

EE2104 Introduction to Electromagnetics
Assignment 2 Solution

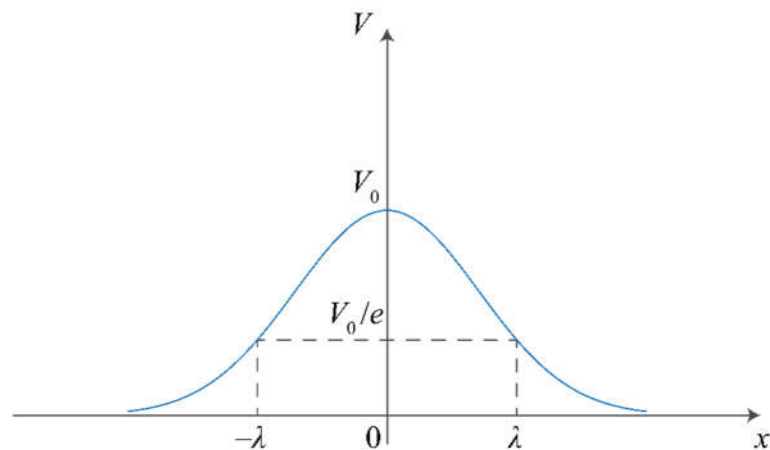
Deadline (for uploading): 7:00 pm, 10 Oct 2020
Please put down your name and university ID.

1. Consider a voltage $V(x, y) = V_0 \exp[-(x^2 + y^2)/\lambda^2]$, where V_0 is a constant in volts and λ is a constant in meters.
- (a) Is $V(x, y)$ a scalar field or a vector field?
 - (b) Plot V versus x at $y = 0$. Label the axes accurately.
 - (c) Plot V versus y at $x = 0$. Label the axes accurately.
 - (d) Plot the contour lines on the xy -plane for $V = 0.2V_0, 0.4V_0, 0.6V_0, 0.8V_0$, and V_0 .
 - (e) Find $\mathbf{E} = -\text{grad}(V)$.

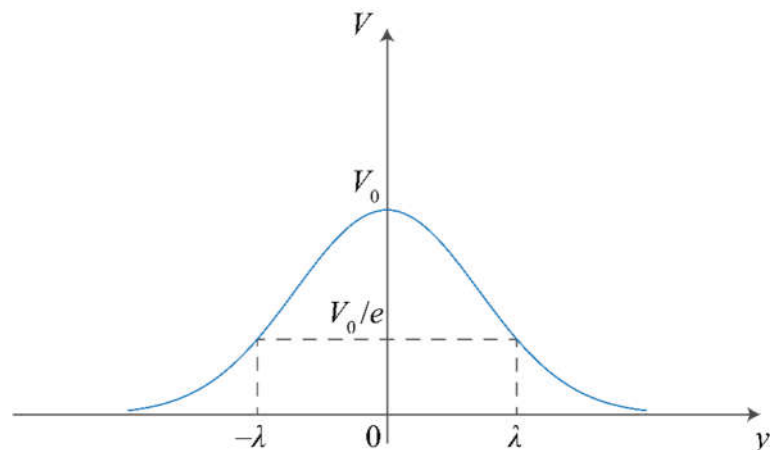
Solution:

(a) $V(x, y)$ is a scalar field.

(b) When $y = 0$, we have $V(x, 0) = V_0 \exp(-x^2/\lambda^2)$



(c) When $x = 0$, we have $V(0, y) = V_0 \exp(-y^2/\lambda^2)$



(d) When $V(x, y) = 0.2V_0$, we have

$$V_0 \exp[-(x^2 + y^2)/\lambda^2] = 0.2V_0$$

$$\exp[-(x^2 + y^2)/\lambda^2] = 0.2$$

$$x^2 + y^2 = 1.61\lambda^2$$

So the contour line on the xy -plane is a circle for $V = 0.2V_0$.

Similarly, we have another three circles and a point on the xy -plane

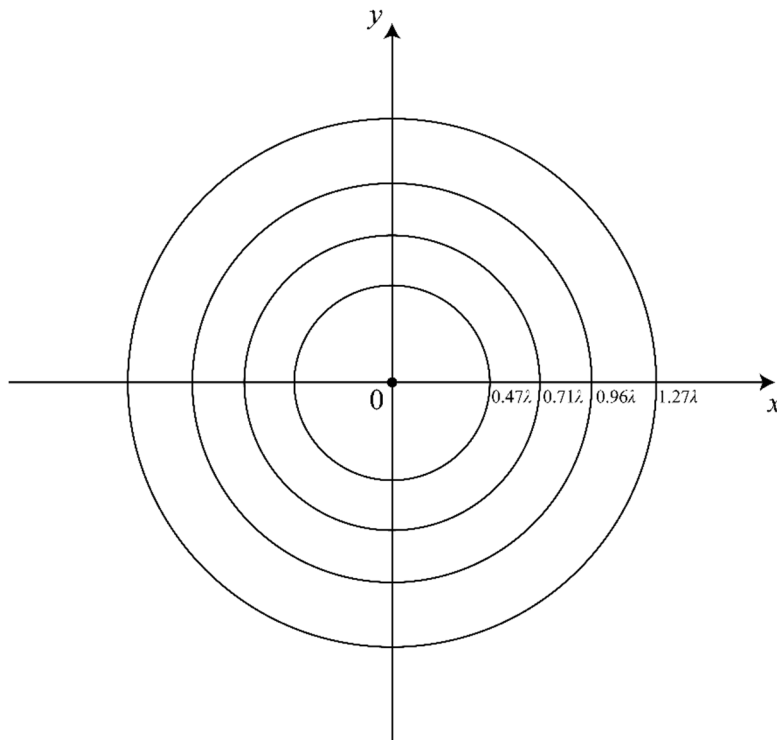
$$x^2 + y^2 = 0.92\lambda^2 \text{ for } V = 0.4V_0$$

$$x^2 + y^2 = 0.51\lambda^2 \text{ for } V = 0.6V_0$$

$$x^2 + y^2 = 0.22\lambda^2 \text{ for } V = 0.8V_0$$

$$x^2 + y^2 = 0 \Rightarrow x = 0, y = 0 \text{ for } V = V_0$$

Therefore, the contour lines on the xy -plane is plotted in the following figure.



(e) $\mathbf{E} = -\text{grad}(V)$

$$= -\left(\frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}\right)$$

$$= \frac{2V_0x}{\lambda^2} \exp[-(x^2 + y^2)/\lambda^2] \hat{\mathbf{x}} + \frac{2V_0y}{\lambda^2} \exp[-(x^2 + y^2)/\lambda^2] \hat{\mathbf{y}}$$

2. Consider a line at $x = 0$ to L . It contains electric charges of density $\rho_L = \rho_{L0} (x/L)^2$ for some constant ρ_{L0} in C/m. Find the total charge.

Solution:

The total charge is

$$\begin{aligned} Q &= \int_0^L \rho_L dx \\ &= \int_0^L \rho_{L0} (x/L)^2 dx \\ &= \int_0^L \rho_{L0} (x^2/L^2) dx \\ &= \frac{\rho_{L0}}{L^2} \int_0^L x^2 dx \\ &= \frac{\rho_{L0}}{L^2} \cdot \left(\frac{x^3}{3} \right) \Big|_{x=0}^L \\ &= \frac{\rho_{L0}}{L^2} \cdot \left(\frac{L^3}{3} - \frac{0}{3} \right) \\ &= \frac{\rho_{L0}}{L^2} \cdot \frac{L^3}{3} \\ &= \frac{\rho_{L0} L}{3} \text{ (C)} \end{aligned}$$

3. (a) Point charges of 50 nC each are located at $A(1, 0, 0)$, $B(-1, 0, 0)$, $C(0, 1, 0)$, and $D(0, -1, 0)$ in free space. Find the total force on the charge at A .
 (b) Let a point charge $Q_1 = 25$ nC be located at $P_1(4, -2, 7)$ and a charge $Q_2 = 60$ nC be at $P_2(-3, 4, -2)$ in free space. Find $\mathbf{E}$ at $P_3(1, 2, 3)$.

Solution:

- (a) The force will be

$$\mathbf{F} = \mathbf{F}_{BA} + \mathbf{F}_{CA} + \mathbf{F}_{DA} = \frac{Q^2}{4\pi\epsilon_0|\mathbf{R}_{BA}|^2} \frac{\mathbf{R}_{BA}}{|\mathbf{R}_{BA}|} + \frac{Q^2}{4\pi\epsilon_0|\mathbf{R}_{CA}|^2} \frac{\mathbf{R}_{CA}}{|\mathbf{R}_{CA}|} + \frac{Q^2}{4\pi\epsilon_0|\mathbf{R}_{DA}|^2} \frac{\mathbf{R}_{DA}}{|\mathbf{R}_{DA}|}$$

where $\mathbf{R}_{BA}$, $\mathbf{R}_{CA}$, and $\mathbf{R}_{DA}$ are vectors pointing to A from B , C , and D , respectively.

So we have $\mathbf{R}_{BA} = 2\hat{\mathbf{x}}$, $\mathbf{R}_{CA} = \hat{\mathbf{x}} - \hat{\mathbf{y}}$, and $\mathbf{R}_{DA} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$

By substituting $Q = 50$ nC, we have

$$\mathbf{F} = \frac{Q^2}{4\pi\epsilon_0} \left[\frac{\mathbf{R}_{BA}}{|\mathbf{R}_{BA}|^3} + \frac{\mathbf{R}_{CA}}{|\mathbf{R}_{CA}|^3} + \frac{\mathbf{R}_{DA}}{|\mathbf{R}_{DA}|^3} \right] = 21.5\hat{\mathbf{x}} \text{ (}\mu\text{N)}$$

- (b) The field $\mathbf{E}$ at $P_3(1, 2, 3)$ is

$$\mathbf{E} = \mathbf{E}_{13} + \mathbf{E}_{23} = \frac{Q_1}{4\pi\epsilon_0|\mathbf{R}_{13}|^2} \frac{\mathbf{R}_{13}}{|\mathbf{R}_{13}|} + \frac{Q_2}{4\pi\epsilon_0|\mathbf{R}_{23}|^2} \frac{\mathbf{R}_{23}}{|\mathbf{R}_{23}|}$$

where $\mathbf{R}_{13}$ and $\mathbf{R}_{23}$ are vectors pointing to P_3 from P_1 and P_2 , respectively.

So we have $\mathbf{R}_{13} = -3\hat{\mathbf{x}} + 4\hat{\mathbf{y}} - 4\hat{\mathbf{z}}$ and $\mathbf{R}_{23} = 4\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 5\hat{\mathbf{z}}$

Then we have

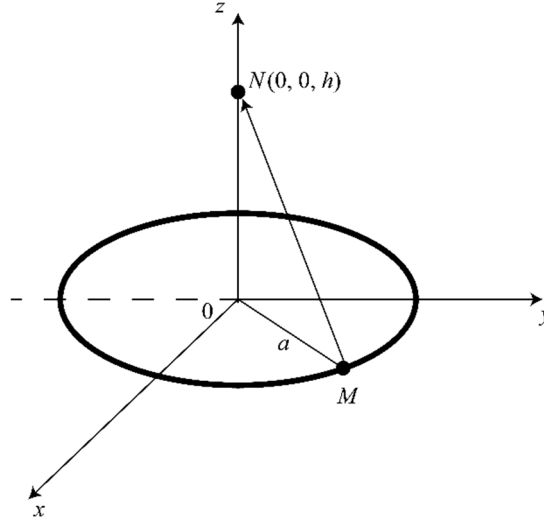
$$\mathbf{E} = \frac{Q_1\mathbf{R}_{13}}{4\pi\epsilon_0|\mathbf{R}_{13}|^3} + \frac{Q_2\mathbf{R}_{23}}{4\pi\epsilon_0|\mathbf{R}_{23}|^3} = 4.58\hat{\mathbf{x}} - 0.15\hat{\mathbf{y}} + 5.51\hat{\mathbf{z}} \text{ (V/m)}$$

4. From the Coulomb's law, derive the electric field measured at $(x, y, z) = (0, 0, h)$ from a charged circle centered at the origin. The circle is oriented along the xy -plane with radius a . The line charge density is given by

$$\rho_L(\phi) = \rho_{L0} \cos^2 2\phi$$

where ϕ is measured from the x -axis and ρ_{L0} (C/m) is a constant.

Solution:



Considering a differential angle $d\phi$, a differential arc length on the charged circle is expressed as

$$dl = a \cdot d\phi$$

Therefore, at position $M(a\cos\phi, a\sin\phi, 0)$ on the circle, the differential arc dl carries a charge of

$$dQ = \rho_L(\phi) \cdot dl = \rho_{L0} \cos^2 2\phi \cdot dl = \rho_{L0} \cos^2 2\phi \cdot (a \cdot d\phi) = a\rho_{L0} \cos^2 2\phi d\phi$$

By Coulomb's law, the differential electric field at $N(0, 0, h)$ is

$$d\mathbf{E} = \frac{dQ}{4\pi\epsilon_0 |\mathbf{R}_{MN}|^2} \frac{\mathbf{R}_{MN}}{|\mathbf{R}_{MN}|} = \frac{dQ \mathbf{R}_{MN}}{4\pi\epsilon_0 |\mathbf{R}_{MN}|^3} = \frac{a\rho_{L0} \cos^2 2\phi d\phi \cdot (-a\cos\phi \hat{x} - a\sin\phi \hat{y} + h\hat{z})}{4\pi\epsilon_0 (\sqrt{a^2 + h^2})^3}$$

where $\mathbf{R}_{MN}$ is the vector pointing to N from M .

(i) The x -component of the total electric field at position N is

$$E_x = \int_{\phi=0}^{2\pi} dE_x = \int_{\phi=0}^{2\pi} \frac{a\rho_{L0} \cos^2 2\phi \cdot (-a\cos\phi)}{4\pi\epsilon_0 (\sqrt{a^2 + h^2})^3} d\phi = \frac{-a^2 \rho_{L0}}{4\pi\epsilon_0 (a^2 + h^2)^{\frac{3}{2}}} \int_{\phi=0}^{2\pi} \cos^2 2\phi \cos\phi d\phi$$

where

$$\cos^2 2\phi \cos\phi = \frac{1}{2} \cos\phi + \frac{1}{4} (\cos 5\phi + \cos 3\phi)$$

Then we have

$$\int_{\phi=0}^{2\pi} \cos^2 2\phi \cos\phi d\phi = \int_{\phi=0}^{2\pi} \left(\frac{1}{2} \cos\phi + \frac{1}{4} (\cos 5\phi + \cos 3\phi) \right) d\phi = 0$$

Thus, we have

$$E_x = 0$$

(ii) Similarly, the y -component of the total electric field at position N is

$$E_y = \int_{\phi=0}^{2\pi} dE_y = \int_{\phi=0}^{2\pi} \frac{a\rho_{L0} \cos^2 2\phi \cdot (-a \sin \phi)}{4\pi\epsilon_0(\sqrt{(a^2 + h^2)})^3} d\phi = \frac{-a^2\rho_{L0}}{4\pi\epsilon_0(a^2 + h^2)^{\frac{3}{2}}} \int_{\phi=0}^{2\pi} \cos^2 2\phi \sin \phi d\phi$$

where

$$\cos^2 2\phi \sin \phi = \frac{1}{2} \sin \phi + \frac{1}{4} (\sin 5\phi - \sin 3\phi)$$

Then we have

$$\int_{\phi=0}^{2\pi} \cos^2 2\phi \sin \phi d\phi = \int_{\phi=0}^{2\pi} \frac{1}{2} \sin \phi + \frac{1}{4} (\sin 5\phi - \sin 3\phi) d\phi = 0$$

Thus,

$$E_y = 0$$

(iii) The z -component of the total electric field at position N is

$$E_z = \int_{\phi=0}^{2\pi} dE_z = \int_{\phi=0}^{2\pi} \frac{a\rho_{L0} \cos^2 2\phi \cdot h}{4\pi\epsilon_0(\sqrt{(a^2 + h^2)})^3} d\phi = \frac{ah\rho_{L0}}{4\pi\epsilon_0(a^2 + h^2)^{\frac{3}{2}}} \int_{\phi=0}^{2\pi} \cos^2 2\phi d\phi$$

Therefore, we have

$$E_z = \frac{ah\rho_{L0}}{4\epsilon_0(a^2 + h^2)^{\frac{3}{2}}}$$

So the electric field at $(x, y, z) = (0, 0, h)$ is

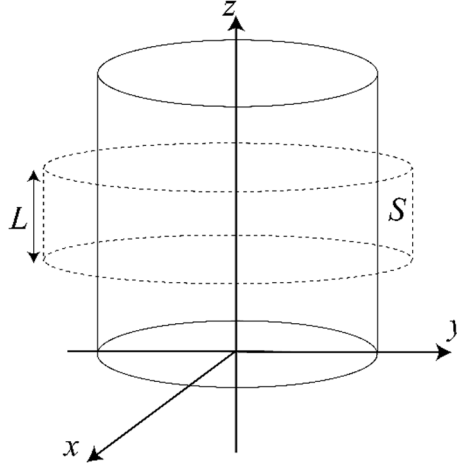
$$\mathbf{E} = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} + E_z \hat{\mathbf{z}} = \frac{ah\rho_{L0}}{4\epsilon_0(a^2 + h^2)^{\frac{3}{2}}} \hat{\mathbf{z}}$$

5. Consider a rod of radius a centered along the z -axis with charge of density

$$\rho_v(\rho, \phi, z) = \rho_0(1 - e^{-\rho^2/a^2})$$

where ρ_0 (C/m³) is a constant. Using Gauss's law, find $\mathbf{E}$ as a function of ρ .

Solution:



By symmetry, the $\mathbf{D}$ -field points away from the z -axis, in parallel to xy -plane. Also, $\mathbf{D}$ -field is independent of (ϕ, z) .

Thus,

$$\mathbf{D}(\rho, \phi, z) = D(\rho)\hat{\boldsymbol{\rho}}$$

Consider a closed cylinder of height L and radius ρ as the Gaussian surface S . Therefore, the differential surface on S can be expressed as:

For the top surface: $d\mathbf{S} = \hat{\mathbf{z}}dS$

For the curved surface: $d\mathbf{S} = \hat{\boldsymbol{\rho}}dS$

For the bottom surface: $d\mathbf{S} = -\hat{\mathbf{z}}dS$

From Gauss's law, we have

$$Q_{\text{enc}} = \oint_S \mathbf{D}(\rho, \phi, z) \cdot d\mathbf{S}$$

Thus,

$$Q_{\text{enc}} = \int_{\text{Top Surf.}} \mathbf{D}(\rho) \cdot (\hat{\mathbf{z}}dS) + \int_{\text{Curved Surf.}} \mathbf{D}(\rho) \cdot (\hat{\boldsymbol{\rho}}dS) + \int_{\text{Bottom Surf.}} \mathbf{D}(\rho) \cdot (-\hat{\mathbf{z}}dS)$$

So we have

$$Q_{\text{enc}} = \int_{\text{Curved Surf.}} \mathbf{D}(\rho) \cdot (\hat{\boldsymbol{\rho}}dS) = \int_{\text{Curved Surf.}} D(\rho)(\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\rho}})dS = \int_{\text{Curved Surf.}} D(\rho)dS$$

Then, we have

$$D(\rho) = \frac{Q_{\text{enc}}}{2\pi\rho L}$$

For $0 < \rho < a$, the enclosed charge is

$$Q_{\text{enc}} = \rho_L L$$

where

$$\begin{aligned}\rho_L &= \int_{\rho=0}^{\rho} \int_{\phi=0}^{2\pi} \rho_v(\rho, \phi, z) \rho d\phi d\rho \\ &= \int_{\rho=0}^{\rho} \int_{\phi=0}^{2\pi} \rho_0 \left(1 - e^{\frac{-\rho^2}{a^2}}\right) \rho d\phi d\rho \\ &= 2\pi \rho_0 \int_{\rho=0}^{\rho} \frac{a^2}{2} \left(1 - e^{\frac{-\rho^2}{a^2}}\right) d\left(\frac{\rho^2}{a^2}\right) \\ &= \pi \rho_0 a^2 \left(\frac{\rho^2}{a^2} + e^{\frac{-\rho^2}{a^2}}\right) \Big|_{\rho=0}^{\rho} \\ &= \pi \rho_0 a^2 \left(\frac{\rho^2}{a^2} + e^{\frac{-\rho^2}{a^2}} - 1\right)\end{aligned}$$

Therefore,

$$\begin{aligned}Q_{\text{enc}} &= L\pi\rho_0 a^2 \left(\frac{\rho^2}{a^2} + e^{\frac{-\rho^2}{a^2}} - 1\right) \\ D(\rho) &= \frac{Q_{\text{enc}}}{2\pi\rho L} = \frac{\rho_0 a^2}{2\rho} \left(\frac{\rho^2}{a^2} + e^{\frac{-\rho^2}{a^2}} - 1\right)\end{aligned}$$

For $\rho \geq a$,

$$Q_{\text{enc}} = L \int_{\rho=0}^a \int_{\phi=0}^{2\pi} \rho_v(\rho, \phi, z) \rho d\phi d\rho = L\pi\rho_0 a^2 / e$$

$$D(\rho) = \frac{Q_{\text{enc}}}{2\pi\rho L} = \frac{\rho_0 a^2}{2\rho e}$$

Because $\mathbf{E}(\rho) = \frac{D(\rho)}{\varepsilon_0} \hat{\boldsymbol{\rho}}$,

We have

$$\mathbf{E}(\rho) = \frac{\rho_0 a^2}{2\varepsilon_0 \rho} \left(\frac{\rho^2}{a^2} + e^{\frac{-\rho^2}{a^2}} - 1\right) \hat{\boldsymbol{\rho}}, \quad \text{for } 0 < \rho < a$$

$$\mathbf{E}(\rho) = \frac{\rho_0 a^2}{2\varepsilon_0 \rho e} \hat{\boldsymbol{\rho}}, \quad \text{for } \rho \geq a$$