

EE 3008. Baseband Modulation Tutorial Solutions

Q1) Using the shift property, we have

$$|S_{01}(f)|^2 = \left\{ A\tau \operatorname{sinc}(\pi f \tau) \left| -e^{-j\pi f \tau} + e^{-j3\pi f \tau} \right| \right\}^2$$

These are two delay factors.

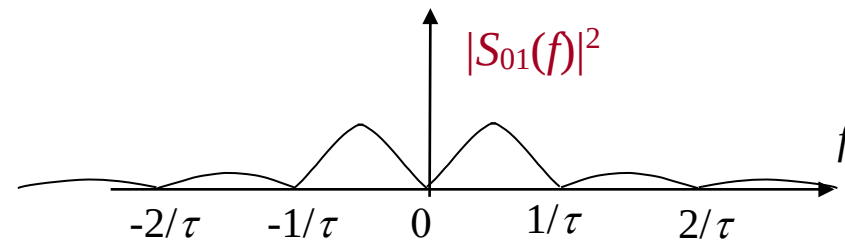
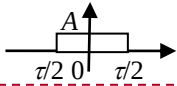
Common factor = 1.

$$= A^2 \tau^2 \operatorname{sinc}^2(\pi f \tau) \left| e^{j\pi f \tau} - e^{-j\pi f \tau} \right|^2 \left| e^{-2j\pi f \tau} \right|^2$$

$$= A^2 \tau^2 \operatorname{sinc}^2(\pi f \tau) \left| e^{j\pi f \tau} - e^{-j\pi f \tau} \right|^2$$

$$= 4A^2 \tau^2 \operatorname{sinc}^2(\pi f \tau) \sin^2(\pi f \tau)$$

This is the spectrum of



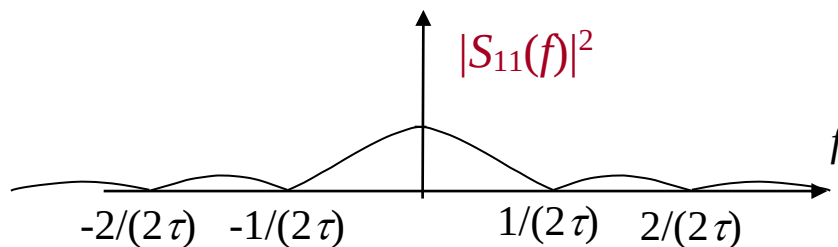
Q2) Similar to Q1, we have

$$|S_{11}(f)|^2 = \left\{ A\tau \operatorname{sinc}(\pi f \tau) \left| e^{-j\pi f \tau} + e^{-j3\pi f \tau} \right| \right\}^2$$

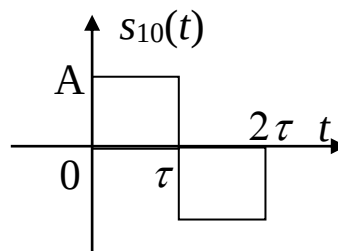
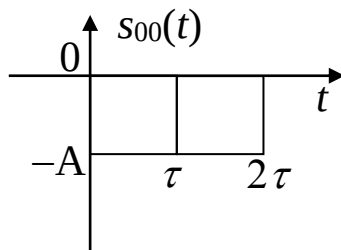
$$= A^2 \tau^2 \operatorname{sinc}^2(\pi f \tau) \left| e^{j\pi f \tau} + e^{-j\pi f \tau} \right|^2 \left| e^{-2j\pi f \tau} \right|^2$$

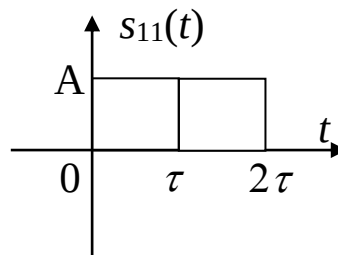
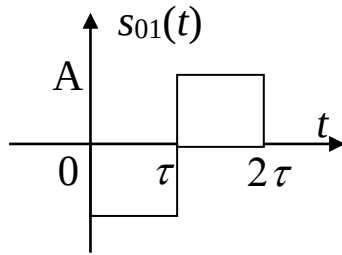
$$= A^2 \tau^2 \operatorname{sinc}^2(\pi f \tau) \left| e^{j\pi f \tau} + e^{-j\pi f \tau} \right|^2$$

$$= 4A^2 \tau^2 \operatorname{sinc}^2(\pi f \tau) \cos^2(\pi f \tau)$$



Q3) (a)





(b) Using the delay property listed in the lecture notes, we have

$$S_{00}(f) = A^2 \tau^2 \text{sinc}^2(\pi f \tau) \left| -e^{-j\pi f \tau} - e^{-j3\pi f \tau} \right|^2,$$

$$S_{10}(f) = A^2 \tau^2 \text{sinc}^2(\pi f \tau) \left| e^{-j\pi f \tau} - e^{-j3\pi f \tau} \right|^2,$$

$$S_{01}(f) = A^2 \tau^2 \text{sinc}^2(\pi f \tau) \left| -e^{-j\pi f \tau} + e^{-j3\pi f \tau} \right|^2,$$

$$S_{11}(f) = A^2 \tau^2 \text{sinc}^2(\pi f \tau) \left| e^{-j\pi f \tau} + e^{-j3\pi f \tau} \right|^2.$$

$$= 4A^2 \tau^2 \text{sinc}^2(\pi f \tau) \cos^2(\pi f \tau)$$

$$= 4A^2 \tau^2 \text{sinc}^2(\pi f \tau) \sin^2(\pi f \tau)$$

Note that $\sin^2(\pi f \tau) + \cos^2(\pi f \tau) = 1$

The average energy spectrum is

$$\frac{1}{4} \left\{ |S_{10}(f)|^2 + |S_{01}(f)|^2 + |S_{11}(f)|^2 + |S_{00}(f)|^2 \right\} = 2A^2 \tau^2 \text{sinc}^2(\pi f \tau)$$

(c) The average power spectrum is

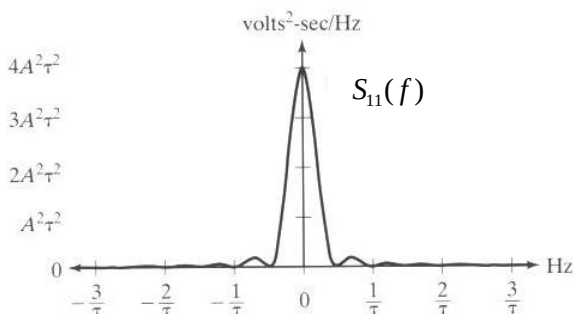
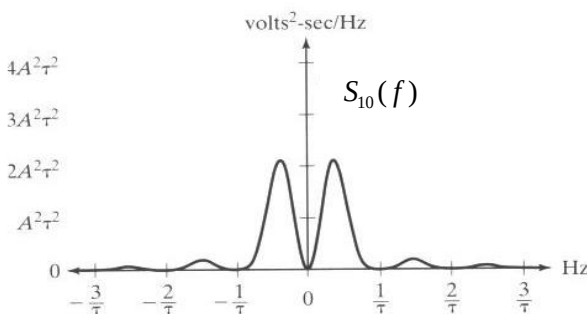
$$\frac{1}{4} \left\{ |S_{10}(f)|^2 + |S_{01}(f)|^2 + |S_{11}(f)|^2 + |S_{00}(f)|^2 \right\} / 2\tau = A^2 \tau \text{sinc}^2(\pi f \tau)$$

(d) The power spectrum for a single pulse is

$$A^2 \tau^2 \text{sinc}^2(\pi f \tau) / \tau = A^2 \tau \text{sinc}^2(\pi f \tau).$$

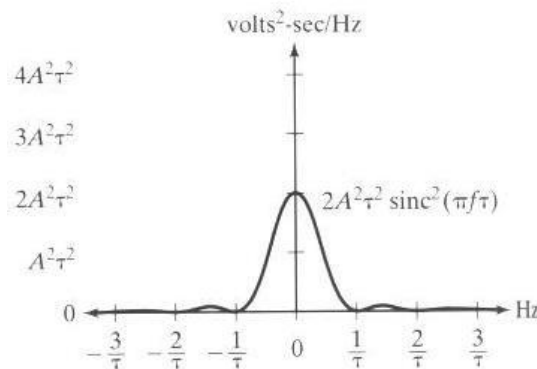
Clearly, this is the same as the result in (c).

Q4) (a) $S_{00}(f) = S_{11}(f)$ and $S_{01}(f) = S_{10}(f)$.



To get the above figures, you need to express $|e^{-j\pi f\tau} + e^{-j3\pi f\tau}|$ into a cos function form and $|e^{-j\pi f\tau} - e^{-j3\pi f\tau}|$ in a sin function form. If you have difficulty, please try to recall the knowledge about complex numbers and their absolute values. (Note that $e^{jx} = \cos(x) + j\sin(x)$.)

$$(b) \bar{S}(f) = \frac{1}{4}(S_{11}(f) + S_{00}(f) + S_{10}(f) + S_{01}(f)) = 2A^2\tau^2 \text{sinc}^2(\pi f\tau).$$



(c) The average power spectrum is $A^2\tau \text{sinc}^2(\pi f\tau)$, that differs from the answer of (b) only by a factor of 2τ .

Q5) (Notes: If you find the solution below too difficult, you may just try to read the conclusion below and skip the details.)

Let $b_1 b_2 b_3$ (each independently and equally taken over $\{\pm 1\}$) be the three bits transmitted. The Fourier transform of this 3-bit PAM waveform can be represented as

$$X_{b_1 b_2 b_3}(f) = A\tau \text{sinc}(\pi f\tau) (b_1 e^{-j\pi f\tau} + b_2 e^{-j3\pi f\tau} + b_3 e^{-j5\pi f\tau}),$$

and the energy spectrum is given by

$$\begin{aligned} \Psi_{b_1 b_2 b_3}(f) &= |X_{b_1 b_2 b_3}(f)|^2 \\ &= A^2\tau^2 \text{sinc}^2(\pi f\tau) (b_1 e^{-j\pi f\tau} + b_2 e^{-j3\pi f\tau} + b_3 e^{-j5\pi f\tau}) (b_1 e^{-j\pi f\tau} + b_2 e^{-j3\pi f\tau} + b_3 e^{-j5\pi f\tau})^* \\ &= A^2\tau^2 \text{sinc}^2(\pi f\tau) (|b_1 e^{-j\pi f\tau}|^2 + |b_2 e^{-j3\pi f\tau}|^2 + |b_3 e^{-j5\pi f\tau}|^2 + b_1 e^{-j\pi f\tau} (b_2 e^{-j3\pi f\tau})^* + b_1 e^{-j\pi f\tau} (b_3 e^{-j5\pi f\tau})^* \\ &\quad + b_2 e^{-j3\pi f\tau} (b_1 e^{-j\pi f\tau})^* + b_2 e^{-j3\pi f\tau} (b_3 e^{-j5\pi f\tau})^* + b_3 e^{-j5\pi f\tau} (b_1 e^{-j\pi f\tau})^* + b_3 e^{-j5\pi f\tau} (b_2 e^{-j3\pi f\tau})^*) \end{aligned}$$

The average energy spectrum is

$$\begin{aligned}\bar{\Psi}(f) &= \frac{1}{8}(\Psi_{+1+1+1}(f) + \Psi_{+1+1-1}(f) + \Psi_{+1-1+1}(f) + \Psi_{+1-1-1}(f) \\ &\quad + \Psi_{-1+1+1}(f) + \Psi_{-1+1-1}(f) + \Psi_{-1-1+1}(f) + \Psi_{-1-1-1}(f)) \\ &= 3A^2\tau^2\text{sinc}^2(\pi f\tau)\end{aligned}$$

Therefore, the average power spectrum is

$$G(f) = \frac{\bar{\Psi}(f)}{3\tau} = A^2\tau\text{sinc}^2(\pi f\tau).$$

Conclusion: This is the same as that of a single rectangular pulse.

Q6) (Notes: If you find the solution below too difficult, you may just try to read the conclusion and skip the details.)

Define the power spectrum of a single pulse as follows.

$$G(f) = E(\Psi_{\text{single pulse}}(f)) / \tau = E|X(f)|^2 / \tau$$

where $X(f)$ is the Fourier transform of a single pulse function. For any two independent bits b_i and b_j equally taken over $\{\pm 1\}$,

$$E(b_i b_j) = \frac{1}{4}((+1)(+1) + (+1)(-1) + (-1)(+1) + (-1)(-1)) = 0.$$

Let n be an arbitrary integer. The average energy spectrum of n -bit PAM waveform is given by

$$\begin{aligned}\bar{\Psi}(f) &= E(\Psi_{b_1 b_2 \dots b_n}(f)) = E(\Psi_{\text{single pulse}}(f)) E(|b_1 e^{-j\pi f\tau} + b_2 e^{-j3\pi f\tau} + \dots + b_n e^{-j(2n-1)\pi f\tau}|^2) \\ &= E(\Psi_{\text{single pulse}}(f)) \left(\sum_i b_i^2 + \sum_{i \neq j} E(b_i b_j) e^{-j\dots} \right) \\ &= E(\Psi_{\text{single pulse}}(f)) \left(\sum_i b_i^2 \right) \\ &= E(\Psi_{\text{single pulse}}(f)) \cdot n\end{aligned}$$

The average power spectrum is given by

$$G(f) = \frac{\bar{\Psi}(f)}{n\tau} = E(\Psi_{\text{single pulse}}(f)) / \tau.$$

Conclusion: The average power spectrum of a string of random pulses (of any shape) is the same as that of a single pulse.

Q7) (a) According to Fourier transform formula, we have

$$S(0) = S(f) \Big|_{f=0} = \int_{-\infty}^{+\infty} s(t) e^{-j2\pi ft} dt \Big|_{f=0} = \int_{-\infty}^{+\infty} s(t) dt = 2A\tau$$

So $S(0)$ is the total area of $s(t)$

(b) Define $g(t) = \begin{cases} A, & -\tau/2 \leq t \leq \tau/2 \\ 0, & \text{others} \end{cases}$. Its spectrum is given by $G(f) = A\tau \text{sinc}(\pi f \tau)$. Hence $s(t)$ can be expressed as

$$s(t) = g\left(t - \frac{\tau}{2}\right) + g\left(t - \frac{3\tau}{2}\right) - g\left(t - \frac{5\tau}{2}\right) + g\left(t - \frac{7\tau}{2}\right)$$

Based on time shift property of Fourier transform, the corresponding spectrum of $s(t)$ is given by

$$\begin{aligned} S(f) &= G(f)(e^{-j\pi f \tau} + e^{-j3\pi f \tau} - e^{-j5\pi f \tau} + e^{-j7\pi f \tau}) \\ &= A\tau \text{sinc}(\pi f \tau)(e^{-j\pi f \tau} + e^{-j3\pi f \tau} - e^{-j5\pi f \tau} + e^{-j7\pi f \tau}) \end{aligned}$$

Since $\text{sinc}(\pi f \tau) \Big|_{f=3/\tau} = \text{sinc}(\pi \cdot 3) = 0$,

we have $S(3/\tau) = 0$.

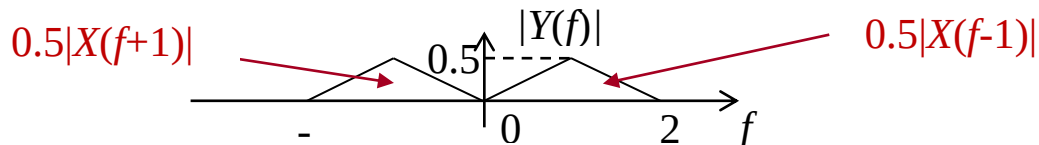
$$(c) \quad S\left(\frac{1}{4\tau}\right) = A\tau \text{sinc}(\pi/4)(e^{-j\pi/4} + e^{-j3\pi/4} - e^{-j5\pi/4} + e^{-j7\pi/4}).$$

Q8) (a) Using the duality property, we have the following:

$$\text{sinc}^2(\pi t) \Leftrightarrow \begin{array}{c} \uparrow 1 \\ \diagdown \quad \diagup \\ -1 \quad \quad 1 \\ \leftarrow f \end{array}$$

Time shift will not change the amplitude spectrum, so the amplitude spectrum of $x(t) = \text{sinc}^2(\pi t - 2\pi)$ is still the same as the above.

(b) Using the modulation property, the spectrum of $\text{sinc}^2(\pi t) \cos(2\pi t)$ is $0.5 \cdot (X(f+1) + X(f-1))$:

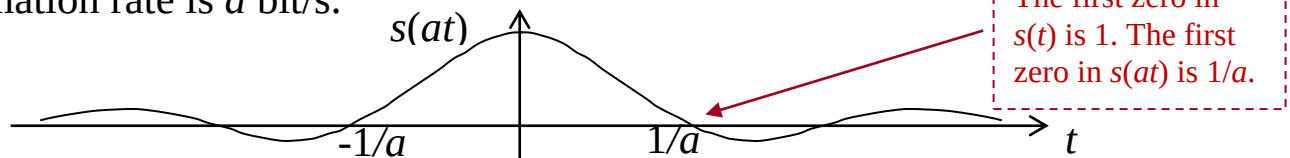


Since time shift does not change amplitude spectrum, the amplitude spectrum of $\text{sinc}^2(\pi(t-2)) \cos(2\pi(t-2))$ is the same as shown above.

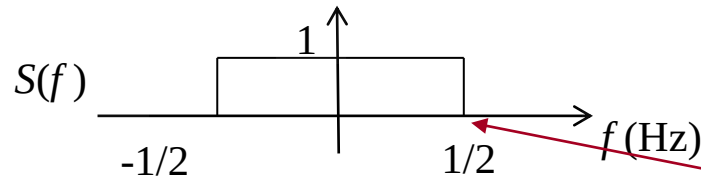
Q9) (a) The curve of $f(t)$ is given below

$$\pm s(at), \pm s(a(t-1/a)), \pm s(a(t-2/a)), \dots$$

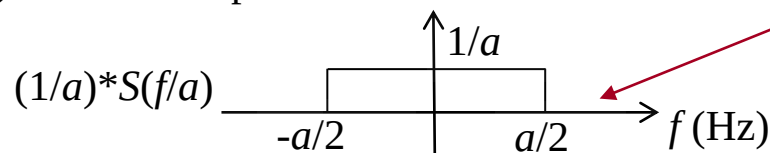
One pulse is transmitted in every $1/a$ second duration, so the information rate is a bit/s.



(c) The spectrum of $s(t)=\text{sinc}(\pi t)$ is shown below.

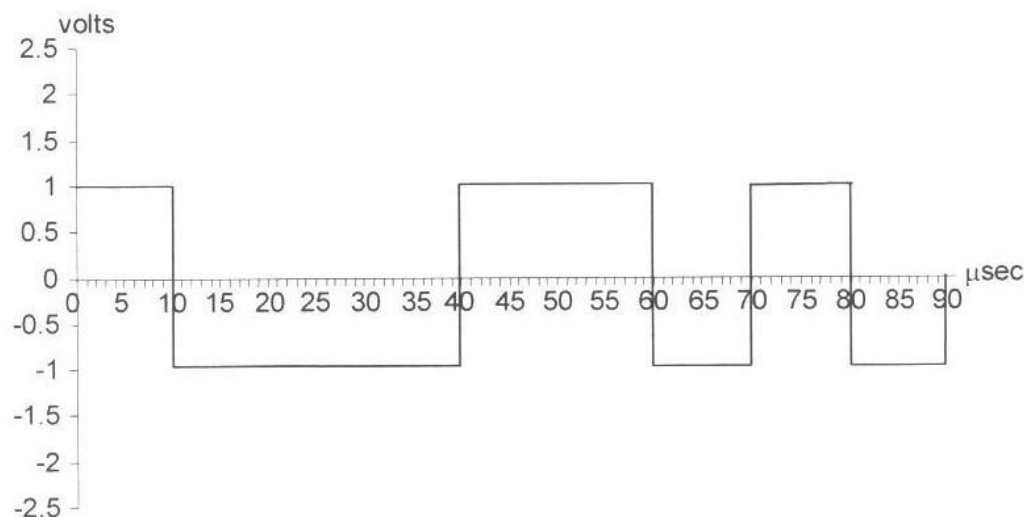


By the time scale property, the spectrum of $s(at)=\text{sinc}(\pi at)$ is $(1/a)*S(f/a)$. So the occupied bandwidth is $a/2$ Hz.



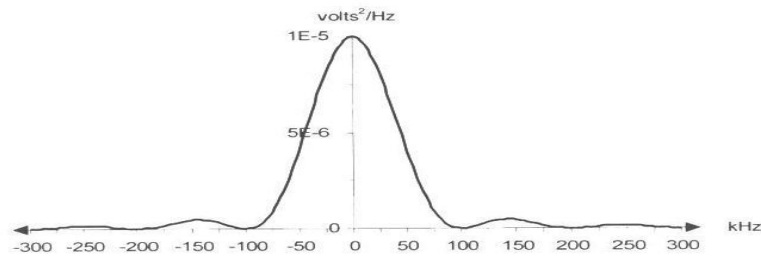
(Note: If $a>1$, transmission speed is increased, pulse duration is reduced and bandwidth is increased. If $a<1$, bandwidth is decreased.)

Q13) (a) (1) The width of each pulse is $1/100000=10^{-5}=10\mu\text{s}$.



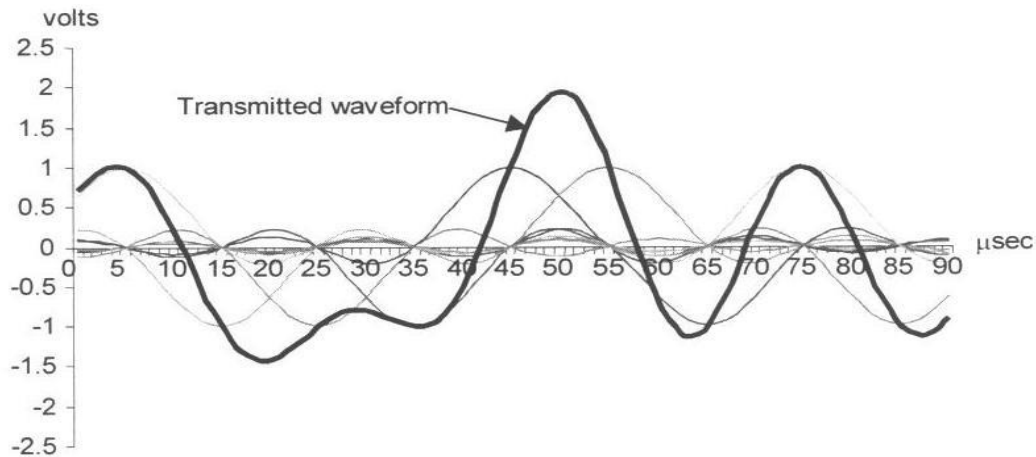
(2) It can be shown that the average power spectrum of a sequence of pulses is the same as that of a single pulse. (You have seen this in previous questions.) Thus

$$G(f) = A^2 \tau \text{sinc}^2(\pi f \tau) = 10^{-5} \text{sinc}^2(10^{-5} \pi f) \text{ V}^2 / \text{Hz}$$

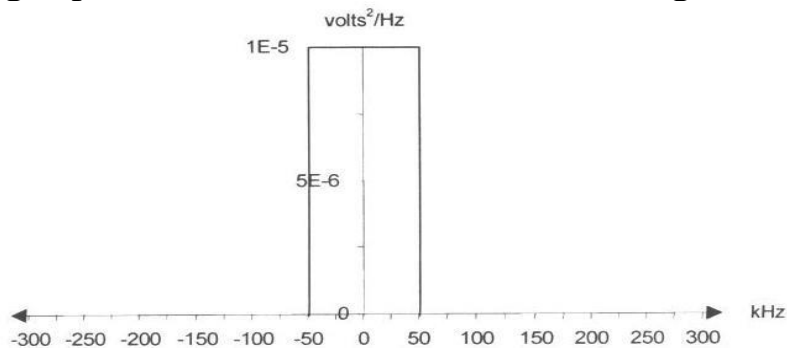


(3) Bandwidth = 200,000 Hz, corresponding to the second zero crossing of the power spectrum.

(b) (1) The following needs some careful drawing. You may use a computer to do it.



(2) The average spectrum is the same as that of a single sinc pulse.



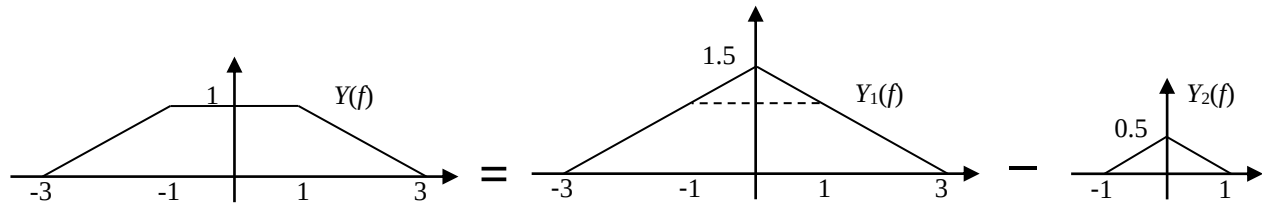
(3) Bandwidth = 50,000 Hz. (Note that this is 1/4 the bandwidth required for Part (a) (rectangular pulses).)

Q14) According to the duality property, we have the following:

$\text{sinc}^2(\pi t) \iff$



We can decompose $Y(f)$ as follows:



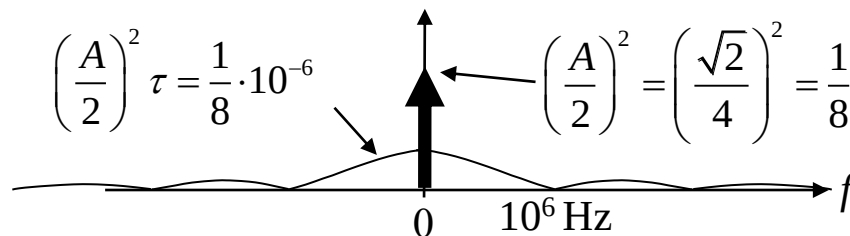
According to the scaling property, we have (please verify):

$$y_1(t) = (9/2) \text{sinc}^2(3\pi t)$$

$$y_2(t) = (1/2) \text{sinc}^2(\pi t).$$

Therefore, $y(t) = y_1(t) - y_2(t) = 4.5 \text{sinc}^2(3\pi t) - 0.5 \text{sinc}^2(\pi t)$.

Q15) (a) For $R = 10^6$ b/s, the duration of a rectangular pulse is $\tau = 10^{-6}$ s. The on-off signal can be decomposed as the sum of a bipolar signal plus a constant $A/2$. Its average power spectrum is given as follows:



(b) The transmit power for “on” is $P_{\text{on}} = A^2 = 0.5\text{W}$. The transmit power for “off” is $P_{\text{off}} = 0\text{W}$. The average power for the on-off scheme system is 0.25W . Therefore, the average energy used to transmit each bit is

$$E_b = P_{\text{avg}} T = 0.25 \times 10^{-6} = 2.5 \times 10^{-7} \text{ Joule}.$$

Q16) The answer is about 10%. This details are left to yourself.

Hints: The distortion power can be found by applying the Parseval’s Theorem to the spectrum of a rectangular pulse beyond its first zero.