

EE2104 Introduction to Electromagnetics
Assignment 2 Solution

Deadline (for uploading): 11:59 pm, 30 Sep 2021
Please put down your name and university ID.

1. Consider a line at $x = -L/4$ to $3L/4$. It contains electric charges of density $\rho_L = \rho_{L0} (x/L)^5$ for some constant ρ_{L0} in C/m. Find the total charge.

Solution:

The total charge is

$$\begin{aligned} Q &= \int_{-L/4}^{3L/4} \rho_L dx \\ &= \int_{-L/4}^{3L/4} \rho_{L0} (x/L)^5 dx \\ &= \int_{-L/4}^{3L/4} \rho_{L0} (x^5/L^5) dx \\ &= \frac{\rho_{L0}}{L^5} \int_{-L/4}^{3L/4} x^5 dx \\ &= \frac{\rho_{L0}}{L^5} \cdot \left(\frac{x^6}{6} \right) \Big|_{x=-L/4}^{3L/4} \\ &= \frac{\rho_{L0}}{L^5} \cdot \frac{1}{6} \cdot \frac{L^6 \cdot (3^6 - 1)}{4^6} \\ &= \frac{\rho_{L0} L \cdot 91}{3 \cdot 1024} \end{aligned}$$

2. A sphere centered at origin has radius b . It contains electric charge of density $\rho_v(r, \theta, \phi)$ in C/m^3 .
- (a) Find the total charge when the density $\rho_v = \rho_0$ for some constant ρ_0 .
- (b) Find the total charge when the density $\rho_v = \rho_0 \exp(-r^3/b^3)$ for some constant ρ_0 .

Solution:

(a) The total charge is

$$\begin{aligned}
 Q &= \int_{\text{sphere } r < b} \rho_v(r, \theta, \phi) dV \\
 &= \int_{\text{sphere } r < b} \rho_0 dV \\
 &= \frac{4\rho_0\pi b^3}{3}
 \end{aligned}$$

(b) The total charge is

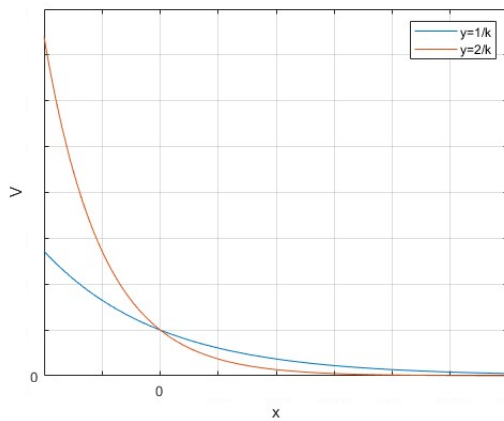
$$\begin{aligned}
 Q &= \int_{\text{sphere } r < b} \rho_v(r, \theta, \phi) dV \\
 &= \int_{\text{sphere } r < b} \rho_0 \exp(-r^3/b^3) dV \\
 &= \rho_0 \int_{\text{sphere } r < b} e^{-r^3/b^3} r^2 \sin\theta d\theta d\phi dr \\
 &= \rho_0 \int_0^{2\pi} d\phi \cdot \int_0^\pi \sin\theta d\theta \cdot \int_0^b e^{-r^3/b^3} r^2 dr \\
 &= 4\pi\rho_0 \frac{b^3}{3} \int_0^b e^{-\frac{r^3}{b^3}} d\left(\frac{r^3}{b^3}\right) \\
 &= -\frac{4\pi\rho_0 b^3}{3} \left(e^{-\frac{r^3}{b^3}} \right) \Big|_{r=0}^b \\
 &= \frac{4\pi\rho_0 b^3}{3} (1 - e^{-1})
 \end{aligned}$$

3. Consider a voltage $V(x, y) = V_0 \exp(-k^2 xy)$, where V_0 is a constant in volts and k is a positive constant in m^{-1} .
- Is $V(x, y)$ a scalar field or a vector field?
 - Plot V versus x at $y = 1/k$ and $y = 2/k$. Label the axes accurately.
 - Plot V versus y at $x = 1/k$ and $x = 2/k$. Label the axes accurately.
 - Plot the contour lines on the xy -plane for $V = 0.25V_0, 0.450V_0, 0.75V_0$, and V_0 .
 - Find $\mathbf{E} = -\text{grad}(V)$.

Solution:

- $V(x, y)$ is a scalar field.
- When $y = 1/k$, we have $V(x, 1/k) = V_0 e^{-kx}$.
When $y = 2/k$, we have $V(x, 2/k) = V_0 e^{-2kx}$.

The plots (or sketches) are exponential decay curves V with respect to x .



- When $x = 1/k$, we have $V(1/k, y) = V_0 e^{-ky}$
When $x = 2/k$, we have $V(2/k, y) = V_0 e^{-2ky}$
The plots (or sketches) are exponential decay curves of V with respect to y

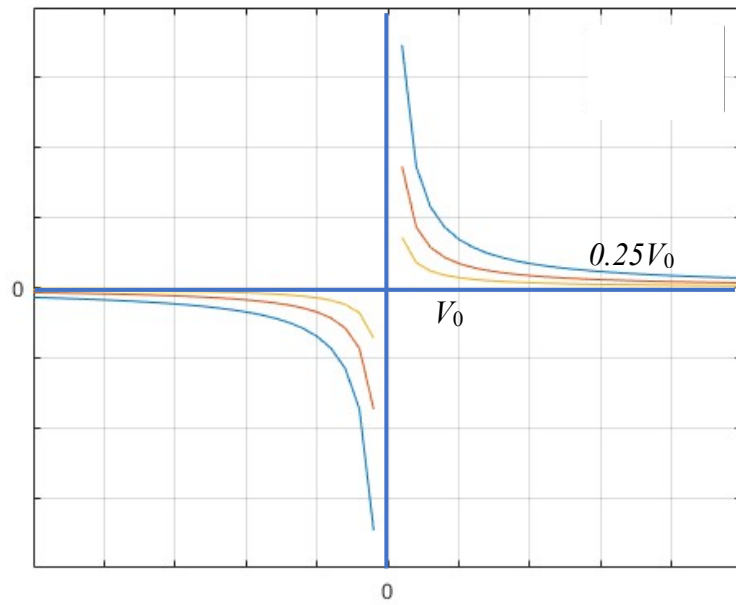
- When $V(x, y) = aV_0$ for some a ,

$$\begin{aligned} V_0 \exp(-k^2 xy) &= aV_0 \\ e^{-k^2 xy} &= a \\ -k^2 xy &= \ln a \\ xy &= k^{-2} \ln\left(\frac{1}{a}\right) \end{aligned}$$

The contour lines for $V = 0.25V_0, 0.450V_0, 0.75V_0$, and V_0 are respectively:

$$xy = 1.38k^{-2}, xy = 0.799k^{-2}, xy = 0.288k^{-2}, \text{ and } xy = 0$$

The contour lines on the xy -plane are plotted in the figure below.



(e) $\mathbf{E} = -\text{grad}(V)$

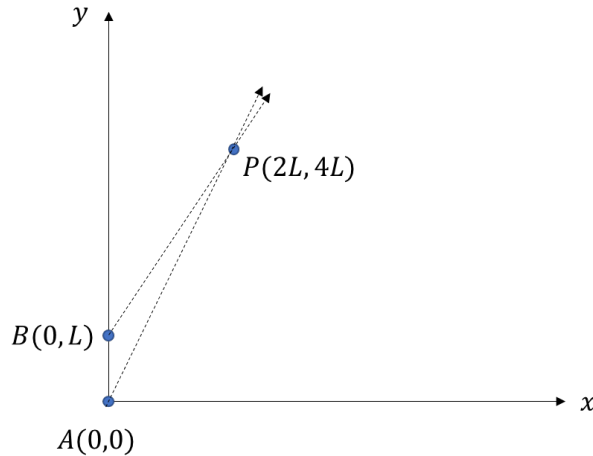
$$= -\left(\frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}\right)$$

$$= k^2 y V_0 e^{-k^2 xy} \hat{\mathbf{x}} + k^2 x V_0 e^{-k^2 xy} \hat{\mathbf{y}}$$

4. Consider two points $A(0, 0)$ and $B(0, L)$ on the xy -plane for some positive distance L . Find the electric field $\mathbf{E}(2L, 4L)$ under the following conditions.
- Point charges $+Q$ and $-Q$ are put at A and B , respectively.
 - Infinite line charges of densities $+\rho_L$ and $-\rho_L$ are put at A and B , respectively. The line charges are parallel to the z -axis.
- Keep all the expressions in the rectangular coordinate system.

Solution:

- (a) On the xy -plane, we have the following:



Unit vector from A to P :

$$\hat{a}_A = \frac{\vec{AP}}{|\vec{AP}|} = \frac{2L\hat{x} + 4L\hat{y}}{[(2L)^2 + (4L)^2]^{1/2}}$$

Unit vector from B to P :

$$\hat{a}_B = \frac{\vec{BP}}{|\vec{BP}|} = \frac{2L\hat{x} + 3L\hat{y}}{[(2L)^2 + (3L)^2]^{1/2}}$$

When A contains charge $+Q$ and B contains $-Q$, the field $\mathbf{E}$ at point P can be obtained by the superposition principle and the Coulomb's Law:

$$\begin{aligned}\vec{E} &= \frac{Q}{4\pi\epsilon_0} \left[\frac{\hat{a}_A}{|\vec{AP}|^2} - \frac{\hat{a}_B}{|\vec{BP}|^2} \right] \\ &= \frac{Q}{4\pi\epsilon_0} \frac{2(13\sqrt{13} - 40\sqrt{5})\hat{x} + (4 \cdot 13\sqrt{13} - 3 \cdot 40\sqrt{5})\hat{y}}{40 \cdot 13 \cdot \sqrt{5 \cdot 13} L^2} \\ &= \frac{Q}{4\pi\epsilon_0 L^2} (-0.0203\hat{x} - 0.0193\hat{y})\end{aligned}$$

- (b) By replacing the point charges by line charges, we have the field $\mathbf{E}$ at point P as

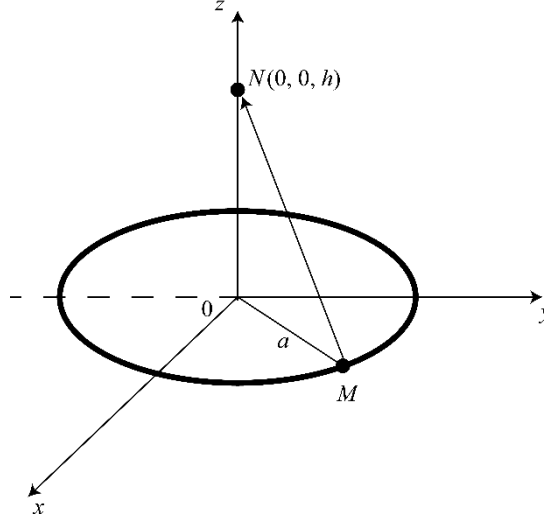
$$\begin{aligned}\vec{E} &= \frac{\rho_L}{2\pi\epsilon_0} \left[\frac{\hat{a}_A}{|\vec{AP}|^1} - \frac{\hat{a}_B}{|\vec{BP}|^1} \right] = \frac{\rho_L}{2\pi\epsilon_0} \cdot \frac{(13 \cdot 2 - 2 \cdot 20)\hat{x} + (4 \cdot 13 - 3 \cdot 20)\hat{y}}{13 \cdot 20 L^1} \\ &= \frac{\rho_L}{2\pi\epsilon_0 L} (-0.0538\hat{x} - 0.0308\hat{y})\end{aligned}$$

5. From the Coulomb's law, derive the electric field measured at $(x, y, z) = (0, 0, h)$ from a charged circle centered at the origin. The circle is oriented along the xy -plane with radius a . The line charge density is given by

$$\rho_L(\phi) = \rho_{L0} \sin^2 2\phi$$

where ϕ is measured from the x -axis and ρ_{L0} (C/m) is a constant.

Solution:



- (i) Considering a differential angle $d\phi$, a differential arc length on the charged circle is expressed as

$$dl = a \cdot d\phi$$

Therefore, at position $M(a \cos\phi, a \sin\phi, 0)$ on the circle, the differential arc dl carries a charge of

$$dQ = \rho_L(\phi)dl = \rho_{L0} \sin^2 2\phi dl = \rho_{L0} \sin^2 2\phi (a d\phi) = a\rho_{L0} \sin^2 2\phi d\phi$$

- (ii) Physically, it noted that $\rho_L(\phi) = \rho_L(\phi + \pi)$, and the electric field has only the z -component.
 (iii) By Coulomb's law, the differential electric field at $N(0, 0, h)$ caused by the differential arc has a z -component of:

$$dE_z = \frac{dQ}{4\pi\epsilon_0 |\mathbf{R}_{MN}|^2} \frac{\mathbf{R}_{MN}}{|\mathbf{R}_{MN}|} \cdot \hat{\mathbf{z}} = \frac{a\rho_{L0} \sin^2 2\phi d\phi \cdot h}{4\pi\epsilon_0 (\sqrt{(a^2 + h^2)})^3}$$

where $\mathbf{R}_{MN}$ is the vector pointing to N from M .

- (iv) By summing over the whole ring, the z -component of the total electric field at position N is

$$E_z = \int_{\phi=0}^{2\pi} dE_z = \int_{\phi=0}^{2\pi} \frac{a\rho_{L0} \sin^2 2\phi \cdot h}{4\pi\epsilon_0 (\sqrt{(a^2 + h^2)})^3} d\phi = \frac{ah\rho_{L0}}{4\pi\epsilon_0 (a^2 + h^2)^{\frac{3}{2}}} \int_{\phi=0}^{2\pi} \sin^2 2\phi d\phi$$

Therefore, we have

$$\mathbf{E} = \frac{ah\rho_{L0}}{4\epsilon_0 (a^2 + h^2)^{\frac{3}{2}}} \hat{\mathbf{z}}$$

Note: For a full mathematical solution without observing the symmetry,

By Coulomb's law, the differential electric field at $N(0, 0, h)$ is

$$d\mathbf{E} = \frac{dQ}{4\pi\epsilon_0|\mathbf{R}_{MN}|^2} \frac{\mathbf{R}_{MN}}{|\mathbf{R}_{MN}|} = \frac{dQ \mathbf{R}_{MN}}{4\pi\epsilon_0|\mathbf{R}_{MN}|^3} = \frac{a\rho_{L0}\sin^2 2\phi d\phi \cdot (-a\cos\phi\hat{\mathbf{x}} - a\sin\phi\hat{\mathbf{y}} + h\hat{\mathbf{z}})}{4\pi\epsilon_0(\sqrt{(a^2 + h^2)})^3}$$

where $\mathbf{R}_{MN}$ is still the vector pointing to N from M .

(a) The x -component of the total electric field at position N is

$$E_x = \int_{\phi=0}^{2\pi} dE_x = \int_{\phi=0}^{2\pi} \frac{a\rho_{L0}\sin^2 2\phi \cdot (-a\cos\phi)}{4\pi\epsilon_0(\sqrt{(a^2 + h^2)})^3} d\phi = \frac{-a^2\rho_{L0}}{4\pi\epsilon_0(a^2 + h^2)^{\frac{3}{2}}} \int_{\phi=0}^{2\pi} \sin^2 2\phi \cos\phi d\phi$$

Since

$$\sin^2 2\phi \cos\phi = \frac{1}{2}\cos\phi - \frac{1}{4}(\cos 5\phi + \cos 3\phi)$$

we have

$$\int_{\phi=0}^{2\pi} \sin^2 2\phi \cos\phi d\phi = \int_{\phi=0}^{2\pi} \left(\frac{1}{2}\cos\phi - \frac{1}{4}(\cos 5\phi + \cos 3\phi) \right) d\phi = 0$$

Thus,

$$E_x = 0$$

(b) Similarly, the y -component of the total electric field at position N is

$$E_y = \int_{\phi=0}^{2\pi} dE_y = \int_{\phi=0}^{2\pi} \frac{a\rho_{L0}\sin^2 2\phi \cdot (-a\sin\phi)}{4\pi\epsilon_0(\sqrt{(a^2 + h^2)})^3} d\phi = \frac{-a^2\rho_{L0}}{4\pi\epsilon_0(a^2 + h^2)^{\frac{3}{2}}} \int_{\phi=0}^{2\pi} \sin^2 2\phi \sin\phi d\phi$$

Since

$$\sin^2 2\phi \sin\phi = \frac{1}{2}\sin\phi - \frac{1}{4}(\sin 5\phi - \sin 3\phi)$$

we have

$$\int_{\phi=0}^{2\pi} \sin^2 2\phi \sin\phi d\phi = \int_{\phi=0}^{2\pi} \left(\frac{1}{2}\sin\phi - \frac{1}{4}(\sin 5\phi - \sin 3\phi) \right) d\phi = 0$$

Thus,

$$E_y = 0$$

Nonetheless, this page of derivation can be omitted by observing the symmetry in the part (ii) of the previous page.