

Test 1 Solution

6:00pm~6:40pm (YEUNG LT-15 or 17)

- As a closed-book test, you are not allowed to use any written materials online/offline.
- Write your answers on the question papers directly.
- Non-graphical calculators are allowed. Laptops/phones must be switched off.
- Unless otherwise specified, SI units are assumed in the questions and 3 significant figures should be used in numerical answers.
- The total score is 100. There are a total of **5 pages**.

The following formulas may or may not be useful.

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \mathbf{E}(x, y, z) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{a}}_R & d\mathbf{B} &= \frac{\mu_0}{4\pi} \frac{I d\mathbf{L} \times \hat{\mathbf{a}}_R}{R^2} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} & \oint_S \mathbf{D} \cdot d\mathbf{S} &= Q & \oint_L \mathbf{H} \cdot d\mathbf{L} &= I \\ \nabla \cdot \mathbf{D} &= \rho_v & \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \epsilon_r \mathbf{E} & \mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 \mu_r \mathbf{H} \\ \nabla \cdot \mathbf{B} &= 0 & \epsilon_0 &= 8.854 \times 10^{-12} \text{ (F/m)} & \mu_0 &= 4\pi \times 10^{-7} \text{ (H/m)}\end{aligned}$$

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \hat{\mathbf{a}}_\rho \quad \mathbf{E} = \frac{P}{4\pi\epsilon_0 r^3} (2\cos\theta \hat{\mathbf{a}}_r + \sin\theta \hat{\mathbf{a}}_\theta) \quad \mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\mathbf{a}}_\phi \quad \mathbf{B} = \mu_0 n I \hat{\mathbf{a}}_z$$

$$\begin{aligned}V_{AB} &= -\int_B^A \mathbf{E} \cdot d\mathbf{L} & \nabla^2 V &= -\frac{\rho_v}{\epsilon} & V &= -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \\ \nabla^2 &= \nabla \cdot \nabla\end{aligned}$$

$$\begin{aligned}\nabla &= \frac{\partial}{\partial x} \hat{\mathbf{a}}_x + \frac{\partial}{\partial y} \hat{\mathbf{a}}_y + \frac{\partial}{\partial z} \hat{\mathbf{a}}_z & \nabla \times \mathbf{F} &= \frac{1}{\rho} \begin{vmatrix} \hat{\mathbf{a}}_\rho & \rho \hat{\mathbf{a}}_\phi & \hat{\mathbf{a}}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} \\ \nabla f &= \frac{\partial f}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial f}{\partial z} \hat{\mathbf{a}}_z & \nabla \times \mathbf{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{a}}_r & r \hat{\mathbf{a}}_\theta & r \sin\theta \hat{\mathbf{a}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin\theta F_\phi \end{vmatrix} \\ \nabla f &= \frac{\partial f}{\partial r} \hat{\mathbf{a}}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\mathbf{a}}_\theta + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi\end{aligned}$$

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{\partial}{\partial \phi} F_\phi + \frac{\partial}{\partial z} (\rho F_z) \right] \\ \nabla \cdot \mathbf{F} &= \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} (r^2 \sin\theta F_r) + \frac{\partial}{\partial \theta} (r \sin\theta F_\theta) + \frac{\partial}{\partial \phi} (r F_\phi) \right]\end{aligned}$$

Question 1 (12 points)

Indicate whether the following statements are true or false.

- (a) An atom is made of negative electron(s) and a positive nucleus.
- (b) An electron carries a charge of -1.6×10^{-19} (C).
- (c) The Maxwell's equations were discovered by Isaac Newton.
- (d) Electrostatics is about electric currents in coils that rotate.
- (e) A voltage of 0.1 mV is the same as 10 kV.
- (f) For a unit vector $\hat{r}$, the length of the vector $|\hat{r}|$ is 1.

Solution: TTFFFT

Question 2 (12 points)

Consider a point charge $Q = +4$ (C) placed at the origin. The electric field is measured at $(x, y) = (0, 21)$ in meters.

- (a) Is the electric field a scalar field or a vector field?
- (b) What is the direction of the electric field?
- (c) What is the magnitude of the electric field.

Solution:

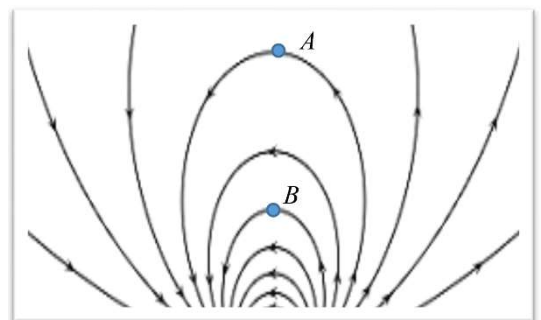
- (a) The electric field is a vector field.
- (b) The direction of the electric field is $\hat{y}$.
- (c) The magnitude of the electric field at $(x, y) = (0, 21)$ is

$$E = \frac{Q}{4\pi\epsilon_0 R^2} = 8.15 \times 10^7 \text{ (V/m)}.$$

Question 3 (12 points)

The picture below shows some electric field lines from charges that are outside the region plotted.

- (a) What is the direction of the electric field at point A ?
- (b) What is the direction of the electric field at point B ?
- (c) When comparing points A and B , which point has a stronger electric field?



Solution:

- (a) The direction of the electric field at point A is left.
- (b) The direction of the electric field at point B is left.
- (c) Point B has a stronger field because the electric field lines around B are denser.

Question 4 (16 points)

Given two position vectors $\mathbf{H} = 50\hat{x}$ and $\mathbf{K} = 20\hat{x} + 21\hat{y}$.

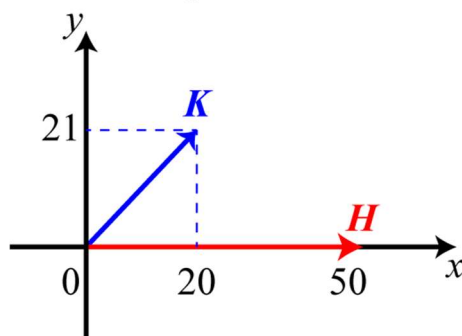
(a) Draw the two vectors as accurately as possible. Label the x -axis and the y -axis.

(b) Find the angle between the vectors $\mathbf{H}$ and $\mathbf{K}$.

(c) Find the unit vector pointing from $\mathbf{H}$ to $\mathbf{K}$.

Solution:

(a) The two vectors are plotted in the figure below.



(b) The angle between $\mathbf{H}$ and $\mathbf{K}$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{H} \cdot \mathbf{K}}{|\mathbf{H}||\mathbf{K}|}\right) = \cos^{-1}\left(\frac{1000}{50 \times \sqrt{20^2 + 21^2}}\right) = 46.4^\circ$$

(c) The vector pointing from $\mathbf{H}$ to $\mathbf{K}$ can be expressed as

$$\mathbf{R} = \mathbf{K} - \mathbf{H} = -30\hat{x} + 21\hat{y}$$

We have

$$|\mathbf{R}| = \sqrt{30^2 + 21^2} = 36.6$$

Therefore, the unit vector pointing from $\mathbf{H}$ to $\mathbf{K}$ is

$$\frac{\mathbf{R}}{|\mathbf{R}|} = \frac{-30\hat{x} + 21\hat{y}}{36.6} = -0.820\hat{x} + 0.574\hat{y}$$

Question 5 (16 points)

Given a field $\mathbf{F}(x, y) = x^2y\hat{\mathbf{x}} + xy^2\hat{\mathbf{y}}$

(a) At point $(x, y) = (5, 3)$, find $\mathbf{F}$.

(b) Also at point $(x, y) = (5, 3)$, find the component of $\mathbf{F}$ along the radial direction (i.e. the direction $\hat{\boldsymbol{\rho}}$ of the cylindrical coordinate system).

(c) Again at point $(x, y) = (5, 3)$, find: $\text{div } \mathbf{F}$.

Solution:

(a) At $(5, 3)$, we have

$$\mathbf{F} = 75\hat{\mathbf{x}} + 45\hat{\mathbf{y}}$$

(b) At $(5, 3)$, the component of $\mathbf{F}$ along the radial direction is

$$\mathbf{F} \cdot \frac{(5\hat{\mathbf{x}} + 3\hat{\mathbf{y}})}{\sqrt{5^2 + 3^2}} = 15\sqrt{5^2 + 3^2} = 87.5$$

(c)

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = 4xy = 60.0$$

Question 6 (16 points)

Consider a rod at $x = 0$ to L (in meters) containing electric charges of density (in C/m):

$$\rho_L(x) = k \left(\frac{x}{L} \right)^7$$

for some constant k .

(a) What is the physical unit of k .

(b) How much charge is contained in a segment at x to $x + dx$?

(c) Find the total charge.

Solution:

(a) The physical unit of k is C/m.

(b) In a segment at x to $x + dx$, the charge is

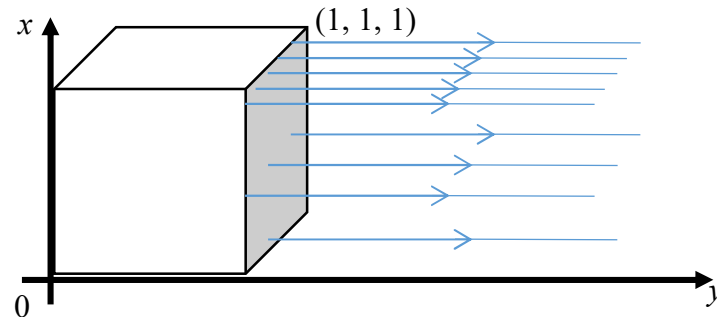
$$\rho_L(x)dx = k \left(\frac{x}{L} \right)^7 dx.$$

(c) The total charge is

$$Q = \int_0^L \rho_L(x)dx = \int_0^L k \left(\frac{x}{L} \right)^7 dx = \frac{k}{L^7} \int_0^L x^7 dx = \frac{kL}{8}.$$

Question 7 (16 points)

Consider a cube of $1 \times 1 \times 1$ with a six-faced surface S . One corner is placed at the origin and another corner is at $(x, y, z) = (1, 1, 1)$ as the diagram shows. At $y = 1$, the face is shaded in gray color and is labeled as surface S_G .



There is an electric field with the density of electric field lines given as:

$$\mathbf{D}(x, y, z) = x \sin(y) \hat{\mathbf{y}}$$

(a) On the diagram, sketch the field lines exiting the surface S_G . The sketch should include the direction and density variation.

(b) Find $\text{div } \mathbf{D}$.

(c) Using the above result, or using direct calculation, find the flux $\psi = \oint_{\text{Cube}} \mathbf{D} \cdot d\mathbf{S}$.

Solution:

(a) The field lines are plotted as the blue arrows in the above diagram.

(b) $\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial y} \hat{\mathbf{y}} \right) \cdot (x \sin y \hat{\mathbf{y}}) = x \cos y.$

(c) The flow is along the y -direction. Also, the face at $y = 0$ has $\mathbf{D} = 0$.

$$\psi = \oint_{\text{Cube}} \mathbf{D} \cdot d\mathbf{S} = \int_{S_G} \mathbf{D} \cdot d\mathbf{S} = \int_{z=0}^1 \int_{x=0}^1 (x \sin y \hat{\mathbf{y}}) \cdot (dx dz \hat{\mathbf{y}}) = \left(\int_{z=0}^1 dz \right) \left(\sin y \int_{x=0}^1 x dx \right) = \frac{1}{2} \sin 1$$

Alternatively, one can use the divergence theorem:

$$\psi = \oint_{\substack{\text{Cube} \\ (\text{Surface})}} \mathbf{D} \cdot d\mathbf{S} = \psi = \int_{\substack{\text{Cube} \\ (\text{Volume})}} \nabla \cdot \mathbf{D} dV = \int_{z=0}^1 \int_{y=0}^1 \int_{x=0}^1 x \cos y dx dy dz = \frac{1}{2} \sin 1$$

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