
EE 3121
Differential Equations for Electrical Engineering

Tutorial 5: Second-Order Linear Ordinary Differential Equations



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Summary

- We have considered in this chapter equations of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t).$$

- Two solutions $y_1(t)$ and $y_2(t)$ to the homogeneous ODE (i.e., when $g(t) = 0$) are called fundamental if any solution to the ODE can be written in the form

$$y(t) = c_1y_1(t) + c_2y_2(t).$$

- Fundamental solutions must satisfy the ODE and have nonzero Wronskian

$$W(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{pmatrix} = y_1(t)\dot{y}_2(t) - y_2(t)\dot{y}_1(t).$$

so that constants c_1 and c_2 can be found to satisfy the given initial conditions

Summary

$$\begin{aligned}y(0) &= c_1 y_1(0) + c_2 y_2(0) = y_0, \\ \dot{y}(0) &= c_1 \dot{y}_1(0) + c_2 \dot{y}_2(0) = \dot{y}_0.\end{aligned}$$

- If the Wronskian is nonzero at one point in time it is nonzero everywhere and if it is zero at one point it is zero everywhere.
- Solutions to the time-invariant homogeneous ODE

$$\frac{d^2 y}{dt^2} + a \frac{dy}{dt} + by = 0$$

are found by trying $y(t) = e^{rt}$, which leads to the characteristic equation

$$r^2 + ar + b = 0$$

which has solutions

$$r_{1,2} = -(a/2) \pm \sqrt{(a/2)^2 - b}.$$

Summary

- If $b - a^2/2 < 0$ then the two fundamental solutions to the ODE are

$$y_1(t) = \exp(r_1 t), \quad y_2(t) = \exp(r_2 t).$$

- If $b - a^2/2 = 0$ then the two solutions are

$$y_1(t) = \exp(r_1 t), \quad y_2(t) = \exp(r_1 t).$$

- If $b - a^2/2 > 0$ then the two solutions are

where $\lambda = -(a/2)$ and $\omega = \sqrt{b - a^2/4}$.

Summary

- A time-invariant system in the aforementioned form is
 - stable if $a \geq 0$ and $b > 0$;
 - asymptotically stable (damped) if $a > 0$ and $b > 0$;
 - unstable if either a or b is less than zero.
- If $y_1(t)$ and $y_2(t)$ are fundamental solutions to the homogeneous ODE

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

then the solution to

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t),$$

with initial conditions $y(t_0) = 0$ and $\dot{y}(t_0) = 0$ is

$$y_P(t) = \int_{t_0}^t K(t, \tau)g(\tau)d\tau$$

Summary

- where:

$$K(t, \tau) = (-y_1(t)y_2(\tau) + y_2(t)y_1(\tau))/W(\tau)$$

is the kernel of the ODE and

$$W(\tau) = y_1(\tau)\dot{y}_2(\tau) - y_2(\tau)\dot{y}_1(\tau)$$

is the Wronskian of $y_1(\tau)$ and $y_2(\tau)$.

- Any solution to the nonhomogeneous ODE can be written as

$$y(t) = y_P(t) + c_1y_1(t) + c_2y_2(t).$$

where the constants c_1 and c_2 are chosen to satisfy initial conditions.

Summary

- A sometimes quicker approach to solving nonhomogeneous time-invariant ODEs is the method of undetermined coefficients.
 - In this method one chooses a trial solution with unknown coefficients based on the form of the input (e.g., if $g(t) = \sin \omega t$, try $y(t) = C \sin \omega_0 t + D \cos \omega_0 t$), substitutes the trial solution in the ODE, and solves for the unknown coefficients.
 - This chapter provided a table of trial solutions for specific inputs and discussed some limitations of the method.
- When the input to a significantly underdamped time-invariant system oscillates at a frequency near the natural frequency of the system, the magnitude of the output significantly increases. This is called **resonance**.
- If the coefficients $p(t)$ and $q(t)$ in a second-order linear ODE are continuous in an open interval, and initial conditions are specified at a point in that interval then the ODE has a solution in that interval and the solution is unique. In particular, LTI ODEs have solutions everywhere.

Summary-Solved Problems

■ **SP5.1** Solve:

$$\frac{d^2y}{dt^2} + 13\frac{dy}{dt} + 12y = 0, \quad y(0) = 1, \quad \dot{y}(0) = -2 \quad (1)$$

■ *Solution:* We try $y(t) = e^{rt}$. Now $dy/dt = re^{rt}$, $d^2y/dt^2 = r^2e^{rt}$.

■ Substituting these into the ODE yields

$$(r^2 + 13r + 12)e^{rt} = 0.$$

■ Since $e^{rt} > 0$ we can divide through by it and obtain the characteristic equation

$$r^2 + 13r + 12 = 0.$$

■ This factors into

$$(r + 1)(r + 12) = 0,$$

and so the roots of the characteristic equation are $r_1 = -1$ and $r_2 = -12$.

Summary-Solved Problems

■ *Solution:*

- Therefore, the fundamental solutions are $y_1(t) = e^{-t}$, $y_2(t) = e^{-12t}$.
- The solution $y(t)$ to Equation (??) must be a linear combination of these: $y(t) = c_1 e^{-t} + c_2 e^{-12t}$.
- We find the coefficients c_1 and c_2 from the initial conditions:

$$\begin{aligned}y(0) &= c_1 e^{(0)} + c_2 e^{(0)} = c_1 + c_2 = 1, \\ \frac{dy}{dt} &= \frac{d}{dt}(c_1 e^{-t} + c_2 e^{-12t}) = -c_1 e^{-t} - 12c_2 e^{-12t}, \\ \frac{dy}{dt}(0) &= -c_1 e^{(0)} - 12c_2 e^{(0)} = -c_1 - 12c_2 = -2.\end{aligned}$$

- We have two linear equations in two unknowns:

$$\begin{aligned}c_1 + c_2 &= 1, \\ -c_1 - 12c_2 &= -2.\end{aligned}$$

Summary-Solved Problems

■ *Solution:*

■ Solving:

$$\begin{aligned}c_1 + c_2 &= 1, \\-c_1 - 12c_2 &= -2. \\c_2 &= 1 - c_1, \\-c_1 - 12(1 - c_1) &= -2, \\11c_1 &= 10, \\c_1 &= 10/11, \\c_2 &= 1/11.\end{aligned}$$

■ Therefore, the solution to Equation (??) is

$$y(t) = (10/11)e^{-t} + (1/11)e^{-12t}.$$

Summary-Solved Problems

■ **SP5.2** Solve:

$$\frac{d^2y}{dt^2} - 12\frac{dy}{dt} + 36y = 0, \quad y(0) = 2, \quad \dot{y}(0) = -3. \quad (2)$$

■ **Solution:** We try $y(t) = e^{rt}$. Now $dy/dt = re^{rt}$, $d^2y/dt^2 = r^2e^{rt}$.

■ Substituting these into the ODE yields

$$(r^2 - 12r + 36)e^{rt} = 0.$$

■ Since $e^{rt} > 0$ we can divide through by it and obtain the characteristic equation

$$r^2 - 12r + 36 = 0.$$

■ This factors into

$$(r - 6)^2 = 0,$$

and so the roots of the characteristic equation are $r_1 = r_2 = 6$.

Summary-Solved Problems

■ *Solution:*

- Therefore, the fundamental solutions are $y_1(t) = e^{6t}$, $y_2(t) = te^{6t}$. (Recall the results of Section 5.1.2.)
- The solution $y(t)$ to Equation (??) must be a linear combination of these: $y(t) = c_1e^{6t} + c_2te^{6t}$.
- We find the coefficients c_1 and c_2 from the initial conditions:

$$\begin{cases} y(0) = c_1e^{(0)} + c_2e^{(0)} = c_1 = 2, \\ \frac{dy}{dt} = \frac{d}{dt}(c_1e^{6t} + c_2te^{6t}) = 6c_1e^{6t} + c_2e^{6t} + 6tc_2e^{6t}, \\ \frac{dy}{dt}(0) = 6c_1e^{(0)} + c_2e^{(0)} + 6(0)c_2e^{(0)} = 6c_1 + 6c_2 = -3. \end{cases}$$

- We have

$$\begin{aligned} c_1 &= 2, \\ 6c_1 + c_2 &= -3. \end{aligned}$$

so, $c_2 = -15$ and the solution to Equation (??) is

$$y(t) = 2e^{6t} - 15te^{6t}.$$

Summary-Solved Problems

■ **SP5.3** Solve:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 0, \quad y(0) = 3, \quad \dot{y}(0) = 2 \quad (3)$$

- *Solution:* Trying $y(t) = e^{rt}$, remembering that $dy/dt = re^{rt}$ and $d^2y/dt^2 = r^2e^{rt}$, substituting these into the ODE and dividing by e^{rt} (which is always nonzero) yields the characteristic equation

$$r^2 + 4r + 20 = 0.$$

■ Completing the square (recall Section 1.1.2.1):

$$\begin{aligned} r^2 + 4r + 20 &= (r + 2)^2 - 4 + 20 = (r + 2)^2 + 16 = 0, \\ (r + 2)^2 &= -16, \\ r + 2 &= \pm\sqrt{-16} = \pm 4\sqrt{-1} = \pm 4i, \\ t &= -2 \pm 4i. \end{aligned}$$

and so the roots of the characteristic equation are $r_1 = -2 + 4i$ and $r_2 = -2 - 4i$.

Summary-Solved Problems

■ *Solution:*

- Therefore, the fundamental solutions are $y_1(t) = e^{-2t} \cos 4t$, $y_2(t) = e^{-2t} \sin 4t$. (Recall the results of Section 5.1.3.)
- The solution $y(t)$ to Equation (??) must be a linear combination of these: $y(t) = c_1 e^{-2t} \cos 4t + c_2 e^{-2t} \sin 4t$.
- We find the coefficients c_1 and c_2 from the initial conditions:

$$\begin{aligned} y(0) &= c_1 e^{(0)} \cos(0) + c_2 e^{(0)} \sin(0) = c_1 \cdot 1 + c_2 \cdot (0) = c_1 = 3, \\ \frac{dy}{dt} &= \frac{d}{dt}(c_1 e^{-2t} \cos 4t + c_2 e^{-2t} \sin 4t) = \\ &\quad -2c_1 e^{-2t} \cos 4t - 4c_1 e^{-2t} \sin 4t - 2c_2 e^{-2t} \sin 4t + 4c_2 e^{-2t} \cos 4t, \\ \frac{dy}{dt}(0) &= -2c_1 e^{(0)} \cos(0) - 4c_1 e^{(0)} \sin(0) - 2c_2 e^{(0)} \sin(0) + 4c_2 e^{(0)} \cos(0), \\ \frac{dy}{dt}(0) &= -2c_1 \cdot 1 - 4c_1 \cdot (0) - 2c_2 \cdot (0) + 4c_2 \cdot (1) = -2c_1 + 4c_2 = 2. \end{aligned}$$

Summary-Solved Problems

■ *Solution:*

■ We have

$$\begin{aligned}c_1 &= 3, \\ -2c_1 + 4c_2 &= 2.\end{aligned}$$

■ So, $c_2 = 2$ and the solution to Equation (??) is

$$y(t) = 3e^{-2t} \cos 4t + 2e^{-2t} \sin 4t.$$

Summary-Solved Problems

- **SP5.4** Show that if $y_1(t)$ and $y_2(t)$ satisfy

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0, \quad (4)$$

then so does $y(t) = c_1y_1(t) + c_2y_2(t)$.

- **Solution:**

- The function $y(t) = c_1y_1(t) + c_2y_2(t)$ satisfies Equation (??) because it is a linear combination of solutions to the ODE and because the ODE is linear:

$$\begin{aligned} \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y &= \frac{d^2}{dt^2}(c_1y_1 + c_2y_2) + p(t)\frac{d}{dt}(c_1y_1 + c_2y_2) + q(t)(c_1y_1 + c_2y_2) \\ &= c_1\left(\frac{d^2y_1}{dt^2} + p(t)\frac{dy_1}{dt} + q(t)y_1\right) + c_2\left(\frac{d^2y_2}{dt^2} + p(t)\frac{dy_2}{dt} + q(t)y_2\right) \\ &= c_1 \cdot (0) + c_2 \cdot (0) = 0 \end{aligned}$$

Summary-Solved Problems

- **SP5.5** Derive the form of the Wronskian and convolution kernel for an (a) overdamped, (b) critically damped, and (c) underdamped LTI second-order system.

■ *Solution:*

- The Wronskian is defined as

$$W(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{pmatrix} = y_1(t)\dot{y}_2(t) - y_2(t)\dot{y}_1(t),$$

and the kernel is defined as

$$K(t - \tau) = (-y_1(t)y_2(\tau) + y_2(t)y_1(\tau))/W(\tau).$$

Summary-Solved Problems

■ *Solution:*

- **(a)** Fundamental solutions for overdamped systems have the form $y_1(t) = e^{-\lambda_1 t}$, and $y_2(t) = e^{-\lambda_2 t}$ where $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_1 \neq \lambda_2$. Hence, for an overdamped system,

$$W(t) = e^{-\lambda_1 t}(-\lambda_2 e^{-\lambda_2 t}) - e^{-\lambda_2 t}(-\lambda_1 e^{-\lambda_1 t}) = (\lambda_1 - \lambda_2)e^{-(\lambda_1 + \lambda_2)t},$$

$$\begin{aligned} K(t - \tau) &= ((-e^{-\lambda_1 t})(e^{-\lambda_2 \tau}) + (e^{-\lambda_2 t})(e^{-\lambda_1 \tau})) / ((\lambda_1 - \lambda_2)e^{-(\lambda_1 + \lambda_2)\tau}) \\ &= ((-e^{-\lambda_1 t})(e^{-\lambda_2 \tau})(e^{+(\lambda_1 + \lambda_2)\tau}) + (e^{-\lambda_2 t})(e^{-\lambda_1 \tau})(e^{+(\lambda_1 + \lambda_2)\tau})) / (\lambda_1 - \lambda_2) \\ &= ((-e^{-\lambda_1 t})(e^{+\lambda_1 \tau}) + (e^{-\lambda_2 t})(e^{+\lambda_2 \tau})) / (\lambda_1 - \lambda_2) \\ &= (e^{-\lambda_1(t-\tau)} - e^{-\lambda_2(t-\tau)}) / (\lambda_2 - \lambda_1). \end{aligned}$$

Summary-Solved Problems

■ *Solution:*

- **(b)** Fundamental solutions for critically damped systems have the form $y_1(t) = e^{-\lambda t}$, and $y_2(t) = te^{-\lambda t}$. Hence, for a critically damped system,

$$W(t) = e^{-\lambda t}(-\lambda te^{-\lambda t} + e^{-\lambda t}) - te^{-\lambda t}(-\lambda e^{-\lambda t}) = e^{-2\lambda t},$$

$$\begin{aligned} K(t - \tau) &= ((-e^{-\lambda t})(\tau e^{-\lambda \tau}) + (te^{-\lambda t})(e^{-\lambda \tau}))/e^{-2\lambda t} \\ &= (-e^{-\lambda t})(\tau e^{-\lambda \tau})(e^{+2\lambda \tau}) + (te^{-\lambda t})(e^{-\lambda \tau})(e^{+2\lambda \tau}) \\ &= (-e^{-\lambda t})(\tau e^{+\lambda \tau}) + (te^{-\lambda t})(e^{+\lambda \tau}) \\ &= (t - \tau)e^{-\lambda(t-\tau)}. \end{aligned}$$

Summary-Solved Problems

■ *Solution:*

- **(c)** Fundamental solutions for underdamped systems have the form $y_1(t) = e^{-\lambda t} \cos \omega t$, $y_2(t) = e^{-\lambda t} \sin \omega t$. Hence, for an underdamped system,

$$\begin{aligned} W(t) &= (e^{-\lambda t} \cos \omega t)(-\lambda e^{-\lambda t} \sin \omega t + \omega e^{-\lambda t} \cos \omega t) \\ &\quad - (e^{-\lambda t} \sin \omega t)(-\lambda e^{-\lambda t} \cos \omega t - \omega e^{-\lambda t} \sin \omega t) \\ &= \omega e^{-2\lambda t} (\cos^2 \omega t + \sin^2 \omega t) = \omega e^{-2\lambda t}, \end{aligned}$$

$$\begin{aligned} K(t - \tau) &= ((-e^{-\lambda t} \cos \omega t)(e^{-\lambda \tau} \sin \omega \tau) + (e^{-\lambda t} \sin \omega t)(e^{-\lambda \tau} \cos \omega \tau)) / (\omega e^{-2\lambda \tau}) \\ &= ((-e^{-\lambda t} \cos \omega t)(e^{-\lambda \tau} \sin \omega \tau)(e^{+2\lambda \tau}) + (e^{-\lambda t} \sin \omega t)(e^{-\lambda \tau} \cos \omega \tau)(e^{+2\lambda \tau})) / \omega \\ &= ((-e^{-\lambda t} \cos \omega t)(e^{+\lambda t} \sin \omega \tau) + (e^{-\lambda t} \sin \omega t)(e^{+\lambda \tau} \cos \omega \tau)) / \omega \\ &= e^{-\lambda(t-\tau)} (\sin \omega \tau \cos \omega t - \cos \omega \tau \sin \omega t) / \omega \\ &= e^{-\lambda(t-\tau)} (\sin \omega(t - \tau)) / \omega. \end{aligned}$$

Summary-Solved Problems

■ **SP5.6** Solve

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 4e^{-t}, \quad y(0) = 0, \quad \dot{y}(0) = 0. \quad (5)$$

■ *Solution:*

- *Step 1:* Find the fundamental solutions to the homogeneous equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0.$$

- Trying $y(t) = e^{rt}$ we find the characteristic equation

$$r^2 + 5r + 6 = 0$$

which factors into

$$(r + 2)(r + 3) = 0,$$

Summary-Solved Problems

■ *Solution:*

- Hence it has roots $r_1 = -2, r_2 = -3$. Therefore the fundamental solutions are

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{-3t}.$$

- *Step 2:* Calculate the Wronskian:

$$\begin{aligned} W(\tau) &= y_1(\tau)\dot{y}_2(\tau) - y_2(\tau)\dot{y}_1(\tau) \\ &= e^{-2\tau}(-3e^{-3\tau}) - e^{-3\tau}(-2e^{-2\tau}) \\ &= (-3 + 2)e^{-5\tau} = -e^{-5\tau}. \end{aligned}$$

- *Step 3:* Calculate the kernel:

$$\begin{aligned} K(t - \tau) &= (-y_1(t)y_2(\tau) + y_2(t)y_1(\tau))/W(\tau) \\ &= (-e^{-2t}(e^{-3\tau}) + e^{-3t}(e^{-2\tau}))/(-e^{-5\tau}) \\ &= e^{-2t}e^{+2\tau} - e^{-3t}e^{+3\tau} \\ &= e^{-2(t-\tau)} - e^{-3(t-\tau)}. \end{aligned}$$

Summary-Solved Problems

■ *Solution:*

■ *Step 4:* Perform the integration:

$$\begin{aligned}y(t) &= \int_0^t K(t, \tau)g(\tau)d\tau = \int_0^t (e^{-2(t-\tau)} - e^{-3(t-\tau)})(4e^{-\tau})d\tau \\&= \int_0^t e^{-2(t-\tau)}(4e^{-\tau})d\tau - \int_0^t e^{-3(t-\tau)}(4e^{-\tau})d\tau \\&= 4e^{-2t} \int_0^t e^{(+2-1)\tau}d\tau - 4e^{-3t} \int_0^t e^{(+3-1)\tau}d\tau \\&= 4 \int_0^t e^{\tau}d\tau - 4e^{-3t} \int_0^t e^{2\tau}d\tau \\&= 4e^{-2t}(e^{-t} - 1) - 4e^{-3t}(e^{-2t} - 1)/2.\end{aligned}$$

■ *Step 5:* Calculate the particular solution

$$\begin{aligned}y_P(t) &= 4e^{-2t}(e^{-t} - 1) - 4e^{-3t}(e^{-2t} - 1)/2 \\&= 4e^{-t} - 4e^{-2t} - 2e^{-t} + 2e^{-3t},\end{aligned}$$

$$y_P(t) = 2e^{-t} - 4e^{-2t} + 2e^{-3t}. \quad (6)$$

Summary-Solved Problems

■ *Solution:*

- *Step 6:* Add in the response due to initial conditions: *In this problem the initial conditions are zero, so the particular solution is the complete solution.*
- *Step 7:* Check the solution by computing its value at $t = 0$. Does it match the initial conditions? Then,

$$\begin{cases} y_P(0) = 2e^{(0)} - 4e^{(0)} + 2e^{(0)} = 2 - 4 + 2 = 0, \\ \dot{y}_P(t) = -2e^{-t} + 8e^{-2t} - 6e^{-3t}, \\ \dot{y}_P(0) = -2e^{(0)} + 8e^{(0)} - 6e^{(0)} = -2 + 8 - 6 = 0. \end{cases}$$

- If we have the time we can also substitute the calculated solution into the ODE and see if the equation is satisfied.

Summary-Solved Problems

- **SP5.7** Find the steady-state solution to

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y = 5te^{2t} \quad (7)$$

- *Solution:*

- Referring to Table 5.1, we try

$$y(t) = (Ct + D)e^{2t}. \quad (8)$$

- Then

$$\frac{dy}{dt} = 2(Ct + D)e^{2t} + Ce^{2t} = 2Cte^{2t} + (2D + C)e^{2t}, \quad (9)$$

$$\frac{d^2y}{dt^2} = 4Cte^{2t} + 2Ce^{2t} + 2(2D + C)e^{2t} = 4Cte^{2t} + (4D + 4C)e^{2t}. \quad (10)$$

Summary-Solved Problems

■ *Solution:*

- Substituting Equations (??), (??), and (??) into Equation (??):

$$[4Cte^{2t} + (4D + 4C)e^{2t}] + 3[2Cte^{2t} + 2(2D + C)e^{2t}] + 4[(Ct + D)e^{2t}] = 5te^{2t},$$

- Dividing through by e^{2t} and collecting coefficients in like powers of t :

$$\begin{aligned} ((4 + 3 \cdot 2 + 4)C)t + ((4 + 3 \cdot 2 + 4)D + (4 + 3)C) &= 5t, \\ (14C)t + (14D + 7C) &= 5t. \end{aligned}$$

- Matching coefficients of similar powers of t :

$$\begin{aligned} 14C &= 5, \\ 14D + 7C &= 0. \end{aligned}$$

Summary-Solved Problems

■ *Solution:*

- The solutions are

$$\begin{aligned}C &= 5/14, \\D &= -5/28.\end{aligned}$$

- Hence the steady-state solution to Equation (??) is

$$y(t) = (5/14)(t - 1/2)e^{2t}.$$

End