

**MA2001      Solution to Problem Set 2      Differential Calculus of Multivariable Functions**

1. Let  $w = f(x, y, z) = z(x^2 + y^2)^{-1}$ . At  $(2, 1, 1)$ , find the rate of change of  $w$  with respect to  $y$ .

Suppose  $x = u + v$ ,  $y = u$  and  $z = uv$ , at  $(1, 1)$ , find the rates of change of  $w$  with respect to  $u$  and

$v$ , respectively. Observe that  $\frac{\partial f}{\partial y} \neq \frac{\partial f}{\partial u}$  even though  $y = u$ . Why is this so?

Solution:

$$w = f(x, y, z) = z(x^2 + y^2)^{-1} \Rightarrow \frac{\partial w}{\partial y} = -z(x^2 + y^2)^{-2} 2y = -\frac{2yz}{(x^2 + y^2)^2}$$

$$\Rightarrow \frac{\partial w}{\partial y}(2, 1, 1) = -\frac{2}{25}$$

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} = -\frac{2xz}{(x^2 + y^2)^2} - \frac{2yz}{(x^2 + y^2)^2} + \frac{v}{x^2 + y^2}$$

$$\left\{ \begin{array}{l} u=1 \\ v=1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x=2 \\ y=1 \\ z=1 \end{array} \right. \Rightarrow \frac{\partial w}{\partial u}(1, 1) = -\frac{2xz}{(x^2 + y^2)^2} \bigg|_{\substack{x=2 \\ y=1 \\ z=1}} - \frac{2yz}{(x^2 + y^2)^2} \bigg|_{\substack{x=2 \\ y=1 \\ z=1}} + \frac{v}{x^2 + y^2} \bigg|_{\substack{x=2 \\ y=1 \\ v=1}} = -\frac{4}{25} - \frac{2}{25} + \frac{1}{5} = -\frac{1}{25}$$

$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial v} = -\frac{2xz}{(x^2 + y^2)^2} + \frac{u}{x^2 + y^2}$$

$$\frac{\partial w}{\partial v}(1, 1) = -\frac{2xz}{(x^2 + y^2)^2} \bigg|_{\substack{x=2 \\ y=1 \\ z=1}} + \frac{u}{x^2 + y^2} \bigg|_{\substack{x=2 \\ y=1 \\ u=1}} = -\frac{4}{25} + \frac{1}{5} = \frac{1}{25}$$

$$\frac{\partial f}{\partial y} - \frac{\partial f}{\partial u} = \frac{2xz}{(x^2 + y^2)^2} - \frac{v}{x^2 + y^2} \neq 0 \text{ in most cases}$$

$\frac{\partial f}{\partial u}$  is used to measure how  $w$  changes as  $u$  changes with  $v$  kept unchanged. We observe as  $u$  changes with  $v$

kept unchanged, all  $x, y, z$  will change although  $y$  will have the same level of change as  $u$  does and the change of  $x, y, z$  all three will affect the change of  $w$ .

However,  $\frac{\partial f}{\partial y}$  is used to measure how  $w$  changes as  $y$  changes with  $x, z$  kept unchanged. In that situation

only the change of  $y$  will affect the change of  $w$ . □

2. Suppose  $w$  is a function of  $u, v$ , that is,  $w = w(u, v)$ . Suppose  $u = x + y$ ,  $v = x - y$ ,  $w = xy - z$ , therefore,  $z$  is a function of  $x, y$ . Transform the following partial differential equation of  $z$  in  $x, y$ ,

$$\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0 \text{ into a partial differential equation of } w \text{ in } u \text{ and } v.$$

Solution:

$$w = xy - z \Rightarrow z = xy - w \Rightarrow \frac{\partial z}{\partial x} = y - \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} - \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} = y - \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}$$

Similarly,

$$w = xy - z \Rightarrow z = xy - w \Rightarrow \frac{\partial z}{\partial y} = x - \frac{\partial w}{\partial u} \frac{\partial u}{\partial y} - \frac{\partial w}{\partial v} \frac{\partial v}{\partial y} = x - \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}.$$

Then

$$\begin{aligned} \frac{\partial z}{\partial x} &= y - \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v} \\ \Rightarrow \frac{\partial^2 z}{\partial x^2} &= -\left(\frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial x}\right) - \left(\frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial x}\right) \\ &= -\frac{\partial^2 w}{\partial u^2} - \frac{\partial^2 w}{\partial v \partial u} - \frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2} = -\frac{\partial^2 w}{\partial u^2} - 2\frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= x - \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \\ \Rightarrow \frac{\partial^2 z}{\partial y^2} &= -\left(\frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial y}\right) + \left(\frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial y}\right) \\ &= -\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v \partial u} + \frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2} = -\frac{\partial^2 w}{\partial u^2} + 2\frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2} \end{aligned}$$

Also

$$\begin{aligned} \frac{\partial z}{\partial x} &= y - \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v} \\ \Rightarrow \frac{\partial^2 z}{\partial y \partial x} &= 1 - \left(\frac{\partial^2 w}{\partial u^2} \frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v \partial u} \frac{\partial v}{\partial y}\right) - \left(\frac{\partial^2 w}{\partial u \partial v} \frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2} \frac{\partial v}{\partial y}\right) = 1 - \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v \partial u} - \frac{\partial^2 w}{\partial u \partial v} + \frac{\partial^2 w}{\partial v^2} = 1 - \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} \end{aligned}$$

Finally

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} &= 0 \Rightarrow -\frac{\partial^2 w}{\partial u^2} - 2\frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2} + 2\left(1 - \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right) - \frac{\partial^2 w}{\partial u^2} + 2\frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2} = 0 \\ \Rightarrow -4\frac{\partial^2 w}{\partial u^2} + 2 &= 0 \Rightarrow \frac{\partial^2 w}{\partial u^2} = \frac{1}{2} \end{aligned}$$

□

3. Let  $z = f\left(x, \frac{x}{y}\right)$  and  $s = x, t = \frac{x}{y}$ . Assume  $f$  has continuous second order partial derivatives. Find  $\frac{\partial z}{\partial x}$

and show that 
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial s^2} + \frac{2}{y} \frac{\partial^2 f}{\partial s \partial t} + \frac{1}{y^2} \frac{\partial^2 f}{\partial t^2}.$$

Solution:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial s} \frac{ds}{dx} + \frac{\partial f}{\partial t} \frac{dt}{dx} = \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \frac{1}{y}.$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \frac{1}{y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial s} \right) + \frac{1}{y} \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial t} \right) \\ &= \frac{\partial^2 f}{\partial s^2} \frac{ds}{dx} + \frac{\partial^2 f}{\partial s \partial t} \frac{\partial t}{\partial x} + \frac{1}{y} \left( \frac{\partial^2 f}{\partial s \partial t} \frac{ds}{dx} + \frac{\partial^2 f}{\partial t^2} \frac{dt}{dx} \right) = \frac{\partial^2 f}{\partial s^2} + \frac{2}{y} \frac{\partial^2 f}{\partial s \partial t} + \frac{1}{y^2} \frac{\partial^2 f}{\partial t^2}\end{aligned}$$

□

4. Suppose  $u = u(x, y)$  satisfies the wave equation  $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$ . Let  $y = y(x) = 2x$ . Along the line

$$y = 2x, \text{ we have } u(x, y(x)) = x \text{ and } \left. \frac{\partial u}{\partial x} \right|_{\substack{x=x \\ y=2x}} = x^2. \text{ Find along the line } y = 2x, \left. \frac{\partial u}{\partial y} \right|_{\substack{x=x \\ y=2x}}, \left. \frac{\partial^2 u}{\partial y \partial x} \right|_{\substack{x=x \\ y=2x}},$$

$$\left. \frac{\partial^2 u}{\partial x^2} \right|_{\substack{x=x \\ y=2x}}, \left. \frac{\partial^2 u}{\partial y^2} \right|_{\substack{x=x \\ y=2x}}.$$

Solution:

$$\text{Let } g(x) = u(x, y(x)), y = y(x) = 2x, \text{ then } \frac{dg}{dx} = \left. \frac{\partial u}{\partial x} \right|_{\substack{x=x \\ y=2x}} + \left. \frac{\partial u}{\partial y} \right|_{\substack{x=x \\ y=2x}} \frac{dy}{dx}.$$

$$\text{We have } g(x) = u(x, 2x) = x, \left. \frac{\partial u}{\partial x} \right|_{\substack{x=x \\ y=2x}} = x^2, \text{ so}$$

$$\frac{dg}{dx} = \left. \frac{\partial u}{\partial x} \right|_{\substack{x=x \\ y=2x}} + \left. \frac{\partial u}{\partial y} \right|_{\substack{x=x \\ y=2x}} \frac{dy}{dx} \Rightarrow 1 = x^2 + 2 \left. \frac{\partial u}{\partial y} \right|_{\substack{x=x \\ y=2x}} \Rightarrow \left. \frac{\partial u}{\partial y} \right|_{\substack{x=x \\ y=2x}} = \frac{1-x^2}{2}$$

$$\text{Let } h(x) = \left. \frac{\partial u}{\partial y} \right|_{\substack{x=x \\ y=2x}} = \frac{1-x^2}{2}, \text{ then } \frac{dh(x)}{dx} = \left. \frac{\partial^2 u}{\partial x \partial y} \right|_{\substack{x=x \\ y=2x}} + \left. \frac{\partial^2 u}{\partial y^2} \right|_{\substack{x=x \\ y=2x}} \frac{dy}{dx}.$$

Again,

$$\frac{dh(x)}{dx} = \left. \frac{\partial^2 u}{\partial x \partial y} \right|_{\substack{x=x \\ y=2x}} + \left. \frac{\partial^2 u}{\partial y^2} \right|_{\substack{x=x \\ y=2x}} \frac{dy}{dx} \Rightarrow -x = \left. \frac{\partial^2 u}{\partial x \partial y} \right|_{\substack{x=x \\ y=2x}} + 2 \left. \frac{\partial^2 u}{\partial y^2} \right|_{\substack{x=x \\ y=2x}} \dots\dots(1)$$

$$\text{In addition, let } k(x) = \left. \frac{\partial u}{\partial x} \right|_{\substack{x=x \\ y=2x}} = x^2, \text{ then}$$

$$2x = \frac{dk(x)}{dx} = \left. \frac{\partial^2 u}{\partial x^2} \right|_{\substack{x=x \\ y=2x}} + \left. \frac{\partial^2 u}{\partial x \partial y} \right|_{\substack{x=x \\ y=2x}} \frac{dy}{dx} = \left. \frac{\partial^2 u}{\partial x^2} \right|_{\substack{x=x \\ y=2x}} + 2 \left. \frac{\partial^2 u}{\partial x \partial y} \right|_{\substack{x=x \\ y=2x}} \Rightarrow 2x = \left. \frac{\partial^2 u}{\partial x^2} \right|_{\substack{x=x \\ y=2x}} + 2 \left. \frac{\partial^2 u}{\partial x \partial y} \right|_{\substack{x=x \\ y=2x}} \dots\dots(2)$$

(1) - 2 × (2) we have

$$-5x = -3 \left. \frac{\partial^2 u}{\partial x \partial y} \right|_{\substack{x=x \\ y=2x}} \Rightarrow \left. \frac{\partial^2 u}{\partial x \partial y} \right|_{\substack{x=x \\ y=2x}} = \frac{5x}{3}.$$

$$\text{Thus, from (1) } -x = \frac{5x}{3} + 2 \left. \frac{\partial^2 u}{\partial x^2} \right|_{\substack{x=x \\ y=2x}} \Rightarrow \left. \frac{\partial^2 u}{\partial x^2} \right|_{\substack{x=x \\ y=2x}} = \frac{-x - \frac{5x}{3}}{2} = -\frac{4x}{3}.$$

Also, we have  $\left. \frac{\partial^2 u}{\partial x^2} \right|_{x=x, y=2x} = \left. \frac{\partial^2 u}{\partial y^2} \right|_{x=x, y=2x} = -\frac{4x}{3}$

□

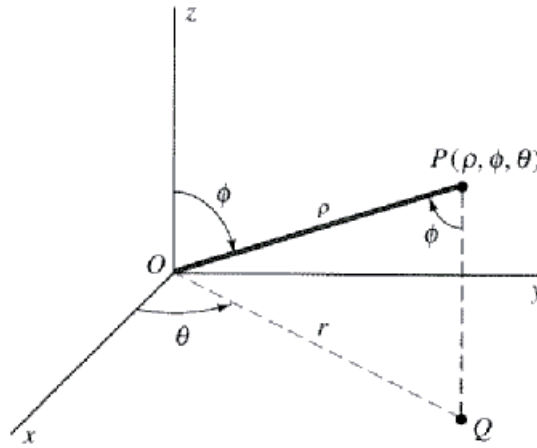
5. One of the most popular alternative coordinate systems to Cartesian coordinates is Spherical polar coordinates. Spherical polar coordinates represent a point  $P(x, y, z)$  in space by ordered triples

$(\rho, \theta, \phi)$  in which

1.  $\rho$  is the distance from  $P$  to the origin.
2.  $\phi$  is the angle  $\overrightarrow{OP}$  makes with the positive  $z$ -axis.
3.  $\theta$  is the angle measured counterclockwise from the positive  $x$ -axis to  $\overrightarrow{OQ}$  which is the projection of  $\overrightarrow{OP}$  on  $xy$ -plane.

The equations relating spherical polar coordinates to Cartesian coordinates are:

$x = \rho \sin \phi \cos \theta$ ,  $y = \rho \sin \phi \sin \theta$ ,  $z = \rho \cos \phi$ , where  $\rho \geq 0$ ,  $0 \leq \phi \leq \pi$ ,  $0 \leq \theta \leq 2\pi$ .



(i) Show that

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial V}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial V}{\partial \phi} \frac{\partial \phi}{\partial x} = \frac{\partial V}{\partial \rho} \sin \phi \cos \theta - \frac{\partial V}{\partial \theta} \frac{\sin \theta}{\rho \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\cos \theta \cos \phi}{\rho},$$

and find also  $\frac{\partial V}{\partial y}$ ,  $\frac{\partial V}{\partial z}$ .

(ii) (optional) Show that in spherical coordinates  $(\rho, \theta, \phi)$  Laplace's equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{takes the form}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\cot \phi}{\rho^2} \frac{\partial V}{\partial \phi} = 0.$$

Proof:

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases} \quad \text{Then we have } x^2 + y^2 + z^2 = \rho^2, \text{ so } \begin{cases} \frac{\partial \rho}{\partial x} = \sin \phi \cos \theta \\ \frac{\partial \rho}{\partial y} = \sin \phi \sin \theta \\ \frac{\partial \rho}{\partial z} = \cos \phi \end{cases}$$

$$\frac{y}{x} = \tan \theta \Rightarrow \frac{y}{-x^2} = \sec^2 \theta \frac{\partial \theta}{\partial x} \Rightarrow \frac{-\rho \sin \phi \sin \theta \cos^2 \theta}{\rho^2 \sin^2 \phi \cos^2 \theta} = \frac{\partial \theta}{\partial x} \Rightarrow \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{\rho \sin \phi}$$

$$\frac{y}{x} = \tan \theta \Rightarrow \frac{1}{x} = \sec^2 \theta \frac{\partial \theta}{\partial y} \Rightarrow \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{\rho \sin \phi} \text{ and } \frac{\partial \theta}{\partial z} = 0$$

$$\text{Thus, } \begin{cases} \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{\rho \sin \phi} \\ \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{\rho \sin \phi} \\ \frac{\partial \theta}{\partial z} = 0 \end{cases},$$

and

$$z = \rho \cos \phi \Rightarrow 0 = -\rho \sin \phi \frac{\partial \phi}{\partial x} + \frac{\partial \rho}{\partial x} \cos \phi \Rightarrow \frac{\partial \phi}{\partial x} = \frac{\frac{\partial \rho}{\partial x} \cos \phi}{\rho \sin \phi} = \frac{\sin \phi \cos \theta \cos \phi}{\rho \sin \phi} = \frac{\cos \theta \cos \phi}{\rho}$$

$$z = \rho \cos \phi \Rightarrow 0 = -\rho \sin \phi \frac{\partial \phi}{\partial y} + \frac{\partial \rho}{\partial y} \cos \phi \Rightarrow \frac{\partial \phi}{\partial y} = \frac{\frac{\partial \rho}{\partial y} \cos \phi}{\rho \sin \phi} = \frac{\sin \phi \sin \theta \cos \phi}{\rho \sin \phi} = \frac{\sin \theta \cos \phi}{\rho}$$

$$z = \rho \cos \phi \Rightarrow 1 = -\rho \sin \phi \frac{\partial \phi}{\partial z} + \frac{\partial \rho}{\partial z} \cos \phi \Rightarrow \frac{\partial \phi}{\partial z} = \frac{\frac{\partial \rho}{\partial z} \cos \phi - 1}{\rho \sin \phi} = \frac{\cos^2 \phi - 1}{\rho \sin \phi} = -\frac{\sin \phi}{\rho}$$

$$\begin{cases} \frac{\partial \phi}{\partial x} = \frac{\cos \theta \cos \phi}{\rho} \\ \frac{\partial \phi}{\partial y} = \frac{\sin \theta \cos \phi}{\rho} \\ \frac{\partial \phi}{\partial z} = -\frac{\sin \phi}{\rho} \end{cases}$$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial V}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial V}{\partial \phi} \frac{\partial \phi}{\partial x} = \frac{\partial V}{\partial \rho} \sin \phi \cos \theta - \frac{\partial V}{\partial \theta} \frac{\sin \theta}{\rho \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\cos \theta \cos \phi}{\rho}$$

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} &= \frac{\partial \left( \frac{\partial V}{\partial x} \right)}{\partial x} = \frac{\partial}{\partial \rho} \left( \frac{\partial V}{\partial x} \right) \sin \phi \cos \theta - \frac{\partial}{\partial \theta} \left( \frac{\partial V}{\partial x} \right) \frac{\sin \theta}{\rho \sin \phi} + \frac{\partial}{\partial \phi} \left( \frac{\partial V}{\partial x} \right) \frac{\cos \theta \cos \phi}{\rho} \\
&= \frac{\partial}{\partial \rho} \left( \frac{\partial V}{\partial \rho} \sin \phi \cos \theta - \frac{\partial V}{\partial \theta} \frac{\sin \theta}{\rho \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\cos \theta \cos \phi}{\rho} \right) \sin \phi \cos \theta \\
&\quad - \frac{\partial}{\partial \theta} \left( \frac{\partial V}{\partial \rho} \sin \phi \cos \theta - \frac{\partial V}{\partial \theta} \frac{\sin \theta}{\rho \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\cos \theta \cos \phi}{\rho} \right) \frac{\sin \theta}{\rho \sin \phi} \\
&\quad + \frac{\partial}{\partial \phi} \left( \frac{\partial V}{\partial \rho} \sin \phi \cos \theta - \frac{\partial V}{\partial \theta} \frac{\sin \theta}{\rho \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\cos \theta \cos \phi}{\rho} \right) \frac{\cos \theta \cos \phi}{\rho} \\
&= \frac{\partial^2 V}{\partial \rho^2} \sin^2 \phi \cos^2 \theta - \frac{\partial^2 V}{\partial \theta \partial \rho} \frac{\sin \theta \cos \theta}{\rho} + \frac{\partial V}{\partial \theta} \frac{\sin \theta \cos \theta}{\rho^2} + \frac{\partial^2 V}{\partial \rho \partial \phi} \frac{\cos^2 \theta \cos \phi \sin \phi}{\rho} - \frac{\partial V}{\partial \phi} \frac{\cos^2 \theta \sin \phi \cos \phi}{\rho^2} \\
&\quad - \frac{\partial^2 V}{\partial \theta \partial \rho} \frac{\cos \theta \sin \theta}{\rho} + \frac{\partial V}{\partial \rho} \frac{\sin^2 \theta}{\rho} + \frac{\partial^2 V}{\partial \theta^2} \frac{\sin^2 \theta}{\rho^2 \sin^2 \phi} + \frac{\partial V}{\partial \theta} \frac{\cos \theta \sin \theta}{\rho^2 \sin^2 \phi} - \frac{\partial^2 V}{\partial \theta \partial \phi} \frac{\cos \theta \cos \phi \sin \theta}{\rho^2 \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\sin^2 \theta \cos \phi}{\rho^2 \sin \phi} \\
&\quad + \frac{\partial^2 V}{\partial \phi \partial \rho} \frac{\sin \phi \cos^2 \theta \cos \phi}{\rho^2} + \frac{\partial V}{\partial \rho} \frac{\cos^2 \phi \cos^2 \theta}{\rho} - \frac{\partial^2 V}{\partial \phi \partial \theta} \frac{\sin \theta \cos \theta \cos \phi}{\rho^2 \sin \phi} + \frac{\partial V}{\partial \theta} \frac{\sin \theta \cos \theta \cos^2 \phi}{\rho^2 \sin^2 \phi} \\
&\quad + \frac{\partial^2 V}{\partial \phi^2} \frac{\cos^2 \theta \cos^2 \phi}{\rho^2} - \frac{\partial V}{\partial \phi} \frac{\cos^2 \theta \sin \phi \cos \phi}{\rho^2}
\end{aligned}$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial \rho} \frac{\partial \rho}{\partial y} + \frac{\partial V}{\partial \theta} \frac{\partial \theta}{\partial y} + \frac{\partial V}{\partial \phi} \frac{\partial \phi}{\partial y} = \frac{\partial V}{\partial \rho} \sin \phi \sin \theta + \frac{\partial V}{\partial \theta} \frac{\cos \theta}{\rho \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\sin \theta \cos \phi}{\rho}$$

$$\begin{aligned}
\frac{\partial^2 V}{\partial y^2} &= \frac{\partial \left( \frac{\partial V}{\partial y} \right)}{\partial y} = \frac{\partial}{\partial \rho} \left( \frac{\partial V}{\partial y} \right) \sin \phi \sin \theta + \frac{\partial}{\partial \theta} \left( \frac{\partial V}{\partial y} \right) \frac{\cos \theta}{\rho \sin \phi} + \frac{\partial}{\partial \phi} \left( \frac{\partial V}{\partial y} \right) \frac{\sin \theta \cos \phi}{\rho} \\
&= \frac{\partial}{\partial \rho} \left( \frac{\partial V}{\partial \rho} \sin \phi \sin \theta + \frac{\partial V}{\partial \theta} \frac{\cos \theta}{\rho \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\sin \theta \cos \phi}{\rho} \right) \sin \phi \sin \theta \\
&+ \frac{\partial}{\partial \theta} \left( \frac{\partial V}{\partial \rho} \sin \phi \sin \theta + \frac{\partial V}{\partial \theta} \frac{\cos \theta}{\rho \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\sin \theta \cos \phi}{\rho} \right) \frac{\cos \theta}{\rho \sin \phi} \\
&+ \frac{\partial}{\partial \phi} \left( \frac{\partial V}{\partial \rho} \sin \phi \sin \theta + \frac{\partial V}{\partial \theta} \frac{\cos \theta}{\rho \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\sin \theta \cos \phi}{\rho} \right) \frac{\sin \theta \cos \phi}{\rho} \\
&= \frac{\partial^2 V}{\partial \rho^2} \sin^2 \phi \sin^2 \theta + \frac{\partial^2 V}{\partial \rho \partial \theta} \frac{\cos \theta \sin \theta}{\rho} - \frac{\partial V}{\partial \theta} \frac{\cos \theta \sin \theta}{\rho^2} + \frac{\partial^2 V}{\partial \rho \partial \phi} \frac{\sin^2 \theta \cos \phi \sin \phi}{\rho} - \frac{\partial V}{\partial \phi} \frac{\sin^2 \theta \cos \phi \sin \phi}{\rho^2} \\
&+ \frac{\partial^2 V}{\partial \theta \partial \rho} \frac{\sin \theta \cos \theta}{\rho} + \frac{\partial V}{\partial \rho} \frac{\cos^2 \theta}{\rho} + \frac{\partial^2 V}{\partial \theta^2} \frac{\cos^2 \theta}{\rho^2 \sin^2 \phi} - \frac{\partial V}{\partial \theta} \frac{\sin \theta \cos \theta}{\rho^2 \sin^2 \phi} + \frac{\partial^2 V}{\partial \theta \partial \phi} \frac{\sin \theta \cos \phi \cos \theta}{\rho^2 \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\cos^2 \theta \cos \phi}{\rho^2 \sin \phi} \\
&+ \frac{\partial^2 V}{\partial \phi \partial \rho} \frac{\sin \phi \sin^2 \theta \cos \phi}{\rho} + \frac{\partial V}{\partial \rho} \frac{\sin^2 \theta \cos^2 \phi}{\rho} + \frac{\partial^2 V}{\partial \phi \partial \theta} \frac{\cos \theta \sin \theta \cos \phi}{\rho^2 \sin \phi} - \frac{\partial V}{\partial \theta} \frac{\cos^2 \phi \cos \theta \sin \theta}{\rho^2 \sin^2 \phi} \\
&+ \frac{\partial^2 V}{\partial \phi^2} \frac{\sin^2 \theta \cos^2 \phi}{\rho^2} - \frac{\partial V}{\partial \phi} \frac{\sin^2 \theta \sin \phi \cos \phi}{\rho^2}
\end{aligned}$$

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial \rho} \frac{\partial \rho}{\partial z} + \frac{\partial V}{\partial \theta} \frac{\partial \theta}{\partial z} + \frac{\partial V}{\partial \phi} \frac{\partial \phi}{\partial z} = \frac{\partial V}{\partial \rho} \cos \phi - \frac{\partial V}{\partial \phi} \frac{\sin \phi}{\rho}$$

$$\begin{aligned}
\frac{\partial^2 V}{\partial z^2} &= \frac{\partial}{\partial \rho} \left( \frac{\partial V}{\partial z} \right) \cos \phi - \frac{\partial}{\partial \phi} \left( \frac{\partial V}{\partial z} \right) \frac{\sin \phi}{\rho} \\
&= \frac{\partial}{\partial \rho} \left( \frac{\partial V}{\partial \rho} \cos \phi - \frac{\partial V}{\partial \phi} \frac{\sin \phi}{\rho} \right) \cos \phi - \frac{\partial}{\partial \phi} \left( \frac{\partial V}{\partial \rho} \cos \phi - \frac{\partial V}{\partial \phi} \frac{\sin \phi}{\rho} \right) \frac{\sin \phi}{\rho} \\
&= \frac{\partial^2 V}{\partial \rho^2} \cos^2 \phi - \frac{\partial^2 V}{\partial \rho \partial \phi} \frac{\sin \phi \cos \phi}{\rho} + \frac{\partial V}{\partial \phi} \frac{\sin \phi \cos \phi}{\rho^2} - \frac{\partial^2 V}{\partial \phi \partial \rho} \frac{\cos \phi \sin \phi}{\rho} + \frac{\partial V}{\partial \rho} \frac{\sin^2 \phi}{\rho} + \frac{\partial^2 V}{\partial \phi^2} \frac{\sin^2 \phi}{\rho^2} \\
&+ \frac{\partial V}{\partial \phi} \frac{\cos \phi \sin \phi}{\rho^2}
\end{aligned}$$

$$\begin{aligned}
\nabla^2 V &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \\
&= \frac{\partial^2 V}{\partial \rho^2} \sin^2 \phi \cos^2 \theta - \frac{\partial^2 V}{\partial \theta \partial \rho} \frac{\sin \theta \cos \theta}{\rho} + \frac{\partial V}{\partial \theta} \frac{\sin \theta \cos \theta}{\rho^2} + \frac{\partial^2 V}{\partial \rho \partial \phi} \frac{\cos^2 \theta \cos \phi \sin \phi}{\rho} - \frac{\partial V}{\partial \phi} \frac{\cos^2 \theta \sin \phi \cos \phi}{\rho^2} \\
&\quad - \frac{\partial^2 V}{\partial \theta \partial \rho} \frac{\cos \theta \sin \theta}{\rho} + \frac{\partial V}{\partial \rho} \frac{\sin^2 \theta}{\rho} + \frac{\partial^2 V}{\partial \theta^2} \frac{\sin^2 \theta}{\rho^2 \sin^2 \phi} + \frac{\partial V}{\partial \theta} \frac{\cos \theta \sin \theta}{\rho^2 \sin^2 \phi} - \frac{\partial^2 V}{\partial \theta \partial \phi} \frac{\cos \theta \cos \phi \sin \theta}{\rho^2 \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\sin^2 \theta \cos \phi}{\rho^2 \sin \phi} \\
&\quad + \frac{\partial^2 V}{\partial \phi \partial \rho} \frac{\sin \phi \cos^2 \theta \cos \phi}{\rho^2} + \frac{\partial V}{\partial \rho} \frac{\cos^2 \phi \cos^2 \theta}{\rho} - \frac{\partial^2 V}{\partial \phi \partial \theta} \frac{\sin \theta \cos \theta \cos \phi}{\rho^2 \sin \phi} + \frac{\partial V}{\partial \theta} \frac{\sin \theta \cos \theta \cos^2 \phi}{\rho^2 \sin^2 \phi} \\
&\quad + \frac{\partial^2 V}{\partial \phi^2} \frac{\cos^2 \theta \cos^2 \phi}{\rho^2} - \frac{\partial V}{\partial \phi} \frac{\cos^2 \theta \sin \phi \cos \phi}{\rho^2} \\
&\quad + \frac{\partial^2 V}{\partial \rho^2} \sin^2 \phi \sin^2 \theta + \frac{\partial^2 V}{\partial \rho \partial \theta} \frac{\cos \theta \sin \theta}{\rho} - \frac{\partial V}{\partial \theta} \frac{\cos \theta \sin \theta}{\rho^2} + \frac{\partial^2 V}{\partial \rho \partial \phi} \frac{\sin^2 \theta \cos \phi \sin \phi}{\rho} - \frac{\partial V}{\partial \phi} \frac{\sin^2 \theta \cos \phi \sin \phi}{\rho^2} \\
&\quad + \frac{\partial^2 V}{\partial \theta \partial \rho} \frac{\sin \theta \cos \theta}{\rho} + \frac{\partial V}{\partial \rho} \frac{\cos^2 \theta}{\rho} + \frac{\partial^2 V}{\partial \theta^2} \frac{\cos^2 \theta}{\rho^2 \sin^2 \phi} - \frac{\partial V}{\partial \theta} \frac{\sin \theta \cos \theta}{\rho^2 \sin^2 \phi} + \frac{\partial^2 V}{\partial \theta \partial \phi} \frac{\sin \theta \cos \phi \cos \theta}{\rho^2 \sin \phi} + \frac{\partial V}{\partial \phi} \frac{\cos^2 \theta \cos \phi}{\rho^2 \sin \phi} \\
&\quad + \frac{\partial^2 V}{\partial \phi \partial \rho} \frac{\sin \phi \sin^2 \theta \cos \phi}{\rho} + \frac{\partial V}{\partial \rho} \frac{\sin^2 \theta \cos^2 \phi}{\rho} + \frac{\partial^2 V}{\partial \phi \partial \theta} \frac{\cos \theta \sin \theta \cos \phi}{\rho^2 \sin \phi} - \frac{\partial V}{\partial \theta} \frac{\cos^2 \phi \cos \theta \sin \theta}{\rho^2 \sin^2 \phi} \\
&\quad + \frac{\partial^2 V}{\partial \phi^2} \frac{\sin^2 \theta \cos^2 \phi}{\rho^2} - \frac{\partial V}{\partial \phi} \frac{\sin^2 \theta \sin \phi \cos \phi}{\rho^2} \\
&\quad + \frac{\partial^2 V}{\partial \rho^2} \cos^2 \phi - \frac{\partial^2 V}{\partial \rho \partial \phi} \frac{\sin \phi \cos \phi}{\rho} + \frac{\partial V}{\partial \phi} \frac{\sin \phi \cos \phi}{\rho^2} - \frac{\partial^2 V}{\partial \phi \partial \rho} \frac{\cos \phi \sin \phi}{\rho} + \frac{\partial V}{\partial \rho} \frac{\sin^2 \phi}{\rho} + \frac{\partial^2 V}{\partial \phi^2} \frac{\sin^2 \phi}{\rho^2} \\
&\quad + \frac{\partial V}{\partial \phi} \frac{\cos \phi \sin \phi}{\rho^2} \\
&= \left( \sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta + \cos^2 \phi \right) \frac{\partial^2 V}{\partial \rho^2} + \left( -\frac{\sin \theta \cos \theta}{\rho} - \frac{\cos \theta \sin \theta}{\rho} + \frac{\sin \theta \cos \theta}{\rho} + \frac{\cos \theta \sin \theta}{\rho} \right) \frac{\partial^2 V}{\partial \theta \partial \rho} \\
&\quad + \left( \frac{\cos^2 \theta \cos \phi \sin \phi}{\rho} + \frac{\sin \phi \cos^2 \theta \cos \phi}{\rho} + \frac{\sin^2 \theta \cos \phi \sin \phi}{\rho} + \frac{\sin \phi \sin^2 \theta \cos \phi}{\rho} - \frac{\sin \phi \cos \phi}{\rho} - \frac{\cos \phi \sin \phi}{\rho} \right) \frac{\partial^2 V}{\partial \phi \partial \rho} \\
&\quad + \left( \frac{\sin^2 \theta}{\rho^2 \sin^2 \phi} + \frac{\cos^2 \theta}{\rho^2 \sin^2 \phi} \right) \frac{\partial^2 V}{\partial \theta^2} + \left( -\frac{\cos \theta \cos \phi \sin \theta}{\rho^2 \sin \phi} - \frac{\sin \theta \cos \theta \cos \phi}{\rho^2 \sin \phi} + \frac{\sin \theta \cos \phi \cos \theta}{\rho^2 \sin \phi} + \frac{\cos \theta \sin \theta \cos \phi}{\rho^2 \sin \phi} \right) \frac{\partial^2 V}{\partial \phi \partial \theta} \\
&\quad + \left( \frac{\cos^2 \theta \cos^2 \phi}{\rho^2} + \frac{\sin^2 \theta \cos^2 \phi}{\rho^2} + \frac{\sin^2 \phi}{\rho^2} \right) \frac{\partial^2 V}{\partial \phi^2} + \left( \frac{\sin^2 \theta}{\rho} + \frac{\cos^2 \phi \cos^2 \theta}{\rho} + \frac{\cos^2 \theta}{\rho} + \frac{\sin^2 \theta \cos^2 \phi}{\rho} + \frac{\sin^2 \phi}{\rho} \right) \frac{\partial V}{\partial \rho} \\
&\quad + \left( \frac{\sin \theta \cos \theta}{\rho^2} + \frac{\cos \theta \sin \theta}{\rho^2 \sin^2 \phi} + \frac{\sin \theta \cos \theta \cos^2 \phi}{\rho^2 \sin^2 \phi} - \frac{\cos \theta \sin \theta}{\rho^2} - \frac{\sin \theta \cos \theta}{\rho^2 \sin^2 \phi} - \frac{\cos^2 \phi \cos \theta \sin \theta}{\rho^2 \sin^2 \phi} \right) \frac{\partial V}{\partial \theta} \\
&\quad + \left( -\frac{\cos^2 \theta \sin \phi \cos \phi}{\rho^2} + \frac{\sin^2 \theta \cos \phi}{\rho^2 \sin \phi} - \frac{\cos^2 \theta \sin \phi \cos \phi}{\rho^2} - \frac{\sin^2 \theta \cos \phi \sin \phi}{\rho^2} + \frac{\cos^2 \theta \cos \phi}{\rho^2 \sin \phi} \right. \\
&\quad \left. - \frac{\sin^2 \theta \sin \phi \cos \phi}{\rho^2} + \frac{\sin \phi \cos \phi}{\rho^2} + \frac{\cos \phi \sin \phi}{\rho^2} \right) \frac{\partial V}{\partial \phi}
\end{aligned}$$

$$\Rightarrow \nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\cot \phi}{\rho^2} \frac{\partial V}{\partial \phi} = 0$$

□

6. Suppose the equations  $\begin{cases} x^2 - y^2 - u^3 + v^2 + 4 = 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0 \end{cases}$  determine functions  $u(x, y)$  and  $v(x, y)$

near  $x = 2$  and  $y = -1$  such that  $u(2, -1) = 2$  and  $v(2, -1) = 1$ . Compute  $\frac{\partial u}{\partial x}(2, -1)$ .

Solution:

$$\begin{cases} x^2 - y^2 - u^3 + v^2 + 4 = 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0 \end{cases} \Rightarrow \begin{cases} 2x - 3u^2 \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0 \\ 2y - 4u \frac{\partial u}{\partial x} + 12v^3 \frac{\partial v}{\partial x} = 0 \end{cases}$$

When  $x = 2, y = -1$ , we have  $u = 2, v = 1$

$$\begin{cases} 4 - 12 \frac{\partial u}{\partial x} + 2 \frac{\partial v}{\partial x} = 0 \\ -2 - 8 \frac{\partial u}{\partial x} + 12 \frac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow \frac{\partial u}{\partial x}(2, -1) = \frac{13}{32}$$

□

7. It is given that  $\begin{cases} x^2 + y^2 = \frac{1}{2}z^2 \\ x + y + z = 2 \end{cases}$ . Find  $\frac{dx}{dz}, \frac{dy}{dz}, \frac{d^2x}{dz^2}, \frac{d^2y}{dz^2}$  when  $x = 1, y = -1, z = 2$ .

Solution:

$$\begin{cases} x^2 + y^2 = \frac{1}{2}z^2 \\ x + y + z = 2 \end{cases} \Rightarrow \begin{cases} 2x \frac{dx}{dz} + 2y \frac{dy}{dz} = z \dots (1) \\ \frac{dx}{dz} + \frac{dy}{dz} + 1 = 0 \dots (2) \end{cases} \Rightarrow \begin{cases} 2 \left( \frac{dx}{dz} \right)^2 + 2x \frac{d^2x}{dz^2} + 2 \left( \frac{dy}{dz} \right)^2 + 2y \frac{d^2y}{dz^2} = 1 \dots (3) \\ \frac{d^2x}{dz^2} + \frac{d^2y}{dz^2} = 0 \dots (4) \end{cases}$$

Put  $x = 1, y = -1, z = 2$  in (1) & (2), we have

$$\begin{cases} 2x \frac{dx}{dz} + 2y \frac{dy}{dz} = z \dots (1) \\ \frac{dx}{dz} + \frac{dy}{dz} + 1 = 0 \dots (2) \end{cases} \Rightarrow 4 \frac{dx}{dz} = 0 \text{ \& } 4 \frac{dy}{dz} = -4 \Rightarrow \frac{dx}{dz} = 0 \text{ \& } \frac{dy}{dz} = -1$$

Put  $x = 1, y = -1, z = 2$  &  $\frac{dx}{dz} = 0, \frac{dy}{dz} = -1$  in (3) & (4), we have  $\frac{d^2x}{dz^2} = -\frac{1}{4}, \frac{d^2y}{dz^2} = \frac{1}{4}$

□

8. Use Taylor's theorem to expand  $f(x, y) = \sin xy$  about the point  $\left(1, \frac{\pi}{3}\right)$ , neglecting cubic and higher terms. Hence estimate  $\sin 0.3\pi$ .

Solution:

Expand  $f(x, y) = \sin xy$  about  $\left(1, \frac{\pi}{3}\right)$ .

Then

$$f(x, y) = \sin xy = f\left(1, \frac{\pi}{3}\right) + f_x\left(1, \frac{\pi}{3}\right)(x-1) + f_y\left(1, \frac{\pi}{3}\right)\left(y - \frac{\pi}{3}\right) \\ + \frac{1}{2} \left[ f_{xx}\left(1, \frac{\pi}{3}\right)(x-1)^2 + 2f_{xy}\left(1, \frac{\pi}{3}\right)(x-1)\left(y - \frac{\pi}{3}\right) + f_{yy}\left(1, \frac{\pi}{3}\right)\left(y - \frac{\pi}{3}\right)^2 \right] + \dots$$

$$f_x(x, y) = y \cos xy, f_y(x, y) = x \cos xy, f_{xx}(x, y) = -y^2 \sin xy$$

Observe that  $\left(0.9, \frac{\pi}{3}\right)$  is close to  $\left(1, \frac{\pi}{3}\right)$ .

Therefore

$$f\left(0.9, \frac{\pi}{3}\right) = \sin\left(0.9 \times \frac{\pi}{3}\right) = \sin 0.3\pi \approx f\left(1, \frac{\pi}{3}\right) + f_x\left(1, \frac{\pi}{3}\right)(0.9-1) + f_y\left(1, \frac{\pi}{3}\right)\left(\frac{\pi}{3} - \frac{\pi}{3}\right) \\ + \frac{1}{2} \left[ f_{xx}\left(1, \frac{\pi}{3}\right)(0.9-1)^2 + 2f_{xy}\left(1, \frac{\pi}{3}\right)(0.9-1)\left(\frac{\pi}{3} - \frac{\pi}{3}\right) + f_{yy}\left(1, \frac{\pi}{3}\right)\left(\frac{\pi}{3} - \frac{\pi}{3}\right)^2 \right] \\ = \sin \frac{\pi}{3} - 0.1 \times \frac{\pi}{3} \cos \frac{\pi}{3} - (-0.1)^2 \times \frac{\pi^2}{9} \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} - 0.1 \times \frac{\pi}{6} - \frac{1}{2} \times (-0.1)^2 \pi^2 \frac{\sqrt{3}}{18} = 0.80892$$

□

9. In surveying a triangular plot of land, two of its sides were measured as 160m and 210m with maximum possible errors of 0.1m and the included angle was  $\frac{\pi}{3}$  exactly. Estimate the maximum error

in calculating the length of the third side from the cosine rule  $c = (a^2 + b^2 - 2ab \cos C)^{\frac{1}{2}}$ .

Solution:

$$c = c(a, b) = (a^2 + b^2 - 2ab \cos C)^{\frac{1}{2}} \Rightarrow \frac{\partial c}{\partial a} = \frac{a - b \cos C}{\sqrt{a^2 + b^2 - 2ab \cos C}}, \frac{\partial c}{\partial b} = \frac{b - a \cos C}{\sqrt{a^2 + b^2 - 2ab \cos C}}.$$

$$\delta c = c(a, b) - c(a_0, b_0) \underset{\substack{\delta a = a - a_0 \\ \delta b = b - b_0}}{=} \delta a \frac{\partial c}{\partial a}(a_0 + \theta \delta a, b_0 + \theta \delta b) + \delta b \frac{\partial c}{\partial b}(a_0 + \theta \delta a, b_0 + \theta \delta b).$$

Since  $\frac{\partial c}{\partial a}, \frac{\partial c}{\partial b}$  are continuous functions, if  $\delta a, \delta b$  are quite small, then

$$\frac{\partial c}{\partial a}(a_0 + \theta \delta a, b_0 + \theta \delta b) \approx \frac{\partial c}{\partial a}(a_0, b_0), \frac{\partial c}{\partial b}(a_0 + \theta \delta a, b_0 + \theta \delta b) \approx \frac{\partial c}{\partial b}(a_0, b_0).$$

Therefore,

$$\delta c = c(a, b) - c(a_0, b_0) \underset{\substack{\delta a = a - a_0 \\ \delta b = b - b_0}}{=} \delta a \frac{\partial c}{\partial a}(a_0 + \theta \delta a, b_0 + \theta \delta b) + \delta b \frac{\partial c}{\partial b}(a_0 + \theta \delta a, b_0 + \theta \delta b) \\ \approx \frac{a_0 - b_0 \cos C}{\sqrt{a_0^2 + b_0^2 - 2a_0b_0 \cos C}} \delta a + \frac{b_0 - a_0 \cos C}{\sqrt{a_0^2 + b_0^2 - 2a_0b_0 \cos C}} \delta b \\ = \frac{a_0 - b_0 \cos C}{\sqrt{a_0^2 + b_0^2 - 2a_0b_0 \cos C} = c_0} \delta a + \frac{b_0 - a_0 \cos C}{c_0} \delta b$$

Observe that  $c_0 = \left(a_0^2 + b_0^2 - 2a_0b_0 \cos C\right)^{\frac{1}{2}} = 190$ .  
 $a_0=160$   
 $b_0=210$   
 $C=\frac{\pi}{3}$

$$\begin{aligned} |\delta c| &\leq \left| \frac{a_0 - b_0 \cos C}{c_0} \delta a \right| + \left| \frac{b_0 - a_0 \cos C}{c_0} \delta b \right| = \frac{|a_0 - b_0 \cos C|}{|c_0|} |\delta a| + \frac{|b_0 - a_0 \cos C|}{|c_0|} |\delta b| \\ &\leq \frac{|a_0 - b_0 \cos C|}{|c_0|} \times 0.1 + \frac{|b_0 - a_0 \cos C|}{|c_0|} \times 0.1 = \frac{55}{190} \times 0.1 + \frac{130}{190} \times 0.1 = 0.09736 \end{aligned}$$

□

10. Find and classify the stationary points of

(a)  $f(x, y) = x^3 + 3x^2 - 3y^2 + 6xy$ ,

(b)  $f(x, y) = x^3 + 3xy - 3x^2 - 3y^2 + 4$ ,

(c)  $z = f(x, y) = x^4 + y^4 - x^2 - 2xy - y^2$ .

Solution:

(a)

$$z = f(x, y) = x^3 + 3x^2 - 3y^2 + 6xy \Rightarrow \begin{cases} \frac{\partial z}{\partial x} = 3x^2 + 6x + 6y = 0 \\ \frac{\partial z}{\partial y} = -6y + 6x = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 + 6x + 6y = 0 \\ x = y \end{cases}$$

$$\Leftrightarrow \begin{cases} 3x^2 + 12x = 0 \\ x = y \end{cases} \Leftrightarrow \begin{cases} x = 0 \text{ or } x = -4 \\ x = y \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x = -4 \\ y = -4 \end{cases}$$

$$\text{Also } \frac{\partial^2 z}{\partial x^2} = 6x + 6 \text{ \& } \frac{\partial^2 z}{\partial x \partial y} = 6 \text{ \& } \frac{\partial^2 z}{\partial y^2} = -6.$$

So

$$\Delta(x, y) = \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 = (6x + 6)(-6) - 36 = -36x - 72$$

And

$$\Delta(0, 0) = -72 < 0, \Delta(-4, -4) = 72 > 0, \left. \frac{\partial^2 z}{\partial x^2} \right|_{x=0, y=0} = 6 > 0, \left. \frac{\partial^2 z}{\partial x^2} \right|_{x=-4, y=-4} = -18 < 0$$

Therefore,  $(-4, -4)$  is a local maximum point and  $(0, 0)$  is a saddle point.

(b)

$$z = f(x, y) = x^3 + 3xy - 3x^2 - 3y^2 + 4 \Rightarrow \begin{cases} \frac{\partial z}{\partial x} = 3x^2 + 3y - 6x = 0 \\ \frac{\partial z}{\partial y} = 3x - 6y = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 + 3y - 6x = 0 \\ x = 2y \end{cases}$$

$$\Leftrightarrow \begin{cases} 12y^2 - 9y = 0 \\ x = 2y \end{cases} \Leftrightarrow \begin{cases} y = 0 \text{ or } y = \frac{3}{4} \\ x = 2y \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x = \frac{3}{2} \\ y = \frac{3}{4} \end{cases}$$

Also  $\frac{\partial^2 z}{\partial x^2} = 6x - 6$  &  $\frac{\partial^2 z}{\partial x \partial y} = 3$  &  $\frac{\partial^2 z}{\partial y^2} = -6$ .

So

$$\Delta(x, y) = \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 = (6x - 6)(-6) - 9 = -36x + 27$$

And

$$\Delta(0, 0) = 27 > 0, \Delta\left(\frac{3}{2}, \frac{3}{4}\right) = -27 < 0, \left. \frac{\partial^2 z}{\partial x^2} \right|_{x=0, y=0} = -6 < 0, \left. \frac{\partial^2 z}{\partial x^2} \right|_{x=3/2, y=3/4} = 3 > 0$$

Therefore,  $(0, 0)$  is a local maximum point and  $\left(\frac{3}{2}, \frac{3}{4}\right)$  is a saddle point.

(c)

$$z = f(x, y) = x^4 + y^4 - x^2 - 2xy - y^2 \Rightarrow \begin{cases} \frac{\partial z}{\partial x} = 4x^3 - 2x - 2y = 0 \\ \frac{\partial z}{\partial y} = 4y^3 - 2x - 2y = 0 \end{cases}$$

$$\begin{cases} \frac{\partial z}{\partial x} = 4x^3 - 2x - 2y = 0 \\ \frac{\partial z}{\partial y} = 4y^3 - 2x - 2y = 0 \end{cases} \Rightarrow 4x^3 - 4y^3 = 0$$

$$\text{And } 4x^3 - 4y^3 = 0 \Leftrightarrow 4(x - y)(x^2 + xy + y^2) = 0$$

$$4(x - y)(x^2 + xy + y^2) = 0 \Leftrightarrow x - y = 0 \text{ or } x^2 + xy + y^2 = 0$$

$$\text{So } \begin{cases} 4x^3 - 2x - 2y = 0 \\ x = y \end{cases} \Leftrightarrow \begin{cases} 4x^3 - 4x = 0 \\ x = y \end{cases} \Leftrightarrow \begin{cases} 4x(x - 1)(x + 1) = 0 \\ x = y \end{cases}$$

$$\text{So we have the roots for } \begin{cases} \frac{\partial z}{\partial x} = 4x^3 - 2x - 2y = 0 \\ \frac{\partial z}{\partial y} = 4y^3 - 2x - 2y = 0 \end{cases}, \text{ they are } \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x = 1 \\ y = 1 \end{cases} \text{ or } \begin{cases} x = -1 \\ y = -1 \end{cases}$$

Note: Only  $(0, 0)$  satisfies  $x^2 + xy + y^2 = 0$ .

$$\text{Also } \frac{\partial z}{\partial x} = 4x^3 - 2x - 2y \Rightarrow \frac{\partial^2 z}{\partial x^2} = 12x^2 - 2 \text{ & } \frac{\partial^2 z}{\partial x \partial y} = -2 \text{ & } \frac{\partial z}{\partial y} = 4y^3 - 2x - 2y \Rightarrow \frac{\partial^2 z}{\partial y^2} = 12y^2 - 2.$$

So

$$\Delta(x, y) = \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial x \partial y} \right)^2 = (12x^2 - 2)(12y^2 - 2) - 4$$

And

$$\Delta(1, 1) > 0, \Delta(-1, -1) > 0, \left. \frac{\partial^2 z}{\partial x^2} \right|_{x=1, y=1} > 0, \left. \frac{\partial^2 z}{\partial x^2} \right|_{x=-1, y=-1} > 0$$

Therefore,  $(1, 1)$ ,  $(-1, -1)$  are local minimum points.

As for  $(0,0)$ ,  $\Delta(0,0) = 0$  so the test does not work but we observe that  $f(0,0) = 0$  and for  $x = \frac{1}{n}, y = \frac{1}{n}$

where  $n$  is a positive integer. we have  $f\left(\frac{1}{n}, \frac{1}{n}\right) - f(0,0) = 2\left(\frac{1}{n}\right)^4 - 4\left(\frac{1}{n}\right)^2 < 0$

and  $\left(\frac{1}{n}, \frac{1}{n}\right)$  is close to  $(0,0)$  when  $n$  is sufficiently large.

Also  $x = \frac{1}{n}, y = -\frac{1}{n}$ ,  $f\left(\frac{1}{n}, -\frac{1}{n}\right) - f(0,0) = 2\left(\frac{1}{n}\right)^4 > 0$  and  $\left(\frac{1}{n}, -\frac{1}{n}\right)$  is close to  $(0,0)$  when  $n$  is sufficiently large.

Thus  $(0,0)$  is a saddle point. □

11. Find the minimum distance between the origin and the surface  $z^2 = x^2y + 4$ .

Solution:

Let the square of the distance between the origin and a point on the surface  $z^2 = x^2y + 4$  be  $w$ .

Then  $w = x^2 + y^2 + z^2$  where  $(x, y, z)$  is a point on the surface  $z^2 = x^2y + 4$ .

Thus  $w = x^2 + y^2 + x^2y + 4$

$$w = f(x, y) = x^2 + y^2 + x^2y + 4 \Rightarrow \begin{cases} \frac{\partial w}{\partial x} = 2x + 2xy = 0 \\ \frac{\partial w}{\partial y} = 2y + x^2 = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - x^3 = 0 \\ 2y + x^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x(\sqrt{2} - x)(\sqrt{2} + x) = 0 \\ 2y + x^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \text{ or } x = \sqrt{2} \text{ or } x = -\sqrt{2} \\ 2y + x^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x = \sqrt{2} \\ y = -1 \end{cases} \text{ or } \begin{cases} x = -\sqrt{2} \\ y = -1 \end{cases}$$

$$\Delta(x, y) = \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 = 2(2 + 2y) - 4x^2 = 4 + 4y - 4x^2.$$

$\Delta(\pm\sqrt{2}, -1) = -8$ . Neither  $(\sqrt{2}, -1)$  nor  $(-\sqrt{2}, -1)$  yields an extremum.

$\Delta(0,0) = 4$ ,  $\frac{\partial^2 w}{\partial x^2} \Big|_{x=0, y=0} = 2 > 0$ . So  $(0,0)$  yields the minimum value of  $w$ .

And  $w = f(0,0) = 4$ .

The minimum distance between the origin and the surface  $z^2 = x^2y + 4$  is 2. □

12. The equation  $x^3 - 3x + y^2 - 2y + z^3 + z + 1 = 0$  implicitly determines an implicit function  $z = z(x, y)$  of  $x$  and  $y$  defined in  $R^2$ . Find the stationary point(s) of  $z = z(x, y)$  and determine the local extreme value(s) of  $z = z(x, y)$  that  $z = z(x, y)$  will attain there (if any) and state what kind of local extreme value(s) it is or they are.

Solution:

$$x^3 - 3x + y^2 - 2y + z^3 + z + 1 = 0 \Rightarrow 3x^2 - 3 + (3z^2 + 1) \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{3 - 3x^2}{3z^2 + 1}$$

$$x^3 - 3x + y^2 - 2y + z^3 + z + 1 = 0 \Rightarrow 2y - 2 + (3z^2 + 1) \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{2 - 2y}{3z^2 + 1}$$

$$\begin{cases} \frac{\partial z}{\partial x} = \frac{3 - 3x^2}{3z^2 + 1} = 0 \\ \frac{\partial z}{\partial y} = \frac{2 - 2y}{3z^2 + 1} = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \text{ or } x = -1 \\ y = 1 \end{cases}$$

So the stationary points of  $z = z(x, y)$  are  $(1, 1), (1, -1)$ .

Put  $\begin{cases} x = 1 \\ y = 1 \end{cases}$  in  $x^3 - 3x + y^2 - 2y + z^3 + z + 1 = 0$ , we have  $z = 1$

Put  $\begin{cases} x = -1 \\ y = 1 \end{cases}$  in  $x^3 - 3x + y^2 - 2y + z^3 + z + 1 = 0$ , we have  $z = -1$

From

$$3x^2 - 3 + (3z^2 + 1) \frac{\partial z}{\partial x} = 0 \Rightarrow 6x + (3z^2 + 1) \frac{\partial^2 z}{\partial x^2} + 6z \left( \frac{\partial z}{\partial x} \right)^2 = 0 \Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{-6x - 6z \left( \frac{\partial z}{\partial x} \right)^2}{3z^2 + 1}$$

From

$$2y - 2 + (3z^2 + 1) \frac{\partial z}{\partial y} = 0 \Rightarrow 2 + (3z^2 + 1) \frac{\partial^2 z}{\partial y^2} + 6z \left( \frac{\partial z}{\partial y} \right)^2 = 0 \Rightarrow \frac{\partial^2 z}{\partial y^2} = \frac{-2 - 6z \left( \frac{\partial z}{\partial y} \right)^2}{3z^2 + 1}$$

Again, from

$$3x^2 - 3 + (3z^2 + 1) \frac{\partial z}{\partial x} = 0 \Rightarrow (3z^2 + 1) \frac{\partial^2 z}{\partial y \partial x} + 6z \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial^2 z}{\partial y \partial x} = \frac{-6z \frac{\partial z}{\partial y} \frac{\partial z}{\partial x}}{3z^2 + 1}$$

$$\left. \frac{\partial^2 z}{\partial x^2} \right|_{\substack{x=1 \\ y=1 \\ z=1}} = \frac{-6x - 6z \left( \frac{\partial z}{\partial x} \right)^2}{3z^2 + 1} \bigg|_{\substack{x=1 \\ y=1 \\ z=1}} = \frac{-6}{4} < 0$$

Note that  $\left. \frac{\partial z}{\partial x} \right|_{\substack{x=1 \\ y=1 \\ z=1}} = 0$ .

$$\left. \frac{\partial^2 z}{\partial y^2} \right|_{\substack{x=1 \\ y=1 \\ z=1}} = \frac{-2 - 6z \left( \frac{dz}{dy} \right)^2}{3z^2 + 1} = \frac{-2}{4} < 0$$

Note that  $\left. \frac{\partial z}{\partial y} \right|_{\substack{x=1 \\ y=1 \\ z=1}} = 0$

$$\left. \frac{\partial^2 z}{\partial y \partial x} \right|_{\substack{x=1 \\ y=1 \\ z=1}} = \frac{-6z \frac{\partial z}{\partial y} \frac{\partial z}{\partial x}}{3z^2 + 1} \bigg|_{\substack{x=1 \\ y=1 \\ z=1}} = 0$$

Therefore,

$$\left[ \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial y \partial x} \right)^2 \right] \bigg|_{\substack{x=1 \\ y=1 \\ z=1}} = \left. \frac{\partial^2 z}{\partial x^2} \right|_{\substack{x=1 \\ y=1 \\ z=1}} \left. \frac{\partial^2 z}{\partial y^2} \right|_{\substack{x=1 \\ y=1 \\ z=1}} - \left( \left. \frac{\partial^2 z}{\partial y \partial x} \right|_{\substack{x=1 \\ y=1 \\ z=1}} \right)^2 = \left( -\frac{6}{4} \right) \left( -\frac{2}{4} \right) - 0 > 0$$

It follows that  $z = z(x, y)$  attains the local maximum value  $z(1, 1) = 1$  at the point  $(1, 1)$

In addition,

$$\left. \frac{\partial^2 z}{\partial x^2} \right|_{\substack{x=-1 \\ y=1 \\ z=-1}} = \frac{-6x - 6z \left( \frac{\partial z}{\partial x} \right)^2}{3z^2 + 1} \bigg|_{\substack{x=-1 \\ y=1 \\ z=-1}} = \frac{6}{4} > 0$$

$$\left. \frac{\partial^2 z}{\partial y^2} \right|_{\substack{x=-1 \\ y=1 \\ z=-1}} = \frac{-2 - 6z \left( \frac{dz}{dy} \right)^2}{3z^2 + 1} = \frac{-2}{4} < 0$$

$$\left. \frac{\partial^2 z}{\partial y \partial x} \right|_{\substack{x=-1 \\ y=1 \\ z=-1}} = \frac{-6z \frac{\partial z}{\partial y} \frac{\partial z}{\partial x}}{3z^2 + 1} \bigg|_{\substack{x=-1 \\ y=1 \\ z=-1}} = 0$$

Note that  $\left. \frac{\partial z}{\partial x} \right|_{\substack{x=-1 \\ y=1 \\ z=-1}} = \left. \frac{\partial z}{\partial y} \right|_{\substack{x=-1 \\ y=1 \\ z=-1}} = 0$

$$\left( \frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left( \frac{\partial^2 z}{\partial y \partial x} \right)^2 \right) \bigg|_{\substack{x=-1 \\ y=1 \\ z=-1}} = \frac{\partial^2 z}{\partial x^2} \bigg|_{\substack{x=-1 \\ y=1 \\ z=-1}} \frac{\partial^2 z}{\partial y^2} \bigg|_{\substack{x=-1 \\ y=1 \\ z=-1}} - \left( \frac{\partial^2 z}{\partial y \partial x} \right)^2 \bigg|_{\substack{x=-1 \\ y=1 \\ z=-1}} = \left( \frac{6}{4} \right) \left( -\frac{2}{4} \right) - 0 < 0$$

Therefore,  $(1, -1)$  is a saddle point for  $z = z(x, y)$ , that means  $z = z(x, y)$  attains neither local maximum nor local minimum at  $(1, -1)$ .

Also, we observe that  $z = z(x, y)$  has neither the global maximum nor the global minimum. □

13. If  $\varphi(x, y, z) = x^2 y^2 z^2$ , find

(a) the maximum rate of change of  $\varphi$  at the point  $(1, 1, 1)$  and the direction in which this occurs;

(b) the rate of change of  $\varphi$  at the point  $(2, 1, 1)$  in the direction of  $3\vec{i} + 4\vec{k}$ .

Solution:

(a)

$$\varphi(x, y, z) = x^2 y^2 z^2 \Rightarrow \text{grad } \varphi = \begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix} = \begin{pmatrix} 2xy^2 z^2 \\ 2yx^2 z^2 \\ 2zx^2 y^2 \end{pmatrix} \& \text{grad } \varphi_{\substack{x=1 \\ y=1 \\ z=1}} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

At the point  $(1, 1, 1)$ ,  $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$  points to the direction in which  $\varphi$  increases most rapidly and the rate of change

$$\text{of } \varphi \text{ at the point } (1, 1, 1) = \left| \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right| = \sqrt{12} = 2\sqrt{3}.$$

(b)

$$\varphi(x, y, z) = x^2 y^2 z^2 \Rightarrow \text{grad } \varphi = \begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix} = \begin{pmatrix} 2xy^2 z^2 \\ 2yx^2 z^2 \\ 2zx^2 y^2 \end{pmatrix} \& \text{grad } \varphi_{\substack{x=2 \\ y=1 \\ z=1}} = \begin{pmatrix} 4 \\ 8 \\ 8 \end{pmatrix}.$$

$$\frac{d\varphi}{ds} = D_v \varphi(2, 1, 1) = \text{grad } \varphi(2, 1, 1) \cdot \frac{3\vec{i} + 4\vec{j}}{|3\vec{i} + 4\vec{j}|} = (4\vec{i} + 8\vec{j} + 8\vec{k}) \cdot \frac{3\vec{i} + 4\vec{j}}{|3\vec{i} + 4\vec{j}|} = \frac{12 + 32}{\sqrt{9 + 12}} = \frac{44}{5}$$

□

14. A bomber is carrying a heat seeking missile which has the property that at any point  $(x, y, z)$  in space it moves in the direction of maximum temperature increase and the temperature at  $(x, y, z)$  is  $T = T(x, y, z) = 2x^2 - xyz$ .

Suppose the bomber has just launched the missile at the point  $P(1, 2, 3)$  and the missile can move at a speed of 50 km/minute in the direction specified.

(a) In what direction will the missile move?

(b) How fast is the temperature experienced by the missile changing in degree Celsius per kilometer at that instant when the missile has just left the bomber?

- (c) How fast is the temperature experienced by the missile changing in degree Celsius per minute at that instant when the missile has just left the bomber?
- (d) Due to the failure of an electronic device, the missile is no longer heat seeking but still can move at a speed of 50 km/minute in the direction specified.  
How fast is the temperature experienced by the missile changing in degree Celsius per minute at that instant when the missile has just left the bomber and moved in the direction specified by  $\vec{v} = \vec{i} + \vec{j}$ ?

Solution:

(a)

The temperature gradient vector  $\nabla T(x, y, z) = (4x - yz)\vec{i} - xz\vec{j} - xy\vec{k}$ .

$$\nabla T(1, 2, 3) = (4x - yz)\vec{i} - xz\vec{j} - xy\vec{k} \Big|_{\substack{x=1 \\ y=2 \\ z=3}} = -2\vec{i} - 3\vec{j} - 2\vec{k}$$

The missile will move in the direction specific by the vector  $-2\vec{i} - 3\vec{j} - 2\vec{k}$  or  $\frac{-2\vec{i} - 3\vec{j} - 2\vec{k}}{\sqrt{17}}$

(b)

The rate of change of temperature with respect to distance experienced by the missile at  $P(1, 2, 3)$  is:

$$\frac{dT}{ds} = |-2\vec{i} - 3\vec{j} - 2\vec{k}| = \sqrt{17} \frac{^{\circ}\text{C}}{\text{km}}.$$

(c)

$$\frac{dw}{dt} = \frac{dw}{ds} \frac{ds}{dt} = \sqrt{17} \frac{^{\circ}\text{C}}{\text{km}} \times 50 \frac{\text{km}}{\text{min}} = 50\sqrt{17} \frac{^{\circ}\text{C}}{\text{min}}.$$

(d)

The unit vector in the direction of the given vector  $\vec{v} = \vec{i} + \vec{j}$  is  $\vec{u} = \frac{\vec{i} + \vec{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$ .

$$\text{The temperature gradient vector } \nabla T(1, 2, 3) = (4x - yz)\vec{i} - xz\vec{j} - xy\vec{k} \Big|_{\substack{x=1 \\ y=2 \\ z=3}} = -2\vec{i} - 3\vec{j} - 2\vec{k}.$$

Therefore the missile's initial rate of change of temperature with respect to distance is:

$$\frac{dT}{ds} \equiv D_{\vec{u}}T(1, 2, 3) = \nabla T(1, 2, 3) \cdot \vec{u} = (-2\vec{i} - 3\vec{j} - 2\vec{k}) \cdot \left( \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} \right) = \frac{-2-3}{\sqrt{2}} \frac{^{\circ}\text{C}}{\text{km}} = -\frac{5}{\sqrt{2}} \frac{^{\circ}\text{C}}{\text{km}}$$

Its speed is  $\frac{ds}{dt} = 50 \frac{\text{km}}{\text{min}}$ , so the time rate of change of temperature experienced by the hawk is

$$\frac{dT}{dt} = \frac{dT}{ds} \frac{ds}{dt} = D_{\vec{u}}T \frac{ds}{dt} = \left( -\frac{5}{\sqrt{2}} \frac{^{\circ}\text{C}}{\text{km}} \right) \left( 50 \frac{\text{km}}{\text{min}} \right) = -\frac{250}{\sqrt{2}} \frac{^{\circ}\text{C}}{\text{min}}.$$

□

15. Let  $u(x, y) = 3x^2 + y^2$ .

- (a) Find the directional derivative of  $u(x, y) = 3x^2 + y^2$  at the point  $(x, y)$  in the direction  $(1, 1)$ .
- (b) Determine the points  $(x, y)$  and directions for which the directional derivative of  $u(x, y) = 3x^2 + y^2$  has its largest value if  $(x, y)$  is restricted to lie on the circle  $x^2 + y^2 = 1$ .

Solution:

(a)

$$u(x, y) = 3x^2 + y^2 \Rightarrow \text{gradu} = \begin{pmatrix} 6x \\ 2y \end{pmatrix}. \text{ The unit vector with the same direction as } (1,1) \text{ is } \vec{n} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Thus, the directional derivative of  $u(x, y) = 3x^2 + y^2$  at the point  $(x, y)$  in the direction  $(1,1)$  is:

$$u_{\vec{n}}(x, y) = \begin{pmatrix} 6x \\ 2y \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{6x + 2y}{\sqrt{2}}.$$

(b)

At  $(x, y)$ ,  $u(x, y) = 3x^2 + y^2$  has its largest directional derivative in the direction as pointed by

$$\text{gradu} = \begin{pmatrix} 6x \\ 2y \end{pmatrix} \text{ and the corresponding value is } |\text{gradu}| = \left| \begin{pmatrix} 6x \\ 2y \end{pmatrix} \right| = \sqrt{36x^2 + 4y^2}.$$

If  $(x, y)$  is restricted to lie on the circle  $x^2 + y^2 = 1$ , then  $x = \cos \theta, y = \sin \theta, 0 \leq \theta \leq 2\pi$ .

$$\text{Then } \left| \begin{pmatrix} 6 \cos \theta \\ 2 \sin \theta \end{pmatrix} \right| = \sqrt{36 \cos^2 \theta + 4 \sin^2 \theta} = \sqrt{4 + 32 \cos^2 \theta} \text{ and } \sqrt{4 + 32 \cos^2 \theta} \text{ has the greatest value } \sqrt{36}$$

when  $\theta = 0, \pi$ .

For  $\theta = 0$ , we have  $x = 1, y = 0$ , for  $\theta = \pi$ , we have  $x = -1, y = 0$ .

Answer: at  $x = 1, y = 0$  and the direction  $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$  and at  $x = -1, y = 0$  and the direction  $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$ ,

$u(x, y) = 3x^2 + y^2$  its largest directional derivative 6.

□

-End-