

Homework #4

Solutions:

Course: EE3121 Differential Equations for Electrical Engineering

Date:

Problem 1

Solution. (a) Step 1: Try the solution e^{rt} .

Step 2: Characteristic polynomial

$$r^2 - 8r + 41 = 0$$

$$r_{1,2} = \frac{8 \pm \sqrt{64 - 4 \cdot 41}}{2} = \frac{8 \pm \sqrt{-100}}{2} = \frac{8 \pm i10}{2} = 4 \pm i5 = \lambda + i\omega$$

$$y(t) = c_1 e^{\lambda t} \cos \omega t + c_2 e^{\lambda t} \sin \omega t = c_1 e^{4t} \cos 5t + c_2 e^{4t} \sin 5t$$

Step 3: Solve for c_1 and c_2

$$6 = y(0) = c_1$$

$$\frac{dy}{dt} = c_1 4e^{4t} \cos 5t + c_1 e^{4t} (-5 \sin 5t) + c_2 4e^{4t} \sin 5t + c_2 e^{4t} 5 \cos 5t$$

$$-2 = \left. \frac{dy}{dt} \right|_{t=0} = 4c_1 + 5c_2$$

$$\Rightarrow \begin{cases} c_1 = 6 \\ 4c_1 + 5c_2 = -2 \end{cases} \Rightarrow \begin{cases} c_1 = 6 \\ c_2 = -\frac{26}{5} \end{cases}$$

$$\Rightarrow y(t) = 6e^{4t} \cos 5t - \frac{26}{5}e^{4t} \sin 5t$$

(b) Step 1: Find two fundamental solutions:

$$\frac{d^2 y}{dt^2} + 6 \frac{dy}{dt} + 8y = 0$$

Characteristic polynomial

$$r^2 + 6r + 8 = 0$$

$$\Rightarrow (r+2)(r+4) = 0 \Rightarrow r_1 = -2, r_2 = -4$$

Fundamental solutions

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{-4t}$$

Step 2: Find the kernel function

$$K(t-\tau) = \frac{e^{-\lambda_1(t-\tau)} - e^{-\lambda_2(t-\tau)}}{\lambda_2 - \lambda_1}$$

$$\lambda_1 = 2, \lambda_2 = 4$$

$$\Rightarrow K(t - \tau) = \frac{e^{-2(t-\tau)} - e^{-4(t-\tau)}}{4 - 2} = \frac{e^{-2(t-\tau)} - e^{-4(t-\tau)}}{2}$$

Step 3: Calculate the convolution integral

$$\begin{aligned} y(t) &= \int_0^t K(t - \tau)g(\tau)d\tau = \frac{1}{2} \int_0^t [e^{-2(t-\tau)} - e^{-4(t-\tau)}] 3\tau d\tau = \frac{3}{2} \int_0^t e^{-2(t-\tau)} \tau d\tau - \frac{3}{2} \int_0^t e^{-4(t-\tau)} \tau d\tau \\ &= \frac{3}{2} e^{-2t} \int_0^t \tau e^{2\tau} d\tau - \frac{3}{2} e^{-4t} \int_0^t \tau e^{4\tau} d\tau = \frac{3}{2} e^{-2t} \cdot \frac{1}{2} \int_0^t \tau d e^{2\tau} - \frac{3}{2} e^{-4t} \cdot \frac{1}{4} \int_0^t \tau d e^{4\tau} \\ &= \frac{3}{4} e^{-2t} \left(\tau e^{2\tau} \Big|_0^t - \int_0^t e^{2\tau} d\tau \right) - \frac{3}{8} e^{-4t} \left(\tau e^{4\tau} \Big|_0^t - \int_0^t e^{4\tau} d\tau \right) \\ &= -\frac{3}{32} e^{-4t} + \frac{3}{8} e^{-2t} + \frac{3}{8} t - \frac{9}{32} \end{aligned}$$

(c) Step 1: Trial solution

$$y(t) = (C \sin 2t + D \cos 2t) e^{-4t}$$

Step 2: Substitute the trial solution into the differential equation

$$\begin{aligned} \frac{dy}{dt} &= -4e^{-4t} (C \sin 2t + D \cos 2t) + e^{-4t} (2C \cos 2t - D \sin 2t) \\ &= e^{-4t} [(-4C - 2D) \sin 2t + (-4D + 2C) \cos 2t] \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dt^2} &= -4e^{-4t} [(-4C - 2D) \sin 2t + (-4D + 2C) \cos 2t] \\ &\quad + e^{-4t} [2(-4C - 2D) \cos 2t - 2(-4D + 2C) \sin 2t] \\ &= e^{-4t} \{ [-4(-4C - 2D) - 2(-4D + 2C)] \sin 2t \\ &\quad + [2(-4C - 2D) - 4(-4D + 2C)] \cos 2t \} \end{aligned}$$

Step 3: Evaluate the D.E.

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 8e^{-4t} \sin 2t$$

$$a_1 = -4(-4C - 2D) - 2(-4D + 2C) = 20C + 16D$$

$$a_2 = 2(-4C - 2D) - 4(-4D + 2C) = -16C + 12D$$

$$a_3 = -4C - 2D$$

$$a_4 = -4D + 2C$$

$$a_1 \sin 2t + a_2 \cos 2t + 4(a_3 \sin 2t + a_4 \cos 2t) + 4(C \sin 2t + D \cos 2t) = 8 \sin 2t$$

$$\Rightarrow \begin{cases} a_1 + 4a_3 + 4C = 8 \\ a_2 + 4a_4 + 4D = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 20C + 16D + 4(-4C - 2D) + 4C = 8 \\ -16C + 12D + 4(-4D + 2C) + 4D = 0 \end{cases}$$

$$\Rightarrow \begin{cases} C = 0 \\ D = 1 \end{cases}$$

$$y(t) = e^{-4t} \cos 2t$$

Since $y(0) = 1 \neq 0$

$$r^2 + 4r + 4 = 0$$

$$\Rightarrow r_{1,2} = -2$$

$$\Rightarrow y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + e^{-4t} \cos 2t$$

$$\frac{dy}{dt} = -2c_1 e^{-2t} + c_2(e^{-2t} - 2t e^{-2t}) + (-4e^{-4t} \cos 2t - 2e^{-4t} \sin 2t)$$

$$\Rightarrow \begin{cases} c_1 + 1 = 0 \\ -2c_1 + c_2 - 4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = -1 \\ c_2 = 2 \end{cases}$$

$$y(t) = -e^{-2t} + 2t e^{-2t} + e^{-4t} \cos 2t$$
