot \vec{a}) - 2(\vec{a} \cdot \vec{b}) + (\vec{b} \cdot \vec{b})} = \sqrt{(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})} = |\vec{a} - \vec{b}|.$$

(c) If $\vec{a}$ and $\vec{b}$ are parallel, the angle between these two vectors is either 0° or 180° . For either case, we always have $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \underbrace{\sin \theta}_{=0} \hat{n} = \vec{0}$.

We consider

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) \stackrel{\text{from (a)}}{\cong} 2(\vec{a} \times \vec{b}) = \vec{0}.$$

Since the cross product is $\vec{0}$, the vector $\vec{a} - \vec{b}$ and $\vec{a} + \vec{b}$ may be collinear and hence are parallel.

$$(d) \quad \frac{|\vec{a} \times \vec{b}|}{\vec{a} \cdot \vec{b}} = \frac{|\vec{a}||\vec{b}| \sin \theta}{|\vec{a}||\vec{b}| \cos \theta} = \tan \theta.$$

$$(e) \quad |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2 \theta = |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2 \theta) = |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2 \theta \\ = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2.$$