

CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I

Session : Semester B, 2018/2019

Time allowed : Three hours

This paper has **SIX** pages (including this cover page).

A brief table of derivatives is attached on pages 5 and 6.

Instructions to candidates:

1. This paper has **EIGHT** questions.
 2. Attempt **ALL** questions.
 3. Each question carries 13 marks.
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*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

NOT TO BE TAKEN AWAY

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BUT FORWARDED TO LIB

Question 1

Differentiate with respect to x :

(a) $\left(\frac{2x-1}{x+3}\right)^4$; (2 marks)

(b) $e^{\cos(3x)}$; (2 marks)

(c) $(\cosh x)^x$; (3 marks)

(d) $\tan^{-1}\left(\frac{9+x^2}{9-x^2}\right)$; (3 marks)

(e) $\log_e \log_e \cos x$. (3 marks)

(Hint: $-\frac{\tan x}{\log_e \cos x}$ is a wrong answer for Question 1 (e) .)

Question 2

(a) Let $f(x) = |\tan x|$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Is $f(x)$ differentiable at $x = 0$?
Give your reason.

(Hint: $|\tan x| = \begin{cases} \tan x & , \text{ if } x \in \left[0, \frac{\pi}{2}\right) \\ -\tan x & , \text{ if } x \in \left(-\frac{\pi}{2}, 0\right] \end{cases}$.) (6 marks)

(b) A curve has parametric equations

$$x = \cos t + \log_e \tan\left(\frac{t}{2}\right) ,$$

$$y = \sin t ,$$

where t is the parameter and $0 < t < \pi$.

Show that

i) $\frac{dy}{dx} = \tan t$,

ii) $\frac{d^2y}{dx^2} = \frac{\tan t}{\cos^3 t}$. (7 marks)

Question 3

(a) Evaluate the following limits:

(i) $\lim_{x \rightarrow 0} \frac{3-x^3-3\cos x}{x^4}$; (4 marks)

(ii) $\lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$. (4 marks)

(b) Show from first principles that $\frac{d}{dx}(\sin x) = \cos x$.

(Hint: $\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$.) (5 marks)

Question 4

Given a unit circle, we try to calculate the perimeters of inscribed regular 6-sided, 12-sided, 24-sided, ... polygons as shown in Figure 1, thus obtaining lower bound for the perimeter of the unit circle ($= 2\pi$ units).

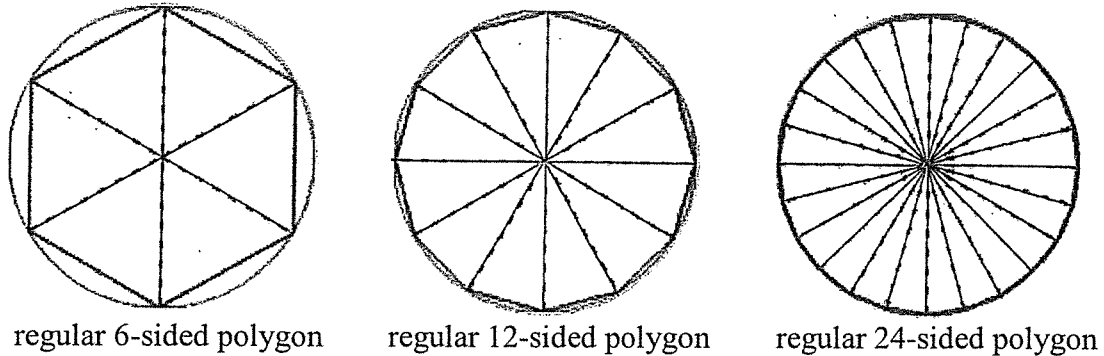


Figure 1

Let P_n denotes the perimeter of inscribed regular n -sided polygon as shown in Figure 1.

(a) Find the values of P_6 , P_{12} , P_{24} and P_{48} , correct your answers to 6 decimal places. (10 marks)

(b) Find the value of $\frac{64P_{48}-56P_{24}+14P_{12}-P_6}{42}$, correct your answer to 5 decimal places. (3 marks)

(Hint: The result in Question 4(b) is an approximation to the value of π .)

Question 5

(a) If $y = (ax + b)^{-p}$, there a and p are positive integers, b is a constant, find the general formula for the n th derivative of y with respect to x . (4 marks)

(b) Express $\frac{4x^2}{(2x+1)^3}$ in partial fractions. (5 marks)

(c) Hence, or otherwise, find $\frac{d^4}{dx^4} \left(\frac{4x^2}{(2x+1)^3} \right)$. You need not simplify your answer. (4 marks)

Question 6

Let $f: \mathbf{R} \rightarrow \mathbf{R}$ and $f(x)$ is defined as $f(x) = (x^2)^{\frac{1}{3}} - (x^2 - 4)^{\frac{1}{3}}$ for $x \in \mathbf{R}$.

(a) Show that $f(x)$ is an even function of x and that $f(x) > 0$ for $x \in \mathbf{R}$. (4 marks)

(b) Expand and simplify $\left[x^{\frac{2}{3}} - (x^2 - 4)^{\frac{1}{3}} \right] \left[x^{\frac{4}{3}} + x^{\frac{2}{3}}(x^2 - 4)^{\frac{1}{3}} + (x^2 - 4)^{\frac{2}{3}} \right]$. (2 marks)

(c) Evaluate $\lim_{x \rightarrow \infty} f(x)$. (3 marks)

(d) Solve the equation $f'(x) = 0$. (4 marks)

Question 7

Let $\frac{x^2}{4} - y^2 = 1$ be the equation of the hyperbola H , and let $\left(t + t^{-1}, \frac{t-t^{-1}}{2}\right)$ be the coordinates of a point P , where $t \neq 0$.

- (a) Find the coordinates of the foci and vertices of the hyperbola H . (5 marks)
- (b) Show that point P lies on the hyperbola H . (3 marks)
- (c) Find the equation of the tangent to the hyperbola H at point P . (5 marks)

Question 8

Let $y = \sin(2 \log_e(1+x))$, for $x > -1$.

- (a) Show that $(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + 4y = 0$. (3 marks)
- (b) Hence, deduce that,
$$(1+x)^2 \frac{d^{n+2} y}{dx^{n+2}} + (2n+1)(1+x) \frac{d^{n+1} y}{dx^{n+1}} + (n^2+4) \frac{d^n y}{dx^n} = 0,$$
for $n = 0, 1, 2, \dots$ (5 marks)
- (c) Hence, or otherwise, find the power series for $\sin(2 \log_e(1+x))$ in ascending powers of x , up to and including the term in x^4 . (5 marks)

(Hint: Leibnitz' rule: For any functions u and v whose derivatives up to the n th order exist, $(uv)^{(n)} = {}_n C_0 u^{(n)} v^{(0)} + {}_n C_1 u^{(n-1)} v^{(1)} + {}_n C_2 u^{(n-2)} v^{(2)} + \dots + {}_n C_r u^{(n-r)} v^{(r)} + \dots + {}_n C_n u^{(0)} v^{(n)}$, where

${}_n C_r = \frac{n!}{(n-r)!r!}$, $u^{(0)} = u$, $v^{(0)} = v$ and $u^{(r)}$, $v^{(r)}$ are the r th derivatives of u and v

respectively, for $r = 1, 2, 3, \dots, n$.)

Short Table of Derivatives of $y = f(u)$ with respect to x , where u is a function of x

Functions, $y = f(u)$	Derivative of y with respect to x
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c \frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1} \frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x .	$\frac{dy}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u} \log_a e \frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a \frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u \frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1} \frac{du}{dx} + u^v \log_e u \frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u \frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u \frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u \frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\operatorname{cosec}^2 u \frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u \frac{du}{dx}$
$y = \operatorname{cosec} u$	$\frac{dy}{dx} = -\operatorname{cosec} u \cot u \frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2} \frac{du}{dx}$

Functions, $y = f(u)$	Derivative of y with respect to x
$y = \cot^{-1} u$	$\frac{dy}{dx} = -\frac{1}{1+u^2} \frac{du}{dx}$
$y = \sec^{-1} u$	$\frac{dy}{dx} = \frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx}$
$y = \operatorname{cosec}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx}$
$y = \sinh u$	$\frac{dy}{dx} = \cosh u \frac{du}{dx}$
$y = \cosh u$	$\frac{dy}{dx} = \sinh u \frac{du}{dx}$
$y = \tanh u$	$\frac{dy}{dx} = \operatorname{sech}^2 u \frac{du}{dx}$
$y = \coth u$	$\frac{dy}{dx} = -\operatorname{cosech}^2 u \frac{du}{dx}$
$y = \operatorname{sech} u$	$\frac{dy}{dx} = -\operatorname{sech} u \tanh u \frac{du}{dx}$
$y = \operatorname{cosech} u$	$\frac{dy}{dx} = -\operatorname{cosech} u \coth u \frac{du}{dx}$
$y = \sinh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$
$y = \cosh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}$
$y = \tanh^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2} \frac{du}{dx}$
$y = \coth^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2} \frac{du}{dx}$
$y = \operatorname{sech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}$
$y = \operatorname{cosech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{ u \sqrt{u^2+1}} \frac{du}{dx}$