

L07: Power and Velocities in EM Waves

EE3109 Applied Electromagnetics

Lecture Notes

Dr. Alex M. H. Wong

Jan – Apr 2023

Outline

- Course Matter
- Power in plane waves
- Velocities in plane waves

Announcements (Week 8)

- Test and Assignment Returns
 - Test 1 and Assignments are collectable during the break
 - Test 1 Solution will be posted by the end of the day
 - Remark Policy:
 - Remark requests (if any) must be submitted within 2 working days of the return of test and postage of solution, whichever comes later
 - Must be submitted in written form, with points for consideration clearly explained
 - We reserve the right to reconsider the entire test / assignment
- Labs will begin on Week 9 (next week) (see L05-06 slides for location and times)
- Assignment 2 due Week 9 (next week)

Announcements Cont'd (Week 8)

Upcoming events:

- [HKSTP Master Class Series: Microelectronics in the GBA](#) (throughout March 2023)
- Area of Excellence Workshop: Meta-Devices (Mar 17, 2023)

The event information PDFs are accessible on Canvas @ Files > Miscellaneous

Tentative Course Schedule

Week #	Week of	Course Coverage
1	Jan 9	Course Introduction, Math Review
2	Jan 16	Introduction to Maxwell Equations, Materials and Boundaries
	Jan 23	<i>Happy Chinese New Year!</i>
3	Jan 30	Wave Equations, Plane Waves, Propagation in Lossy Media
4	Feb 6	Polarizations, <i>Asst. 1a Due</i>
5	Feb 13	EM Potential, Polarizations Cont'd, Potentials <i>Asst. 1b Due</i>
6	Feb 20	Test 1 , Reflection and Transmission
7	Feb 27	Reflection and Transmission Cont'd
8	Mar 6	Power, Velocity, Waveguides and Transmission Lines (TL)
9	Mar 13	VIP in a TL, Terminated and Lossy TLs, <i>Asst. 2 Due, Lab 1 Begins</i>
10	Mar 20	Impedance in a TL, Smith Charts,
11	Mar 27	Smith Charts Cont'd, Impedance Matching, Quarter-wave Transformers, <i>Lab 2 Begins</i>
12	Apr 3	Test 2 , Introduction to Antennas
13	Apr 10	Dipole Antennas, <i>Asst. 3 Due</i>

Review and Preview

- Up till now, we talked about
 - Plane waves and their properties
 - Plane wave propagation, reflection and transmission
- This class (and next class) we will examine
 - Power transfer in plane waves
 - Phase and group velocities

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Poynting Vector

Energy is transported through space to distant receiving points by electromagnetic waves.

Maxwell:

$$\nabla \times \mathbf{E}(t) = -\frac{\partial \mathbf{B}(t)}{\partial t} \quad \nabla \times \mathbf{H}(t) = \frac{\partial \mathbf{D}(t)}{\partial t}$$

Consider: ($\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$ and $\mathbf{D}$ are dependent on time)

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}) &= \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \\ &= -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J} \\ &= -\mathbf{H} \cdot \frac{\partial(\mu \mathbf{H})}{\partial t} - \mathbf{E} \cdot \frac{\partial(\epsilon \mathbf{E})}{\partial t} - \mathbf{E} \cdot (\sigma \mathbf{E}) \\ &= -\frac{\partial}{\partial t} \left(\frac{1}{2} \mu \mathbf{H}^2 \right) - \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon \mathbf{E}^2 \right) - \sigma \mathbf{E}^2 \end{aligned}$$

Integrate over a volume:

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) dV$$
$$= - \underbrace{\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \mu \mathbf{H}^2 + \frac{1}{2} \epsilon \mathbf{E}^2 \right) dV}_{\text{Time rate of change of the energy stored in the electric and magnetic fields}} - \underbrace{\int_V \sigma \mathbf{E}^2 dV}_{\text{Power dissipated in the volume as a result of the flow of conduction current}}$$

Poynting Theorem: The term $\oint_S \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \cdot d\mathbf{s}$ represents the power leaving the enclosed volume through its surface S.

Poynting Vector:

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \text{ (W/m}^2\text{)}$$

Power Density

Poynting vector:

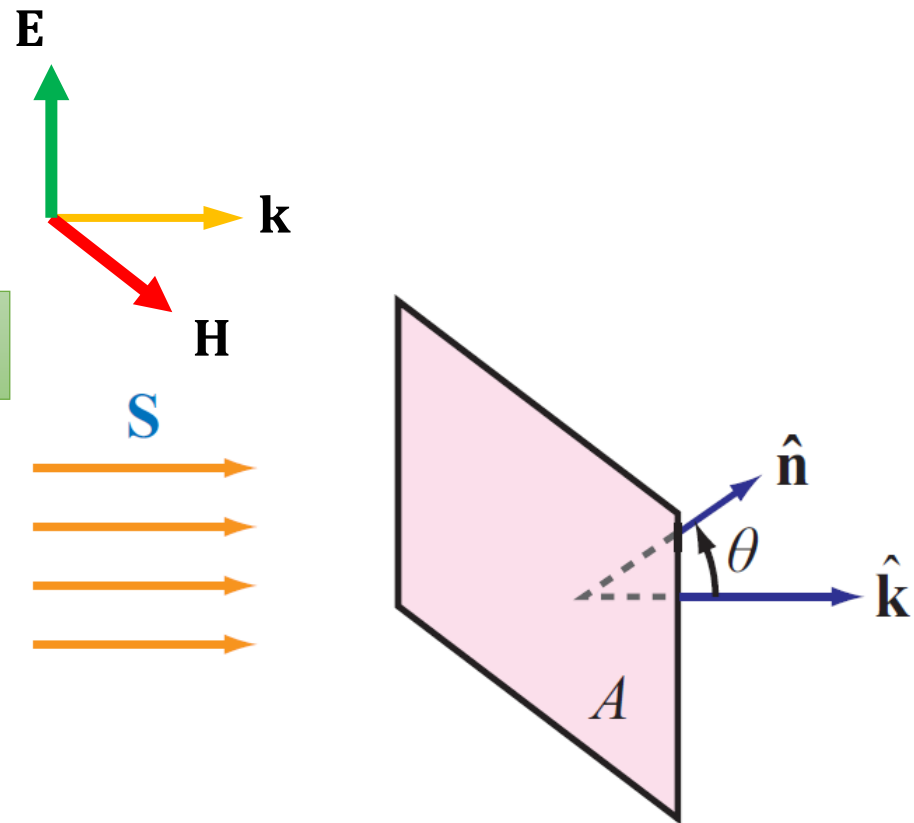
$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \quad (\text{W/m}^2)$$

Total power intercepted by A:

$$P(\mathbf{r}, t) = \int_A \mathbf{S}(\mathbf{r}, t) \cdot \hat{\mathbf{n}} \, dA \quad (\text{W})$$

Time-average power density:

$$\mathbf{s}_{av} = \frac{1}{T} \int_{t_0}^{t_0+T} \mathbf{S}(\mathbf{r}, t) \, dt = \frac{1}{2} \text{Re}\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^*\} \quad (\text{W/m}^2)$$



Power Density Carried by Plane Wave

For a plane wave with **E** field :

$$\begin{aligned}\tilde{\mathbf{E}}(z) &= \hat{\mathbf{x}} \tilde{E}_x(z) + \hat{\mathbf{y}} \tilde{E}_y(z) \\ &= (\hat{\mathbf{x}} E_{x0} + \hat{\mathbf{y}} E_{y0})e^{-jkz},\end{aligned}$$

the average power density carried by the wave is:

$$\begin{aligned}\mathbf{S}_{\text{av}} &= \hat{\mathbf{z}} \frac{1}{2\eta} (|E_{x0}|^2 + |E_{y0}|^2) \\ &= \hat{\mathbf{z}} \frac{|\tilde{\mathbf{E}}|^2}{2\eta} \quad (\text{W/m}^2),\end{aligned}$$

Plane Wave in Lossy Medium

For a plane wave travelling in a lossy medium:

$$\begin{aligned}\tilde{\mathbf{E}}(z) &= \hat{\mathbf{x}} \tilde{E}_x(z) + \hat{\mathbf{y}} \tilde{E}_y(z) \\ &= (\hat{\mathbf{x}} E_{x0} + \hat{\mathbf{y}} E_{y0}) e^{-\alpha z} e^{-j\beta z}, \\ \tilde{\mathbf{H}}(z) &= \frac{1}{\eta_c} (-\hat{\mathbf{x}} E_{y0} + \hat{\mathbf{y}} E_{x0}) e^{-\alpha z} e^{-j\beta z},\end{aligned}$$

the power density is :

$$\begin{aligned}\mathbf{S}_{av}(z) &= \frac{1}{2} \Re \left[\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \right] \\ &= \frac{\hat{\mathbf{z}} (|E_{x0}|^2 + |E_{y0}|^2)}{2} e^{-2\alpha z} \Re \left(\frac{1}{\eta_c^*} \right).\end{aligned}$$

$$\mathbf{S}_{av}(z) = \hat{\mathbf{z}} \frac{|\tilde{E}(0)|^2}{2|\eta_c|} e^{-2\alpha z} \cos \theta_\eta \quad (\text{W/m}^2)$$

where $\eta_c = |\eta_c| e^{j\theta_\eta}$

Whereas $\mathbf{E}$ and $\mathbf{H}$ decay with z as $e^{-\alpha z}$, $\mathbf{S}$ decay with z as $e^{-2\alpha z}$.

Example 7-6: Power Received by a Submarine Antenna

A submarine at a depth of 200 m below the sea surface uses a wire antenna to receive signal transmissions at 1 kHz. Determine the power density incident upon the submarine antenna due to the EM wave of Example 7-4.

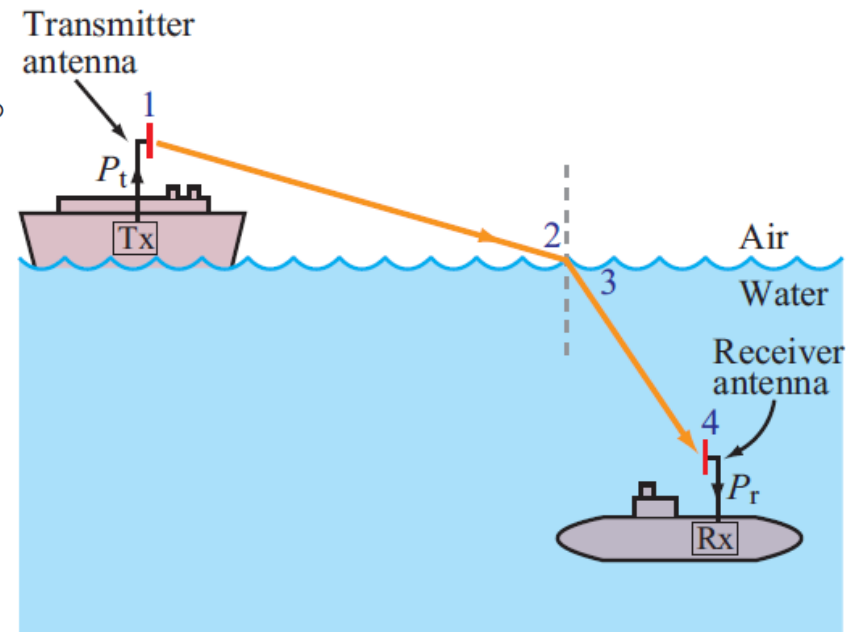
$$\left(\begin{array}{l} |E(z=0)| = 4.44 [\text{mVm}]; \\ \alpha = 0.126 [\text{Npm}]; \\ \eta_c = 0.044 \angle 45^\circ [\Omega] \end{array} \right)$$

Solution: From Example 7-4, $|\vec{E}(0)| = |E_{x0}| = 4.44$ (mV/m), $\alpha = 0.126$ (Np/m), and $\eta_c = 0.044 \angle 45^\circ$ (Ω). Application of Eq. (7.109) gives

$$\begin{aligned} \mathbf{S}_{\text{av}}(z) &= \hat{\mathbf{z}} \frac{|E_0|^2}{2|\eta_c|} e^{-2\alpha z} \cos \theta_\eta \\ &= \hat{\mathbf{z}} \frac{(4.44 \times 10^{-3})^2}{2 \times 0.044} e^{-0.252z} \cos 45^\circ \\ &= \hat{\mathbf{z}} 0.16 e^{-0.252z} \quad (\text{mW/m}^2). \end{aligned}$$

At $z = 200$ m, the incident power density is

$$\begin{aligned} \mathbf{S}_{\text{av}} &= \hat{\mathbf{z}} (0.16 \times 10^{-3} e^{-0.252 \times 200}) \\ &= 2.1 \times 10^{-26} \quad (\text{W/m}^2). \end{aligned}$$



Example 8-6: Wave Incident Obliquely on a Soil Surface

Using the coordinate system of Fig. 8-15, a plane wave radiated by a distant antenna is incident in air upon a plane soil surface located at $z = 0$. The electric field of the incident wave is given by

$$\mathbf{E}^i = \hat{\mathbf{y}}100 \cos(\omega t - \pi x - 1.73\pi z) \quad (\text{V/m}), \quad (8.61)$$

and the soil medium may be assumed to be a lossless dielectric with a relative permittivity of 4.

- (a) Determine k_1 , k_2 , and the incidence angle θ_i .
- (b) Obtain expressions for the total electric fields in air and in the soil.
- (c) Determine the average power density carried by the wave traveling in soil.

Example 8-6 (cont.)

Solution: (a) We begin by converting Eq. (8.61) into phasor form, akin to the expression given by Eq. (8.46a):

$$\tilde{\mathbf{E}}^i = \hat{\mathbf{y}}100e^{-j\pi x - j1.73\pi z} \quad (\text{V/m}),$$

$$\Rightarrow \mathbf{k} = \pi \hat{\mathbf{x}} + 1.73\pi \hat{\mathbf{z}}$$

But we know $k_{ix} = k_1 \sin \theta_i$; $k_{iz} = k_1 \cos \theta_i$

Hence,

$$k_1 \sin \theta_i = \pi,$$

$$k_1 \cos \theta_i = 1.73\pi,$$

which together give

$$k_1 = \sqrt{\pi^2 + (1.73\pi)^2} = 2\pi \quad (\text{rad/m}),$$

$$\theta_i = \tan^{-1} \left(\frac{\pi}{1.73\pi} \right) = 30^\circ.$$

The wavelength in medium 1 (air) is

$$\lambda_1 = \frac{2\pi}{k_1} = 1 \text{ m},$$

and the wavelength in medium 2 (soil) is

$$\lambda_2 = \frac{\lambda_1}{\sqrt{\epsilon_{r2}}} = \frac{1}{\sqrt{4}} = 0.5 \text{ m}.$$

The corresponding wave number in medium 2 is

$$k_2 = \frac{2\pi}{\lambda_2} = 4\pi \quad (\text{rad/m}).$$

Example 8-6 (cont.)

(b) Given that $\theta_i = 30^\circ$, the transmission angle θ_t is obtained with the help of Eq. (8.56):

$$\sin \theta_t = \frac{k_1}{k_2} \sin \theta_i = \frac{2\pi}{4\pi} \sin 30^\circ = 0.25$$

or

$$\theta_t = 14.5^\circ.$$

With $\varepsilon_1 = \varepsilon_0$ and $\varepsilon_2 = \varepsilon_{r2} \varepsilon_0 = 4\varepsilon_0$, the reflection and transmission coefficients for perpendicular polarization are determined with the help of Eqs. (8.59) and (8.60),

$$\Gamma_{\perp} = \frac{\cos \theta_i - \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2 \theta_i}}{\cos \theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2 \theta_i}} = -0.38,$$

$$\tau_{\perp} = 1 + \Gamma_{\perp} = 0.62.$$

Example 8-6 (cont.)

Using Eqs. (8.48a) and (8.49a) with $E_{\perp 0}^i = 100$ V/m and $\theta_i = \theta_r$, the total electric field in medium 1 is

$$\begin{aligned}\tilde{\mathbf{E}}_{\perp}^1 &= \tilde{\mathbf{E}}_{\perp}^i + \tilde{\mathbf{E}}_{\perp}^r \\ &= \hat{\mathbf{y}} E_{\perp 0}^i e^{-jk_1(x \sin \theta_i + z \cos \theta_i)} \\ &\quad + \hat{\mathbf{y}} \Gamma E_{\perp 0}^i e^{-jk_1(x \sin \theta_i - z \cos \theta_i)} \\ &= \hat{\mathbf{y}} 100 e^{-j(\pi x + 1.73\pi z)} - \hat{\mathbf{y}} 38 e^{-j(\pi x - 1.73\pi z)},\end{aligned}$$

and the corresponding instantaneous electric field in medium 1 is

$$\begin{aligned}\mathbf{E}_{\perp}^1(x, z, t) &= \Re \left[\tilde{\mathbf{E}}_{\perp}^1 e^{j\omega t} \right] \\ &= \hat{\mathbf{y}} [100 \cos(\omega t - \pi x - 1.73\pi z) \\ &\quad - 38 \cos(\omega t - \pi x + 1.73\pi z)] \quad (\text{V/m}).\end{aligned}$$

Example 8-6 (cont.)

In medium 2, using Eq. (8.49c) with $E_{\perp 0}^t = \tau_{\perp} E_{\perp 0}^i$ gives

$$\begin{aligned}\tilde{\mathbf{E}}_{\perp}^t &= \hat{\mathbf{y}} \tau E_{\perp 0}^i e^{-jk_2(x \sin \theta_t + z \cos \theta_t)} \\ &= \hat{\mathbf{y}} 62 e^{-j(\pi x + 3.87\pi z)}\end{aligned}$$

and, correspondingly,

$$\begin{aligned}\mathbf{E}_{\perp}^t(x, z, t) &= \Re \left[\tilde{\mathbf{E}}_{\perp}^t e^{j\omega t} \right] \\ &= \hat{\mathbf{y}} 62 \cos(\omega t - \pi x - 3.87\pi z) \quad (\text{V/m}).\end{aligned}$$

(c) In medium 2, $\eta_2 = \eta_0 / \sqrt{\epsilon_{r2}} \simeq 120\pi / \sqrt{4} = 60\pi \text{ } (\Omega)$, and the average power density carried by the wave is

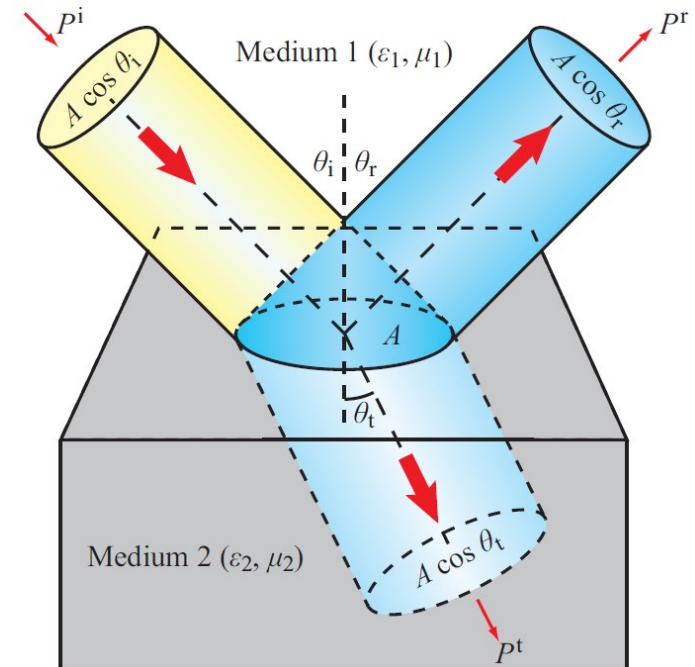
$$S_{\text{av}}^t = \frac{|E_{\perp 0}^t|^2}{2\eta_2} = \frac{(62)^2}{2 \times 60\pi} = 10.2 \quad (\text{W/m}^2).$$

Power Reflectivity and Transmissivity

$$T_{\perp} = \frac{P_{\perp}^t}{P_{\perp}^i} = \frac{|E_{\perp 0}^t|^2}{|E_{\perp 0}^i|^2} \frac{\eta_1}{\eta_2} \frac{A \cos \theta_t}{A \cos \theta_i}$$

$$= |\tau_{\perp}|^2 \left(\frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i} \right), \quad (8.79a)$$

$$T_{\parallel} = \frac{P_{\parallel}^t}{P_{\parallel}^i} = |\tau_{\parallel}|^2 \left(\frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i} \right). \quad (8.79b)$$



The incident, reflected, and transmitted waves do not have to obey any such laws as conservation of electric field, conservation of magnetic field, or conservation of power density, but they do have to obey the law of conservation of power.

Summary For Reflection and Transmission

Table 8-2: Expressions for Γ , τ , R , and T for wave incidence from a medium with intrinsic impedance η_1 onto a medium with intrinsic impedance η_2 . Angles θ_i and θ_t are the angles of incidence and transmission, respectively.

Property	Normal Incidence $\theta_i = \theta_t = 0$	Perpendicular Polarization	Parallel Polarization
Reflection coefficient	$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$	$\Gamma_{\perp} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$	$\Gamma_{\parallel} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$
Transmission coefficient	$\tau = \frac{2\eta_2}{\eta_2 + \eta_1}$	$\tau_{\perp} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$	$\tau_{\parallel} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$
Relation of Γ to τ	$\tau = 1 + \Gamma$	$\tau_{\perp} = 1 + \Gamma_{\perp}$	$\tau_{\parallel} = (1 + \Gamma_{\parallel}) \frac{\cos \theta_i}{\cos \theta_t}$
Reflectivity	$R = \Gamma ^2$	$R_{\perp} = \Gamma_{\perp} ^2$	$R_{\parallel} = \Gamma_{\parallel} ^2$
Transmissivity	$T = \tau ^2 \left(\frac{\eta_1}{\eta_2} \right)$	$T_{\perp} = \tau_{\perp} ^2 \frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i}$	$T_{\parallel} = \tau_{\parallel} ^2 \frac{\eta_1 \cos \theta_t}{\eta_2 \cos \theta_i}$
Relation of R to T	$T = 1 - R$	$T_{\perp} = 1 - R_{\perp}$	$T_{\parallel} = 1 - R_{\parallel}$
Notes: (1) $\sin \theta_t = \sqrt{\mu_1 \epsilon_1 / \mu_2 \epsilon_2} \sin \theta_i$; (2) $\eta_1 = \sqrt{\mu_1 / \epsilon_1}$; (3) $\eta_2 = \sqrt{\mu_2 / \epsilon_2}$; (4) for nonmagnetic media, $\eta_2 / \eta_1 = n_1 / n_2$.			

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Phase and Group Velocities

1. Phase Velocity

The phase velocity is defined as the velocity of the sinusoidal pattern of the wave

$$v_p = \frac{\omega}{k}$$

2. Group Velocity

The velocity with which the **envelope**—or equivalently the wave group—travels through the medium is called the **group velocity** v_g . As such, v_g is the velocity of the energy carried by the wave-group, and of the information encoded in it. Depending on whether or not the propagation medium is dispersive, v_g may or may not be equal to the phase velocity v_p .

$$v_g = \left(\frac{dk}{d\omega} \right)^{-1}$$

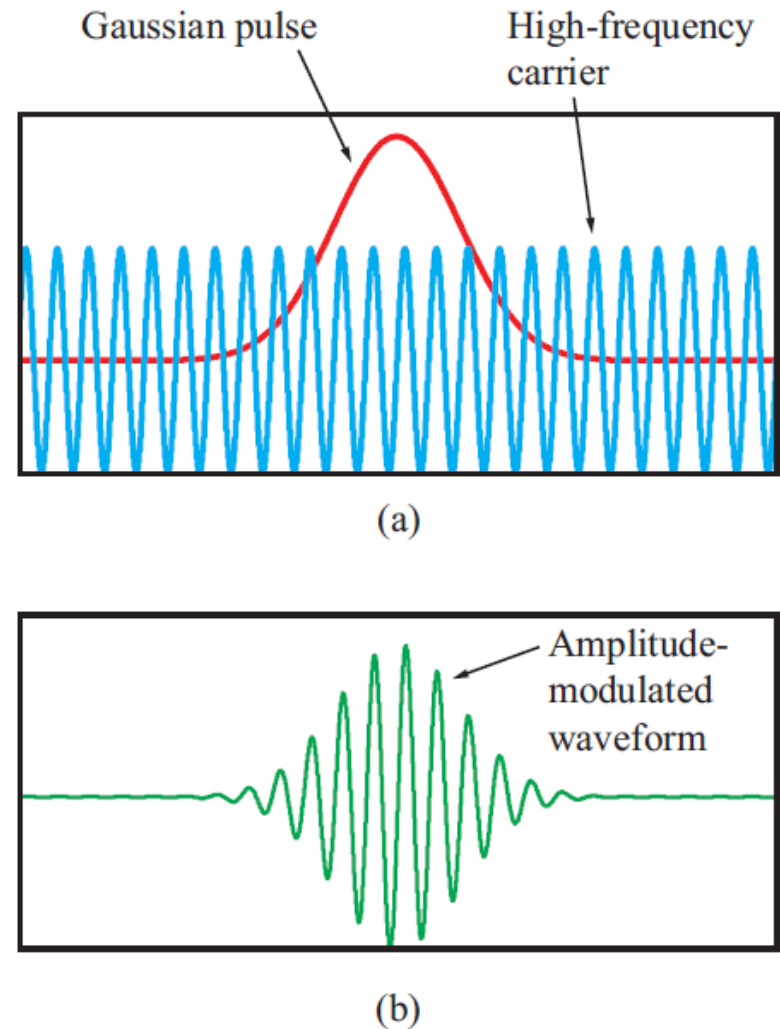


Figure 8-25: The amplitude-modulated high-frequency waveform in (b) is the product of the Gaussian-shaped pulse with the sinusoidal high-frequency carrier in (a).

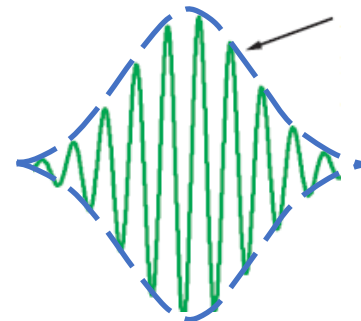
Why is the group velocity $v_g = \left(\frac{dk}{d\omega}\right)^{-1}$?

Consider 2 travelling waves, where $\Delta\omega \ll \omega_0$:

$$\begin{aligned} E(z, t) &= E_0 \cos[(\omega_0 + \Delta\omega)t - (\beta + \Delta\beta)z] \\ &\quad + E_0 \cos[(\omega_0 - \Delta\omega)t - (\beta - \Delta\beta)z] \\ &= 2E_0 \cos(t\Delta\omega - z\Delta\beta) \cos(\omega_0 t - \beta_0 z) \end{aligned}$$

The wave inside the envelope propagates with phase velocity :

$$\begin{aligned} \omega_0 t - \beta_0 z &= \text{Constant} \\ v_p &= \frac{dz}{dt} = \frac{\omega_0}{\beta_0} \end{aligned}$$

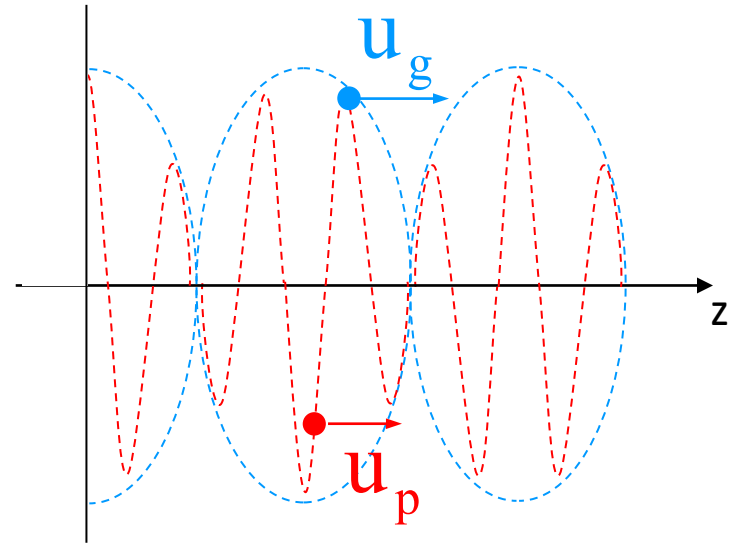


velocity of the envelope is determined by:

$$t\Delta\omega - z\Delta\beta = \text{Constant}$$

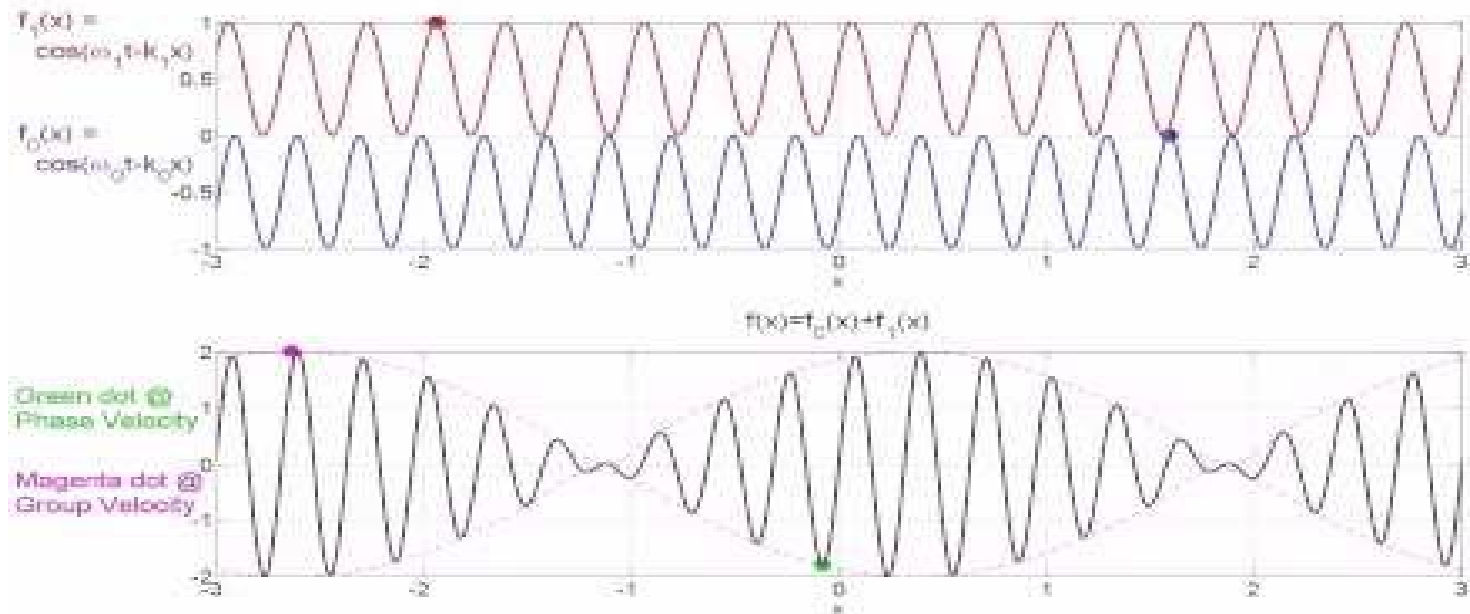
$$v_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta}$$

$$\text{As } \Delta\omega \rightarrow 0, v_g = \frac{dz}{dt} = \left(\frac{d\beta}{d\omega}\right)^{-1}$$



⇒ The group velocity is the velocity of the envelope of the wave packet

Visualizing Phase and Group Velocities



<https://youtu.be/tlM9vq-bepA>

Summary

- In this class, we looked at
 - The Poynting vector and how power is transferred by electromagnetic waves
 - The phase and group velocities of an electromagnetic wave