

EE3121

Differential Equations for Engineers

Lecture 6: Higher-Order Linear Ordinary Differential Equations



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Summary

- Nonlinear ODEs are typically difficult to solve analytically. This chapter presented three techniques for doing so that have a modest range of applicability:
 - Separating variables;
 - Transforming the nonlinear ODE into a linear one;
 - For almost linear systems, successive approximations.

Summary

Separable equations have the form

$$\frac{dy}{dt} = f(t)g(y), \quad y(t_0) = y_0.$$

We separate variables by dividing through by $g(y)$:

$$\frac{1}{g(y)} \frac{dy}{dt} = f(t).$$

Then we employ the shortcut procedure introduced in Lecture 2:

$$\int_{y(t_0)}^{y(t)} \frac{dy}{g(y)} = \int_{t_0}^t f(\tau) d\tau. \quad (1)$$

Summary

Sometimes we can perform the integration on both sides and arrive at an implicit equation

$$G(y) = F(t).$$

- In rare circumstances we are then able to solve for y as an explicit function of t ;
- The transformations can involve changing either the independent or dependent variable or both;
- Successive approximations on almost linear systems turns a nonlinear problem into a sequence of linear ones.

Summary

If

$$\frac{dy}{dt} + p(t)y + f(t, y) = g(t), \quad y(t_0) = y_0,$$

and the nonlinear term $f(t, y)$ is not too large then solving the sequence

$$\frac{dy_1}{dt} + p(t)y_1 = g(t) - f(t, y_0), \quad y_1(t_0) = y_0$$

$$\frac{dy_2}{dt} + p(t)y_2 = g(t) - f(t, y_1), \quad y_2(t_0) = y_0$$

...

$$\frac{dy_n}{dt} + p(t)y_n = g(t) - f(t, y_{n-1}), \quad y_n(t_0) = y_0$$

can sometimes yield a solution approximating the nonlinear answer.

Solved Problems

S.P3.1 Solve

$$\frac{dy}{dt} = 3(2 + t)\sqrt{4 - y^2}, \quad y(1) = 0$$

Solution:

We separate variables, putting everything dealing with y on the left-hand side of the equation and everything dealing with t on the right:

$$\frac{dy}{\sqrt{4 - y^2}} = 3(2 + t)dt.$$

Solved Problems

Then we integrate both sides from beginning to endpoint. These are definite integrals.

- On the right-hand side the beginning is 1 and its endpoint is the independent variable t .
- On the left-hand side the beginning is $y(1) = 0$ and the endpoint the dependent variable y .
- To avoid confusion we use the dummy integration variables u and τ underneath the integral signs to distinguish them from the endpoint variables:

Solved Problems

$$\int_0^y \frac{du}{\sqrt{4-u^2}} = 3 \int_1^t (2+\tau) d\tau. \quad (2)$$

Now, on the left, let $u = 2v$. Then

$$\int_0^y \frac{du}{\sqrt{4-u^2}} = \int_0^{y/2} \frac{2dv}{\sqrt{4-4v^2}} = \int_0^{y/2} \frac{dv}{\sqrt{1-v^2}} = \arcsin\left(\frac{y}{2}\right). \quad (3)$$

And on the right

$$3 \int_1^t (2+\tau) d\tau = 3 \left(2t + \frac{t^2}{2} - 2(1) - \frac{(1)^2}{2} \right) = 6t + \frac{3t^2}{2} - \frac{15}{2}. \quad (4)$$

Solved Problems

Using Equations (2), (3), and (4):

$$\arcsin\left(\frac{y}{2}\right) = 6t + \frac{3t^2}{2} - \frac{15}{2}.$$

Hence

$$y(t) = 2 \sin\left(6t + \frac{3t^2}{2} - \frac{15}{2}\right).$$

Solved Problems

S.P3.2 Solve

$$\frac{dy}{dt} = 5e^{2t+4y}, \quad y(t_0) = y_0$$

Solution:

Separating variables:

$$e^{-4y} dy = 5e^{2t} dt$$

Integrating both sides and changing to dummy variables under the integral signs:

$$\int_{y_0}^y e^{-4u} du = 5 \int_{t_0}^t e^{2\tau} d\tau$$

Solved Problems

$$\left(\frac{1}{4}\right) (e^{-4y_0} - e^{-4y}) = 5 \left(\frac{1}{4}\right) (e^{2t} - e^{2t_0})$$

$$e^{-4y} = e^{-4y_0} - 10 (e^{2t} - e^{2t_0})$$

$$-4y = \ln (e^{-4y_0} - 10 (e^{2t} - e^{2t_0}))$$

$$y = - \left(\frac{1}{4}\right) \ln (e^{-4y_0} - 10 (e^{2t} - e^{2t_0})) .$$

Solved Problems

S.P3.3 Solve

$$\frac{dy}{dt} = a \cos^2 y \cos^2 t, \quad y(t_0) = y_0$$

Solution:

Separating variables:

$$\frac{dy}{\cos^2 y} = a \cos^2 t dt.$$

Integrating both sides and changing to dummy variables under the integral sign:

$$\int_{y_0}^y \frac{du}{\cos^2 u} = \int_{t_0}^t a \cos^2 \tau d\tau. \quad (5)$$

Solved Problems

On the left-hand side, recalling our trigonometric identities:

$$\int_{y_0}^y \frac{du}{\cos^2 u} = \int_{y_0}^y \sec^2 u du. \quad (6)$$

Now remembering the derivatives of common functions in [Table 1.1](#):

$$\frac{d}{du}(\tan u) = \sec^2 u,$$

so Equation (6) becomes, by the fundamental theorem of calculus

$$\int_{y_0}^y \frac{du}{\cos^2 u} = \int_{y_0}^y \sec^2 u du = \int_{y_0}^y \frac{d}{du}(\tan u) du = \tan y - \tan y_0. \quad (7)$$

Solved Problems

On the right-hand side of Equation (5), we recall the trigonometric identity

$$\cos^2 \tau = (1 + \cos 2\tau)/2, \quad (8)$$

and then

$$\int_{t_0}^t a \cos^2 \tau d\tau = \frac{a}{2} \int_{t_0}^t (1 + \cos 2\tau) d\tau = \frac{a}{2}(t - t_0) + \frac{a}{4}(\sin 2t - \sin 2t_0). \quad (9)$$

From Equations (5), (7), and (9):

$$\tan y - \tan y_0 = \frac{a}{2}(t - t_0) + \frac{a}{4}(\sin 2t - \sin 2t_0).$$

$$y = \arctan \left(\tan y_0 + \frac{a}{2}(t - t_0) + \frac{a}{4}(\sin 2t - \sin 2t_0) \right).$$

Solved Problems

S.P3.4 Solve

$$\frac{dy}{dt} + 8y + y^2 = -15, y(0) = 0$$

Solution:

Rewriting:

$$\frac{dy}{dt} = -(y^2 + 8y + 15).$$

Separating variables:

$$\frac{dy}{y^2 + 8y + 15} = -dt.$$

Solved Problems

Integrating both sides, changing to dummy variables under the integral sign,

$$\int_0^y \frac{du}{u^2 + 8u + 15} = - \int_0^t d\tau = -t. \quad (10)$$

Using partial fraction expansion:

$$\frac{1}{u^2 + 8u + 15} = \frac{1}{(u + 3)(u + 5)} = \frac{c_1}{u + 3} + \frac{c_2}{u + 5}. \quad (11)$$

Recombining:

$$\frac{c_1}{u + 3} + \frac{c_2}{u + 5} = \frac{c_1(u + 5) + c_2(u + 3)}{(u + 3)(u + 5)}. \quad (12)$$

Solved Problems

Collecting coefficients in powers of u in the numerator:

$$\frac{c_1(u + 5) + c_2(u + 3)}{(u + 3)(u + 5)} = \frac{(c_1 + c_2)u + 5c_1 + 3c_2}{(u + 3)(u + 5)}. \quad (13)$$

From Equations (11), (12), and (13):

$$\frac{1}{(u + 3)(u + 5)} = \frac{(c_1 + c_2)u + 5c_1 + 3c_2}{(u + 3)(u + 5)},$$

which implies that

$$1 = (c_1 + c_2)u + 5c_1 + 3c_2 \quad (14)$$

Solved Problems

- We can consider both sides of Equation (14) as a polynomial in u . As discussed in [Section 1.1.2.2](#):
 - If two polynomials are everywhere equal then their coefficients are equal. Then $5c_1 + 3c_2$, the coefficient of $u^0 = 1$ on the right-hand side, equals the coefficient of u^0 on the left, which is 1.
 - Similarly, $c_1 + c_2$, the coefficient of $u^1 = u$ on the right-hand side, equals the coefficient of u^1 on the left, which is zero. This gives us two equations in two unknowns:

$$5c_1 + 3c_2 = 1$$

$$c_1 + c_2 = 0$$

Solved Problems

Solving:

$$\begin{aligned}c_1 &= \frac{1}{2} \\c_2 &= -\frac{1}{2}\end{aligned}$$

Then Equation (10) can be written as

$$\int_0^y \frac{(1/2)du}{u+3} - \int_0^y \frac{(1/2)du}{u+5} = -t$$

$$\begin{aligned}\int_0^y \frac{du}{u+3} - \int_0^y \frac{du}{u+5} &= -2t \\ \ln(y+3) - \ln 3 - \ln(y+5) + \ln 5 &= -2t \\ \ln \left(\frac{y+3}{3} \cdot \frac{5}{y+5} \right) &= -2t\end{aligned}$$

Solved Problems

$$\exp \left(\ln \left(\frac{y+3}{3} \cdot \frac{5}{y+5} \right) \right) = \exp(-2t)$$

$$\frac{y+3}{3} \cdot \frac{5}{y+5} = \exp(-2t)$$

$$5(y+3) = 3(y+5)e^{-2t}$$

$$(5 - 3e^{-2t})y = -15(1 - e^{-2t})$$

$$y = -\frac{3(1 - e^{-2t})}{1 - (3/5)e^{-2t}}$$

Solved Problems

S.P3.5 Find the second in the sequence of successive approximations to the solution of

$$\frac{dy}{dt} + 3y + \varepsilon y^2 = 3, \quad y(0) = 0$$

given that the first is the solution when $\varepsilon = 0$.

Solution:

We first find the solution $y_1(t)$ when $\varepsilon = 0$. The equations are

$$\frac{dy_1}{dt} + 3y_1 = 3, \quad y_1(0) = 0$$

Solved Problems

Using the integrating factor,

$$e^{3t} \left(\frac{dy_1}{dt} + 3y_1 \right) = 3e^{3t}$$

$$\frac{d}{dt}(e^{3t}y_1) = 3e^{3t}$$

$$\int_0^t \frac{d}{d\tau}(e^{3\tau}y_1)d\tau = \int_0^t 3e^{3\tau}d\tau$$

$$e^{3t}y_1(t) = e^{3t} - 1$$

$$y_1(t) = 1 - e^{-3t}$$

Solved Problems

We find the second approximation $y_2(t)$ by solving the equations

$$\frac{dy_2}{dt} + 3y_2 = 3 - \varepsilon y_1^2(t), \quad y_2(0) = 0$$

$$\frac{dy_2}{dt} + 3y_2 = 3 - \varepsilon(1 - e^{-3t})^2$$

$$\frac{dy_2}{dt} + 3y_2 = 3 - \varepsilon(1 - 2e^{-3t} + e^{-6t})$$

Solved Problems

Again using the integration factor,

$$e^{3t} \left(\frac{dy_2}{dt} + 3y_2 \right) + 3e^{3t} - \varepsilon e^{3t} (1 - 2e^{-3t} + e^{-6t})$$

$$\frac{d}{dt}(e^{3t}y_2) = 3e^{3t} - \varepsilon(e^{3t} - 2 + e^{-3t})$$

$$\int_0^t \frac{d}{d\tau}(e^{3\tau}y_2) = \int_0^t [3e^{3\tau} - \varepsilon(e^{3\tau} - 2 + e^{-3\tau})] d\tau$$

$$e^{3t}y_2(t) = e^{3t} - 1 - \varepsilon \left[(e^{3t} - 1)/3 - 2t + (1 - e^{-3t})/3 \right]$$

$$y_2(t) = 1 - e^{-3t} - \varepsilon(1 - 6te^{-3t} - e^{-6t})/3$$

END