
EE 3121
Differential Equations for Electrical Engineering

Lecture 5: Second-Order Linear Ordinary Differential Equations



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Second-Order Linear ODE

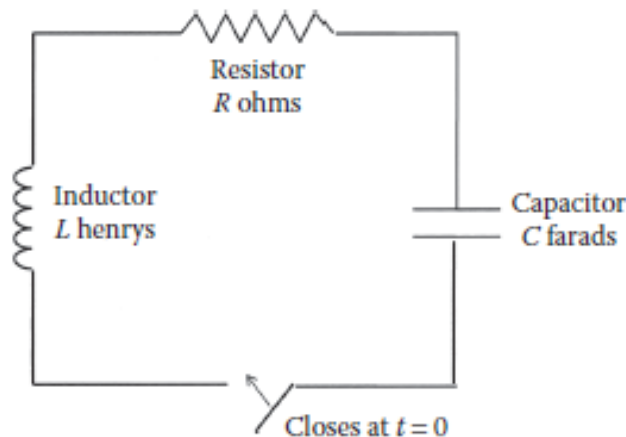
- Lectures 2 through 4 are focused on first-order ordinary differential equations (ODEs).
- In this lecture, we begin our march to higher-order ODEs.
 - A second-order linear ODE has the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t). \quad (1)$$

- Second-order linear ODEs probably command the attention of more engineers more of the time than any other type of differential equation, for many physical systems can be described by second-order ODEs (think of Newton's law).

5.1 Second-Order LTI Homogeneous Ordinary Differential Equations

- In a *time-invariant* linear second-order ODE, the coefficients $p(t)$ and $q(t)$ in Equation (1) are constants.
- A homogeneous equation has input $g(t) = 0$.
- Consider the electrical circuit shown in Fig. 5.1.



• Fig. 5.1 Second-order electrical circuit.

- Recall that the circuits we analyzed in Chapter 2 included a resistor accompanied by either a capacitor or an inductor.

5.1 Second-Order LTI Homogeneous Ordinary Differential Equations

- The circuit in Figure 5.1 combines all three. As discussed in Chapter 2, **Kirchhoffs voltage law** says that the sum of the voltage drops around the circuit equals zero.
- The voltage drop over a resistor is IR , where I is the current through the resistor, the voltage drop over an inductor is LdI/dt ;
- The voltage drop over a capacitor is $V_c = q/C$, where q is the electrical charge on the capacitor. (Note: the terms R , L , and C are known constants.)
- We assume that the quantity of interest is the voltage over the capacitor, $V = V_c$, that a voltage V_0 exists on the capacitor at time $t = 0$ and then a switch is turned on, resulting a response.

5.1 Second-Order LTI Homogeneous Ordinary Differential Equations

- Kirchhoffs loop law:

$$\begin{cases} L\frac{dI}{dt} + RI + V_c = 0, & V_c = V \\ LC\frac{d^2V}{dt^2} + RC\frac{dV}{dt} + V = 0, \\ \frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LC}V = 0. \end{cases} \quad (2)$$

- This is a linear, time-invariant (LTI) second-order homogeneous ODE.

- In this case the initial conditions are

$$\begin{aligned} V(0) &= V_0, \\ I(0) &= I_0. \end{aligned} \quad (3)$$

- Second-order ODEs posed as initial value problems must have two stated initial conditions to yield a unique solution.
- How do we solve Equations (2) and (3)?

5.1 Second-Order LTI Homogeneous Ordinary Differential Equations

- Based on our experience with first-order LTI ODEs we try the solution

$$V(t) = e^{rt}. \quad (4)$$

- Then we have

$$\frac{d}{dt}(e^{rt}) = re^{rt}, \quad (5)$$

$$\frac{d^2}{dt^2}(e^{rt}) = r^2 e^{rt}. \quad (6)$$

- Substituting Equations (4), (5) and (6) into Equation (2) we obtain

$$(r^2 + \frac{R}{L}r + \frac{1}{LC})e^{rt} = 0. \quad (7)$$

- Since $e^{rt} > 0$ we can divide through by e^{rt} and we are left with

$$r^2 + \frac{R}{L}r + \frac{1}{LC} = 0. \quad (8)$$

5.1 Second-Order LTI Homogeneous Ordinary Differential Equations

- Equation (8), a second-order polynomial equation in r , is called the *characteristic equation* for the ODE in Equation (2).

- Its solutions are

$$r = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}. \quad (9)$$

- Solutions to the ODE differ in form and behavior depending on the value of the expression under the radical.
 - If $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} > 0$, the characteristic equation has two real roots;
 - If $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0$, it has repeated roots;
 - If $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0$, it has complex conjugate roots.

5.1.1 Two Real Poles

- If $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} > 0$, the roots of the characteristic equation can be written

$$\begin{aligned} r_1 &= -\alpha - \beta, \\ r_2 &= -\alpha + \beta. \end{aligned} \tag{10}$$

where:

$$\alpha = \frac{R}{2L}, \tag{11}$$

$$\beta = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}. \tag{12}$$

- Note that $\alpha > \beta > 0$.

5.1.1 Two Real Poles

- In Section 5.2 will prove that any solution to the ODE in Equation (2) can be written as

$$V(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \quad (13)$$

where the coefficients c_1 and c_2 are constants that depend on initial conditions.

- Here, to satisfy Equation (3), we must have

$$\begin{aligned} V(0) &= c_1 + c_2 = V_0, \\ \frac{dV}{dt}(0) &= r_1 c_1 + r_2 c_2 = I_0. \end{aligned} \quad (14)$$

- Equations (14) form two linear simultaneous equations in two unknowns.
- The solutions are

$$\begin{aligned} c_1 &= \frac{I_0 - r_2 V_0}{r_1 - r_2}, \\ c_2 &= \frac{I_0 - r_1 V_0}{r_2 - r_1}. \end{aligned} \quad (15)$$

5.1.1 Two Real Poles

- Engineers call this kind of systems **overdamped**, a term we can explain more clearly in Section 5.1.4.

- If the ODE is of the form

$$\frac{d^2V}{dt^2} + a\frac{dV}{dt} + bV = 0. \quad (16)$$

- The system is overdamped if $a > 0$, $b > 0$, and

$$b - a^2/4 < 0. \quad (17)$$

5.1.2 Repeated Roots

- If the electrical circuit in Fig. 5.1 satisfies

$$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0,$$

then $\beta = 0$, $r_1 = r_2 = -\alpha$.

- We determine the solution by taking the limit when $\beta \rightarrow 0$. The two roots

$$\begin{aligned} r_1 &= -\alpha - \beta, \\ r_2 &= -\alpha + \beta, \end{aligned}$$

approach one another. For simplicity, assume $I_0 = 0$.

- As β goes to zero:

$$V(t) = \lim_{\beta \rightarrow 0} V_0 \left(-\frac{\alpha - \beta}{2\beta} e^{(-\alpha - \beta)t} + \frac{\alpha + \beta}{2\beta} e^{(-\alpha + \beta)t} \right). \quad (18)$$

5.1.2 Repeated Roots

- We begin by dividing through by $V_0 e^{-\alpha t}$:

$$\frac{V(t)}{V_0 e^{-\alpha t}} = -\frac{(\alpha - \beta)}{2\beta} e^{-\beta t} + \frac{(\alpha + \beta)}{2\beta} e^{\beta t}. \quad (19)$$

- Next we expand $e^{-\beta t}$ and $e^{\beta t}$ in power series:

$$e^{-\beta t} = 1 - \beta t + \frac{(\beta t)^2}{2} + \dots, \quad (20)$$

$$e^{\beta t} = 1 + \beta t + \frac{(\beta t)^2}{2} + \dots \quad (21)$$

- Substituting Equations (20) and (21) into Equation (19):

$$\frac{V(t)}{V_0 e^{-\alpha t}} = -\frac{(\alpha - \beta)}{2\beta} \left(1 - \beta t + \frac{(\beta t)^2}{2} + \dots \right) + \frac{(\alpha + \beta)}{2\beta} \left(1 + \beta t + \frac{(\beta t)^2}{2} + \dots \right),$$

5.1.2 Repeated Roots

$$\begin{aligned}\Rightarrow \frac{V(t)}{V_0 e^{-\alpha t}} &= \frac{1}{2\beta} \left(-\alpha + \beta + (\alpha - \beta)\beta t - \frac{(\alpha - \beta)(\beta t)^2}{2} + \dots \right) \\ &\quad + \frac{1}{2\beta} \left(\alpha + \beta + (\alpha + \beta)\beta t + \frac{(\alpha + \beta)(\beta t)^2}{2} + \dots \right) \\ &= \frac{1}{2\beta} (2\beta + 2\alpha\beta t + \beta^3 t^2 + \dots) = 1 + \alpha t + \frac{\beta^2 t^2}{2} + \dots\end{aligned}$$

■ Hence,

$$\begin{aligned}\lim_{\beta \rightarrow 0} \frac{V(t)}{V_0 e^{-\alpha t}} &= 1 + \alpha t, \\ V(t) &= V_0 e^{-\alpha t} (1 + \alpha t).\end{aligned}\tag{22}$$

- Equation (22) is the solution to Equation (2) with the specific initial conditions given in Equation (3).
- Note that we can also use L'Hospitale rule to evaluate directly the limit

$$V(t) = \lim_{\beta \rightarrow 0} V_0 \left(-\frac{\alpha - \beta}{2\beta} e^{(-\alpha - \beta)t} + \frac{\alpha + \beta}{2\beta} e^{(-\alpha + \beta)t} \right)$$

5.1.2 Repeated Roots

- One can verify by direct substitution into Equation (??) that $V_1 = e^{-\alpha t}$ and $V_2 = te^{-\alpha t}$ are solutions to Equation (??).
- Any solution to (??) can be written as

$$V(t) = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t}, \quad (23)$$

where coefficients c_1 and c_2 are constants that depend on initial conditions.

- Engineers call a system modeled by Equation (??) **critically damped**, another term we will defer explaining until in a later section.
- If the ODE is of the form

$$\frac{d^2 V}{dt^2} + a \frac{dV}{dt} + bV = 0, \quad (24)$$

- the system is critically damped if $a > 0$, $b > 0$, and

$$b - a^2/4 = 0. \quad (25)$$

5.1.3 Complex Conjugate Roots

- If the electrical circuit in Fig. 5.1 satisfies

$$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0,$$

then according to Equation (9) the roots of the characteristic equation can be written as

$$\begin{aligned} r_1 &= -\lambda + i\omega, \\ r_2 &= -\lambda - i\omega. \end{aligned} \tag{26}$$

where:

$$\lambda = \frac{R}{2L}, \tag{27}$$

$$\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}, \tag{28}$$

and $i = \sqrt{-1}$, the imaginary unit.

5.1.3 Complex Conjugate Roots

- To summarize the situation, in seeking to solve Equation (2) we proposed in Equation (4) the trial solution e^{rt} and Equations (26) through (28) resulted.
- A general solution would be of the form

$$V(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{(-\lambda + i\omega)t} + c_2 e^{(-\lambda - i\omega)t} = c_1 e^{-\lambda t} e^{i\omega t} + c_2 e^{-\lambda t} e^{-i\omega t}. \quad (29)$$

- What does Equation (29) mean? What are $e^{i\omega t}$ and $e^{-i\omega t}$?
- Consider the series expansion

$$e^x = 1 + x + \left(\frac{1}{2}\right)x^2 + \left(\frac{1}{3 \cdot 2}\right)x^3 + \left(\frac{1}{4 \cdot 3 \cdot 2}\right)x^4 + \dots + \left(\frac{1}{n!}\right)x^n + \dots \quad (30)$$

5.1.3 Complex Conjugate Roots

- Let $x = i\omega t$ in Equation (30), remembering that

$$\begin{aligned}i^2 &= -1, \\i^3 &= -i, \\i^4 &= 1, \\i^5 &= i.\end{aligned}$$

- Then Equation (30) becomes

$$\begin{aligned}e^{i\omega t} &= 1 + i\omega t - \left(\frac{1}{2}\right)(\omega t)^2 - i\left(\frac{1}{3 \cdot 2}\right)(\omega t)^3 + \left(\frac{1}{4 \cdot 3 \cdot 2}\right)(\omega t)^4 + i\left(\frac{1}{5 \cdot 4 \cdot 3 \cdot 2}\right)(\omega t)^5 \dots \\&= 1 - \left(\frac{1}{2}\right)(\omega t)^2 + \left(\frac{1}{4 \cdot 3 \cdot 2}\right)(\omega t)^4 + \dots \\&\quad + i\omega t - i\left(\frac{1}{3 \cdot 2}\right)(\omega t)^3 + i\left(\frac{1}{5 \cdot 4 \cdot 3 \cdot 2}\right)(\omega t)^5 - \dots\end{aligned} \quad (31)$$

- We recognize the real part of Equation (31) as

$$\cos(\omega t) = 1 - \left(\frac{1}{2}\right)(\omega t)^2 + \left(\frac{1}{4 \cdot 3 \cdot 2}\right)(\omega t)^4 + \dots + \frac{(-1)^n}{(2n)!}(\omega t)^{2n} + \dots$$

5.1.3 Complex Conjugate Roots

- and the imaginary part as

$$i \sin(\omega t) = i(\omega t - (\frac{1}{3 \cdot 2})(\omega t)^3 + (\frac{1}{5 \cdot 4 \cdot 3 \cdot 2})(\omega t)^5 - \dots + \frac{(-1)^n}{(2n+1)!}(\omega t)^{2n+1} + \dots)$$

- Hence we can write

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t). \quad (32)$$

- Equation (32) is known as the Euler's Identity.

- We have, from Equation (29),

$$V(t) = c_1 e^{-\lambda t} e^{i\omega t} + c_2 e^{-\lambda t} e^{-i\omega t},$$

and using the Euler's identity,

$$\begin{aligned} V(t) &= c_1 e^{-\lambda t} (\cos(\omega t) + i \sin(\omega t)) + c_2 e^{-\lambda t} (\cos(\omega t) - i \sin(\omega t)), \\ V(t) &= (c_1 + c_2) e^{-\lambda t} \cos(\omega t) + i(c_1 - c_2) e^{-\lambda t} \sin(\omega t), \\ V(t) &= c_3 e^{-\lambda t} \cos(\omega t) + c_4 e^{-\lambda t} \sin(\omega t). \end{aligned} \quad (33)$$

5.1.3 Complex Conjugate Roots

- Now if $V(t)$ is a real solution then the coefficients $c_3 = c_1 + c_2$ and $c_4 = i(c_1 - c_2)$ must be real.
- Since the coefficients c_1 and c_2 are arbitrary, we see that they must be complex numbers.
- If we let $c_1 = (c_3 - ic_4)/2$, $c_2 = (c_3 + ic_4)/2$, then

$$V(t) = c_1 e^{-\lambda t} e^{i\omega t} + c_2 e^{-\lambda t} e^{-i\omega t},$$

is equivalent to

$$V(t) = c_3 e^{-\lambda t} \cos \omega t + c_4 e^{-\lambda t} \sin \omega t, \quad (34)$$

where all the constants in Equation (34) are real valued.

5.1.3 Complex Conjugate Roots

- We can verify by direct substitution that $V_1(t) = e^{-\lambda t} \cos(\omega t)$, $V_2(t) = e^{-\lambda t} \sin(\omega t)$ are solutions to

$$\frac{d^2V}{dt^2} + \frac{R}{L} \frac{dV}{dt} + \frac{1}{LC} V = 0, \quad (35)$$

where, again,

$$\lambda = \frac{R}{2L},$$

and

$$\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}.$$

- For the initial conditions $V(0) = V_0$ and $dV/dt = 0$, we have

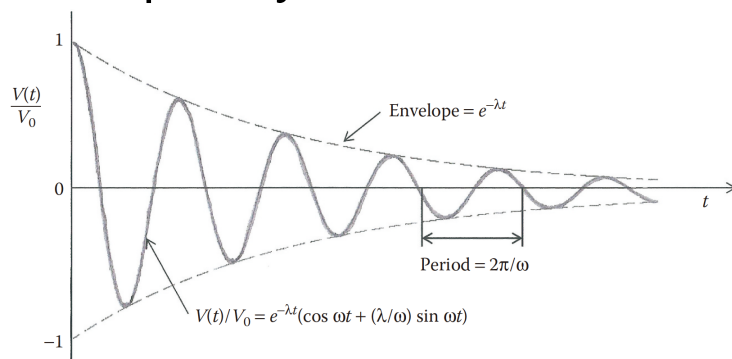
$$\begin{aligned} V(0) &= c_3 = V_0, \\ \frac{dV}{dt}(0) &= -\lambda c_3 + \omega c_4 = 0. \end{aligned} \quad (36)$$

5.1.3 Complex Conjugate Roots

- So, the solution to Equations (2) and (3) is

$$V(t) = V_0(e^{-\lambda t} \cos \omega t + (\lambda/\omega)e^{-\lambda t} \sin \omega t). \quad (37)$$

- Fig.5.2 presents a graph of an example of this function, showing the decreasing envelope due to the damping factor λ and the oscillatory behavior with frequency ω .



- Fig. 5.2 *Solution of a second – order LTI ODE when the characteristic equation has complex roots.*

- Note that, in terms of λ and ω , Equation (35) can be rewritten as

$$\frac{d^2V}{dt^2} + 2\lambda \frac{dV}{dt} + (\lambda^2 + \omega^2)V = 0. \quad (38)$$

5.1.3 Complex Conjugate Roots

- Any equation of the form in Equation (38) has the general solution

$$V(t) = c_1 e^{-\lambda t} e^{i\omega t} + c_2 e^{-\lambda t} e^{-i\omega t}, \quad (39)$$

where c_1 and c_2 are constants determined by initial conditions.

- It is important to note that in solving a new problem we do not have to go through the intermediate steps represented here by Equation (29) and invocation of Eulers equation.
- We can go directly from identification of a complex root of the characteristic equation to the real solution represented by Equation (39).

5.1.3 Complex Conjugate Roots

- Engineers call a system modelled by Equation (38). ***underdamped***, and call w the circuits natural frequency.
- If the ODE is of the form

$$\frac{d^2V}{dt^2} + a\frac{dV}{dt} + bV = 0, \quad (40)$$

the system is underdamped if $a > 0$, $b > 0$, and

$$b - a^2/4 > 0. \quad (41)$$

5.1.4 Under-, Over-, and Critically Damped Systems

- We are now in a position to explain why engineers use the terminology **under-**, **over-**, and **critically damped** systems.
- You will recall that engineers say **an LTI system is damped if, with no external input, it ultimately rests at the origin independent of initial conditions.**
- Consider now a general second-order LTI homogeneous ODE with $a > 0$, $b > 0$, and specific initial conditions:

$$\begin{aligned}\frac{d^2 y}{dt^2} + a \frac{dy}{dt} + by &= 0, \\ y(0) &= 1, \\ \dot{y}(0) &= 0.\end{aligned}\tag{42}$$

- The solution in all detail depends on the two parameters a and b ;
- But the qualitative character of the solution, that is, **the shape of the curve $y(t)$, is determined by just one parameter.**

5.1.4 Under-, Over-, and Critically Damped Systems

- We can scale (nondimensionalize) time with the substitution $t_* = t\sqrt{b}$ and Equations (42) are converted to a form that can be written as

$$\begin{aligned}\ddot{y} + 2\zeta\dot{y} + y &= 0, \\ y(0) &= 1, \\ \dot{y}(0) &= 0,\end{aligned}\tag{43}$$

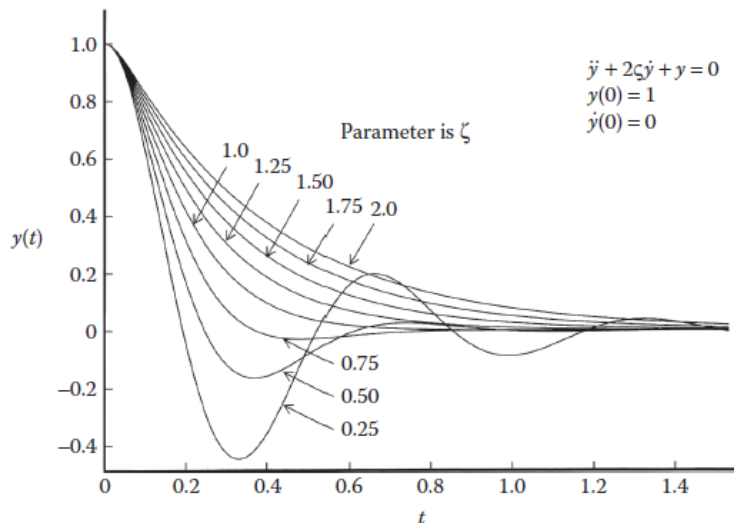
- the derivatives are understood to be with respect to the nondimensionalized time;
- $\zeta = a / (2\sqrt{b})$ is called the **damping ratio**.
- This is the form employed by control system engineers.

5.1.4 Under-, Over-, and Critically Damped Systems

- Communication system engineers prefer the form

$$\ddot{y} + \left(\frac{1}{Q}\right)\dot{y} + y = 0,$$

- where $Q = \sqrt{b}/a$ is called the **quality factor**.
- The point is that the solution as a function of scaled time is governed by just one parameter.



- Fig. 5.3 *Solution of a second – order LTI ODE as a function of time and damping ratio.*

5.1.4 Under-, Over-, and Critically Damped Systems

- Fig. 5.3 graphs the solution of Equation (43) as a function of scaled time and damping ratio.
- We see that solutions for $\zeta < 1$, the underdamped systems, overshoot the origin and oscillate about it before finally settling in. This behavior is more prominent the lower the damping ratio.
- Solutions for $\zeta > 1$, the overdamped systems, do not overshoot and oscillate but progress relatively slowly toward the origin, the higher the damping ratio the slower the motion.
- The system with $\zeta = 1$, the critically damped system, is the fastest to reach any given distance from the origin among the nonoscillating systems. Because it is fast and nonoscillating, critical damping is often sought as a design goal.
- But perhaps more often, systems with $\zeta \cong 0.75$ are implemented because they are faster yet and their overshoot and oscillation are minimal.

5.1.5 Stability of Second-Order Systems

- From the forms the fundamental solutions can take and their dependence on the coefficients a and b in Equations (16), (24), and (40), we can easily see that:
 - second-order time-invariant systems are stable if $a \geq 0$ and $b > 0$ and damped if $a > 0$ and $b > 0$;
 - If either a or b is negative the system is unstable;
 - Any second-order electrical circuit consisting only of inductors and capacitors is stable and any consisting only of resistors, inductors, and capacitors is stable and damped.

5.2 Fundamental Solutions

- Consider the general homogeneous linear second-order ODE:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0, \quad (44)$$

with initial conditions

$$\begin{aligned} y(0) &= y_0, \\ \frac{dy}{dt}(0) &= \dot{y}_0. \end{aligned} \quad (45)$$

- Solutions $y_1(t)$ and $y_2(t)$ to Equation 5.48 are called fundamental solutions if any solution to Equation (44) can be written as

$$y(t) = c_1y_1(t) + c_2y_2(t). \quad (46)$$

- What makes a solution fundamental?

- It is easy to prove that a function in the form Equation (46) satisfies Equation (44) but that is insufficient.

5.2 Fundamental Solutions

- A function in the form Equation (46) must also be able to satisfy the initial conditions, Equation (45).
- That is, we must be able to find coefficients c_1 and c_2 satisfying

$$\begin{aligned}c_1 y_1(0) + c_2 y_2(0) &= y_0, \\c_1 \dot{y}_1(0) + c_2 \dot{y}_2(0) &= \dot{y}_0.\end{aligned}\tag{47}$$

- From algebra we recall that we will be able to do this provided the coefficients of c_1 and c_2 have a nonzero determinant:

$$\det \begin{pmatrix} y_1(0) & y_2(0) \\ \dot{y}_1(0) & \dot{y}_2(0) \end{pmatrix} = y_1(0)\dot{y}_2(0) - \dot{y}_2(0)y_1(0) \neq 0.\tag{48}$$

- Therefore, functions $y_1(t)$ and $y_2(t)$ are ***fundamental solutions*** provided that they satisfy Equations (44) and (48).

5.2 Fundamental Solutions

- The determinant

$$W(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{pmatrix} = y_1(t)\dot{y}_2(t) - \dot{y}_2(t)\dot{y}_1(t), \quad (49)$$

is called the ***Wronskian***.

- Word Problem 5.7 seeks to show that the Wronskian satisfies the differential equation

$$\frac{dW}{dt} + p(t)W = 0, \quad (50)$$

which we know from results of Chapter 2 has solution

$$W(t) = c \exp \left(- \int_0^t p(\tau) d\tau \right), \quad (51)$$

where c is a constant.

5.2 Fundamental Solutions

- From Equation (51) it follows that if $W(0) \neq 0$ then $W(t) \neq 0$ for all t .
- This is an important result because it implies that if $y_1(t)$ and $y_2(t)$ are found to be fundamental solutions at a specific point in time then they are fundamental solutions for all time.
- The Wronskian also proves important in the study of nonhomogeneous linear ODEs, as Section 5.3.2 will discuss.

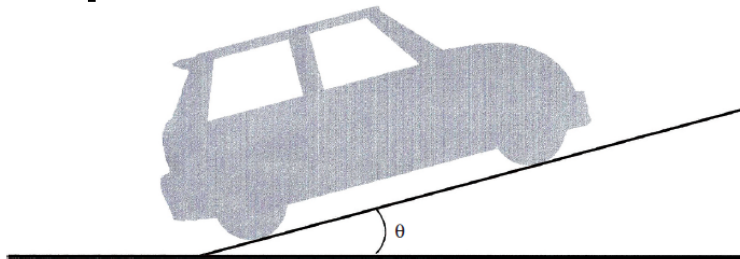
5.3 Second-Order Linear Nonhomogeneous Ordinary Differential Equations

- A nonhomogeneous second-order linear ODE has the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t), \quad (52)$$

where $g(t)$ is nonzero.

- **Example: Automobile Cruise Control**



- Fig. 5.4 Schematic for automobile cruise control

5.3.1 Example: Automobile Cruise Control

- Consider an automobile under cruise (speed)-control transitioning from a level roadway to an ascending grade (See Fig.5.4).
- We seek to derive the equations governing this process to demonstrate an important application but also to illustrate the technique of **linearization** so vital to engineers.
- As discussed in Section 3.1.1.2, Newtons second law of motion is

$$F = ma,$$

where: F is the total force on the vehicle, m is its mass, and a is its acceleration.

- We assume that the vehicle is in steady state (i.e., at a constant speed V_0) on the level roadway before transitioning to the ascending grade at time $t = 0$.

5.3.1 Example: Automobile Cruise Control

- Prior to $t = 0$, on an axis parallel to the level roadway, the cars constant engine thrust T_0 balances the air resistance (drag) D_0 and horizontal road resistance R_0 , which gives to:

$$\begin{aligned} T_0 &= D_0 + R_0 \\ T_0 &= \frac{\rho S C_D}{2} V_0^2 + R_0, \end{aligned} \tag{53}$$

where, ρ is the air density, S the vehicles cross-sectional area, and C_D is its drag coefficient, assumed constant.

- When the car transitions to the ascending grade, in an axis parallel to the ascending grade a component of the cars weight is added to the forces on the car.
- Then the equation of motion on that axis becomes

$$m \frac{dV}{dt} = T - \frac{\rho S C_D}{2} V^2 - R - mg \sin \theta. \tag{54}$$

5.3.1 Example: Automobile Cruise Control

- Now we assume that $R = R_0$ and let

$$\begin{aligned} V &= V_0 + u, \\ T &= T_0 + \Delta T. \end{aligned} \tag{55}$$

- From Equation (55),

$$\frac{dV}{dt} = \frac{du}{dt}. \tag{56}$$

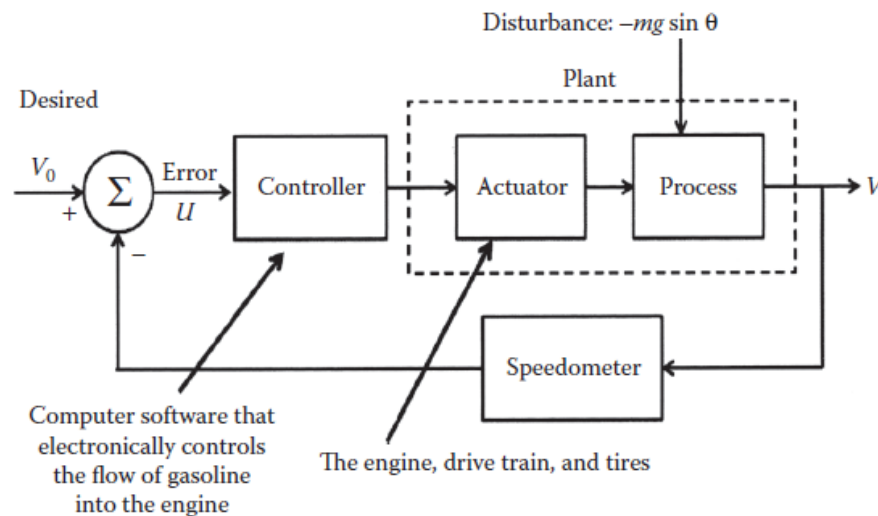
- We assume that u is small relative to V_0 , so that we can approximate

$$V^2 = V_0^2 + 2V_0u + u^2 \cong V_0^2 + 2V_0u. \tag{57}$$

- That is, we **linearize** Equation (54).
- We are examining perturbations about the nominal condition.
- Our approximations will be valid as long as **the perturbations remain small**, which will be the case if **the angle θ of the grade is not too large and cruise control is effective**.

5.3.1 Example: Automobile Cruise Control

- What is the change in thrust ΔT ?
- We assume that the cruise-control system is based on feedback as shown in Fig. 5.5.



● Fig. 5.5 Block diagram for feedback control of vehicle speed.

- that is, that a change in thrust results when the speedometer measures that the speed deviates from that desired.

5.3.1 Example: Automobile Cruise Control

- In Fig. 5.5 we model the response of 'Controller + Actuator' as

$$\Delta T = -k_1 u - k_2 \int^t u(\tau) d\tau. \quad (58)$$

- In doing so we are assuming a *proportional plus integral control law* and that the time required for the vehicles engine to deliver increased horsepower in response to the controller's command for added gasoline flow is small.
- A realistic control-system design would **take into account engine dynamics and be digitally implemented in the form of difference equations.**
- We are presenting here an introductory, top-level view.

5.3.1 Example: Automobile Cruise Control

- Combining Equations (54) through (58) results in our model for the 'Process' in Fig. 5.5:

$$m \frac{du}{dt} = (T_0 - \frac{\rho S C_D}{2} V_0^2 - R_0) - (\rho S C_D V_0 + k_1)u - k_2 \int^t u(\tau) d\tau - mg \sin \theta. \quad (59)$$

- From Equation (53) the first term in parentheses on the right-hand side equals zero, so Equation (59) becomes

$$m \frac{du}{dt} = -(\rho S C_D V_0 + k_1)u - k_2 \int^t u(\tau) d\tau - mg \sin \theta. \quad (60)$$

- Defining a new variable $y(t) = \int^t u(\tau) d\tau$ and rearranging Equation (60), then we obtain that

5.3.1 Example: Automobile Cruise Control

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = -g \sin \theta, \quad (61)$$

where:

$$\begin{aligned} a &= (\rho S C_D V_0 + k_1)/m, \\ b &= k_2/m. \end{aligned} \quad (62)$$

- With knowledge of vehicle characteristics and density of air the designer of the vehicles cruise-control system can achieve a desired response by **scheduling gains** k_1 and k_2 as a function of the speed V_0 set by the driver.
- One would expect the control system designer to seek a rapid, smooth transition between power states:
 - rapid so as not to cause traffic problems,
 - smooth (without overshoot and oscillation) so as not to cause driver and passenger discomfort.

5.3.1 Example: Automobile Cruise Control

- Consistent with these design goals, we assume here that the designer selects a ***critically damped*** response (recall the discussion in Section 5.1.4) with a given time constant, which can be accomplished by setting

$$\begin{aligned}a &= 2\alpha, \\ b &= \alpha^2.\end{aligned}\tag{63}$$

- From Equations (61) and (63), the equation of motion of the vehicle under cruise control is

$$\frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + \alpha^2y = -g \sin \theta.\tag{64}$$

- It follows from Section 5.1.2 that the fundamental solutions to the homogeneous form of Equation (64) are

$$\begin{aligned}y_1(t) &= e^{-\alpha t}, \\ y_2(t) &= te^{-\alpha t}.\end{aligned}\tag{65}$$

5.3.1 Example: Automobile Cruise Control

- Section 5.3.2 will prove that the solution to Equation (64) can be written as the convolution

$$y(t) = \int_0^t K(t - \tau)(-g \sin \theta) d\tau. \quad (66)$$

where the kernel $K(t - \tau)$ is given in this case by

$$K(t - \tau) = (t - \tau)e^{-\alpha(t-\tau)}. \quad (67)$$

- Substituting Equation (67) into Equation (66), we find

$$y(t) = \int_0^t (t - \tau)e^{-\alpha(t-\tau)}(-g \sin \theta) d\tau = -g \sin \theta \cdot e^{-\alpha t} \left(t \int_0^t e^{\alpha\tau} d\tau - \int_0^t \tau e^{\alpha\tau} d\tau \right). \quad (68)$$

- Now

$$\int_0^t e^{\alpha\tau} d\tau = \frac{1}{\alpha}(e^{\alpha t} - 1).$$

5.3.1 Example: Automobile Cruise Control

- Using integration by parts, we have

$$\int_0^t \tau e^{\alpha\tau} d\tau = \int u dv = uv - \int v du.$$

- Let $u = \tau$, which means that $du = d\tau$, $dv = e^{\alpha\tau} d\tau$, $v = e^{\alpha\tau}/\alpha$ and so

$$\int_0^t \tau e^{\alpha\tau} d\tau = \frac{1}{\alpha} \tau e^{\alpha\tau} \Big|_0^t - \frac{1}{\alpha} \int_0^t e^{\alpha\tau} d\tau = \frac{1}{\alpha^2} (\alpha t e^{\alpha\tau} - e^{\alpha\tau} + 1).$$

- Then Equation (68) becomes

$$\begin{aligned} y(t) &= -g \sin \theta \cdot e^{-\alpha t} (t \int_0^t e^{\alpha\tau} d\tau - \int_0^t \tau e^{\alpha\tau} d\tau) \\ &= -g \sin \theta [t e^{-\alpha t} (e^{\alpha t} - 1)/\alpha - e^{-\alpha t} (\alpha t e^{\alpha\tau} - e^{\alpha\tau} + 1)/\alpha^2] \end{aligned}$$

$$y(t) = -g \sin \theta \cdot (1 - (1 + \alpha t)e^{-\alpha t})/\alpha^2. \quad (69)$$

5.3.1 Example: Automobile Cruise Control

- Recall now that the speed error $u(t) = dy/dt$:

$$u(t) = -gt \sin \theta \cdot e^{-\alpha t}. \quad (70)$$

- Figure 5.6a graphs this result for an incline of 5 degree and a time constant $1/\alpha$ as a parameter.

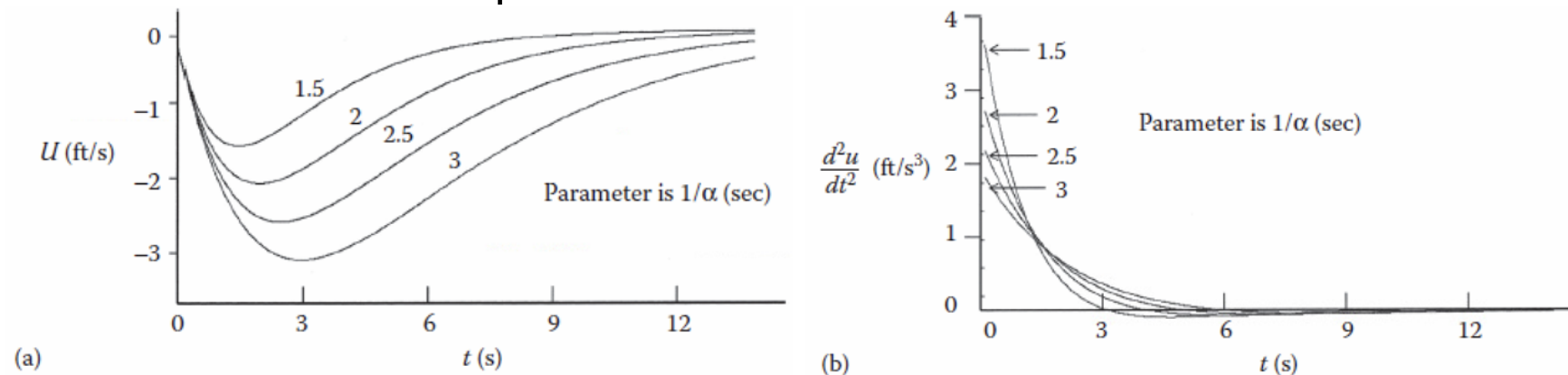


Fig. 5.6 Speed error and 'jerk' due to an abrupt 5 degree change in grade for a vehicle under cruise control: (a) speed error and (b) derivative of acceleration (a measure of rider discomfort.)

5.3.1 Example: Automobile Cruise Control

- Under the same conditions, Figure 5.6b graphs the second derivative of the speed error (the derivative of the acceleration), which is frequently used as a measure of rider discomfort. It is often called 'jerk'.
 - The lower the value of the time constant the smaller the speed error, but the greater the jerk.
 - The control system engineer must make a decision on the design value of α balancing speed error with comfort of the ride.
- In summary, an automobile in cruise control mode transitioning from a level roadway to an ascending grade experiences a speed error given by Equation (70) if it utilizes a proportional plus integral control law set for critically damped response.
- Note that the steady-state error is zero. It is left as an exercise for the reader to show that the steady-state error is not zero if the control law is merely proportional (i.e., if $k_2 = 0$ in Equation (58)).

5.3.2 Variation of Parameters (Kernel) Method

■ **Theorem:**

If $y_1(t)$ and $y_2(t)$ are fundamental solutions to

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0, \quad (71)$$

then the solution to

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t), \quad (72)$$

$$\begin{aligned} y(t_0) &= 0, \\ \dot{y}(t_0) &= 0. \end{aligned} \quad (73)$$

is

$$y(t) = \int_{t_0}^t K(t, \tau)g(\tau)d\tau, \quad (74)$$

5.3.2 Variation of Parameters (Kernel) Method

■ where:

$$K(t, \tau) = (-y_1(t)y_2(\tau) + y_2(t)y_1(\tau))/W(\tau), \quad (75)$$

and

$$W(\tau) = y_1(\tau)\dot{y}_2(\tau) - y_2(\tau)\dot{y}_1(\tau) \quad (76)$$

is the Wronskian of $y_1(\tau)$ and $y_2(\tau)$.

■ *Proof:*

■ We try a solution in the form

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t). \quad (77)$$

Then

$$\dot{y} = \dot{u}_1 y_1 + u_1 \dot{y}_1 + \dot{u}_2 y_2 + u_2 \dot{y}_2. \quad (78)$$

5.3.2 Variation of Parameters (Kernel) Method

■ *Proof:*

■ We set

$$\dot{u}_1 y_1 + \dot{u}_2 y_2 = 0, \quad (79)$$

and are left with

$$\dot{y} = u_1 \dot{y}_1 + u_2 \dot{y}_2. \quad (80)$$

Then

$$\ddot{y} = \dot{u}_1 \dot{y}_1 + u_1 \ddot{y}_1 + \dot{u}_2 \dot{y}_2 + u_2 \ddot{y}_2. \quad (81)$$

■ Substituting equations for $y(t)$ (Equation (77)), $\dot{y}(t)$ (Equation (80)) and $\ddot{y}(t)$ (Equation (81)) into the ODE, it gives

$$\frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = g(t). \quad (82)$$

■ We have, after some rearranging,

$$u_1[\ddot{y}_1 + p(t)\dot{y}_1 + q(t)y_1] + u_2[\ddot{y}_2 + p(t)\dot{y}_2 + q(t)y_2] + \dot{u}_1\dot{y}_1 + \dot{u}_2\dot{y}_2 = g(t). \quad (83)$$

5.3.2 Variation of Parameters (Kernel) Method

■ *Proof:*

■ Now

$$\begin{aligned}\ddot{y}_1 + p(t)\dot{y}_1 + q(t)y_1 &= 0, \\ \ddot{y}_2 + p(t)\dot{y}_2 + q(t)y_2 &= 0.\end{aligned}\tag{84}$$

so

$$\dot{u}_1\dot{y}_1 + \dot{u}_2\dot{y}_2 = g(t).\tag{85}$$

■ Equations (79) and (85) constitute two linear equations in two unknowns, and their solution is

$$\dot{u}_1 = -y_2g/W,\tag{86}$$

$$\dot{u}_2 = y_1g/W,\tag{87}$$

where

$$W = y_1\dot{y}_2 - y_2\dot{y}_1.\tag{88}$$

5.3.2 Variation of Parameters (Kernel) Method

■ *Proof:*

■ Integrating Equations (86) and (87) we have

$$\begin{aligned} u_1(t) &= - \int_{t_0}^t \frac{y_2(\tau)g(\tau)}{W(\tau)} d\tau, \\ u_2(t) &= \int_{t_0}^t \frac{y_1(\tau)g(\tau)}{W(\tau)} d\tau. \end{aligned} \tag{89}$$

■ Incorporating Equation (89) into Equation (77) results in

$$y(t) = -y_1(t) \int_{t_0}^t \frac{y_2(\tau)g(\tau)}{W(\tau)} d\tau + y_2(t) \int_{t_0}^t \frac{y_1(\tau)g(\tau)}{W(\tau)} d\tau. \tag{90}$$

■ Defining

$$K(t, \tau) = (-y_1(t)y_2(\tau) + y_2(t)y_1(\tau))/W(\tau). \tag{91}$$

5.3.2 Variation of Parameters (Kernel) Method

■ *Proof:*

■ We then arrive at

$$y(t) = \int_{t_0}^t K(t, \tau) g(\tau) d\tau. \quad (92)$$

■ It remains to show that $y(t)$ defined by Equation (92) satisfies the initial conditions of Equation (73):

$$\begin{aligned} y(t_0) &= 0, \\ \dot{y}(t_0) &= 0. \end{aligned}$$

It is clear that

$$y(t_0) = \int_{t_0}^{t_0} K(t_0, \tau) g(\tau) d\tau = 0.$$

■ Now, by Leibniz rule:

$$\dot{y}(t) = K(t, t)g(t) + \int_{t_0}^t \frac{\partial K}{\partial t}(t, \tau)g(\tau) d\tau. \quad (93)$$

5.3.2 Variation of Parameters (Kernel) Method

■ *Proof:*

■ By its definition (Equation (91)), $K(t, t) = 0$, so Equation (93) becomes

$$\dot{y}(t) = \int_{t_0}^t \frac{\partial K}{\partial t}(t, \tau) g(\tau) d\tau, \quad (94)$$

■ Then we have

$$\dot{y}(t_0) = \int_{t_0}^{t_0} \frac{\partial K}{\partial t}(t_0, \tau) g(\tau) d\tau = 0, \quad (95)$$

and so $y(t)$ as defined by Equations (92), (91), and (88) satisfies both the ODE Equation (72) and initial conditions Equation (73), as was to be shown.

5.3.2 Variation of Parameters (Kernel) Method

- For LTI nonhomogeneous second-order ODEs Equation (92) becomes the convolution

$$y(t) = \int_{t_0}^t K(t - \tau)g(\tau)d\tau.$$

- When a linear nonhomogeneous ODE has nonzero initial conditions its solution will be of the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t), \quad (96)$$

where $y_p(t)$ is the particular solution (or, recalling Section 2.2.3, the forced response) given by Equations (88), (91), and (92) and the coefficients c_1 and c_2 are determined by the initial conditions.

- Equation (96) follows from the ODEs linearity.

5.3.2 Variation of Parameters (Kernel) Method

- The variation of parameters method is the most powerful of the three methods.
 - We will examine for the solution of second- and higher-order linear nonhomogeneous ODEs.
 - It works for time-varying equations with continuous coefficients and for all continuous inputs $g(t)$, whereas the other methods do not.
 - Its disadvantage compared to the other methods is the time and energy required to execute it.
- We will meet the second solution method in the following Section 5.3.3 and the third in Chapter 7.

5.3.3 Undetermined Coefficients Method

- The simplest of the three approaches we will meet for solving **LTI nonhomogeneous ODEs**, but the most limited in its power, is the **undetermined coefficients method**.
- In this method, we let the form of the input determine a trial solution.

- Table 5.1 lists solutions to try, given a specific input function.

TABLE 5.1 Trial Solutions for Given Input Functions Using the Undetermined Coefficients Method	
Input	Trial Solution
e^{-kt}	Ce^{-kt}
t	$Ct + D$
te^{-kt}	$(Ct + D)e^{-kt}$
t^2	$Ct^2 + Dt + E$
$\sin \omega t$	$C \sin \omega t + D \cos \omega t$
$e^{-kt} \sin \omega t$	$(C \sin \omega t + D \cos \omega t)e^{-kt}$
$t \sin \omega t$	$(Ct + D) \sin \omega t + (Et + F) \cos \omega t$
$te^{-kt} \sin \omega t$	$((Ct + D) \sin \omega t + (Et + F) \cos \omega t)e^{-kt}$
$t^2 e^{-kt} \sin \omega t$	$((Ct^2 + Dt + E) \sin \omega t + (Ft^2 + Gt + H) \cos \omega t)e^{-kt}$

- We substitute the trial solution into the ODE and solve for the undetermined coefficients.

5.3.3.1 Steady-State Solutions

- Recall the definition of steady-state solution in Section 2.2.3: **in a stable, damped system, the steady-state solution remains when transients have died away.**
- The undetermined coefficient method is especially attractive when we seek only the steady-state solution, because we are then spared the effort of factoring the characteristic equation and determining fundamental solutions.
- We must **determine that the system is stable and damped**, but recalling Section 5.1.5 we can do this for second-order ODEs by examining whether the ODEs coefficients are greater than zero.
- **Example:**
 - Problem: Find the steady-state solution to the ODE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 12t. \quad (97)$$

5.3.3.1 Steady-State Solutions

- *Solution:* The ODEs coefficients are greater than zero, so the system is stable and damped.
- Then, following Table 5.1, we try the solution

$$y(t) = Ct + D, \quad (98)$$

and solve for C and D .

- Now, from Equation (98),

$$\frac{dy}{dt} = C, \quad (99)$$

and

$$\frac{d^2y}{dt^2} = 0. \quad (100)$$

- Substituting Equations (98), (99), and (100) into Equation (97) we have

$$0 + 3C + 2(Ct + D) = 12t. \quad (101)$$

5.3.3.1 Steady-State Solutions

■ *Solution:*

- Collecting coefficients of powers of t , it gives

$$(3C + 2D) + (2C)t = 12t. \quad (102)$$

Equating coefficients of powers of t ,

$$\begin{aligned} 3C + 2D &= 0, \\ 2C &= 12. \end{aligned} \quad (103)$$

- Equation (103) are two simultaneous linear equations in two unknowns. The solutions are

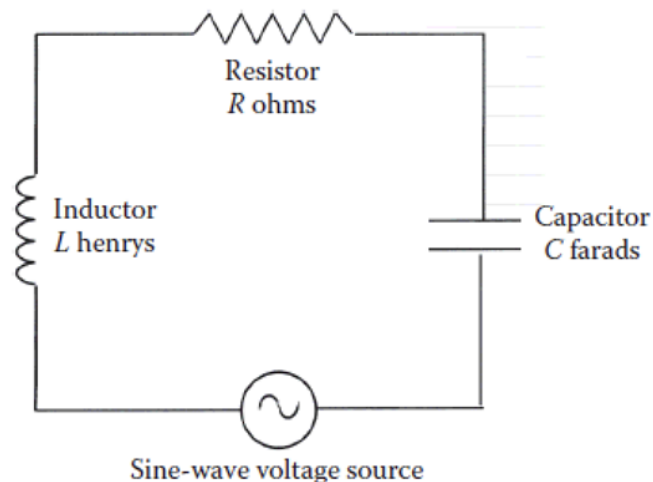
$$\begin{aligned} C &= 6, \\ D &= -3C/2 = -9, \end{aligned}$$

and the steady-state solution to Equation (97) is

$$y(t) = 6t - 9.$$

5.3.3.2 Example: RLC Circuit with Sine-Wave Source

- Consider the electrical circuit shown in Fig. 5.7:



- Fig. 5.7 Second-order RLC electrical circuit with sine-wave voltage source.

- We are interested in the steady-state voltage $V(t)$ over the capacitor when the voltage source has the form $V_0 \sin \omega t$. Following the results in Section 5.1 we can write the circuit equation as

$$\frac{d^2V}{dt^2} + \frac{R}{L} \frac{dV}{dt} + \frac{1}{LC} V = \frac{1}{LC} V_0 \sin \omega t. \quad (104)$$

5.3.3.2 Example: RLC Circuit with Sine-Wave Source

- For simplicity let

$$\begin{aligned} a &= \frac{R}{L}, \\ b &= \frac{1}{LC}. \end{aligned} \tag{105}$$

- Then Equation (104) becomes

$$\frac{d^2V}{dt^2} + a\frac{dV}{dt} + bV = bV_0 \sin \omega t. \tag{106}$$

- Using Table 5.1 we try the solution

$$V(t) = C \sin \omega t + D \cos \omega t. \tag{107}$$

5.3.3.2 Example: RLC Circuit with Sine-Wave Source

- We are then led to

$$\frac{dV}{dt} = \omega C \cos \omega t - \omega D \sin \omega t, \quad (108)$$

$$\frac{d^2V}{dt^2} = -\omega^2 C \sin \omega t - \omega^2 D \cos \omega t. \quad (109)$$

- Substituting Equations (107), (108), and (109) into Equation 5.110 we have, after some rearranging,

$$[(b - \omega^2)C - a\omega D] \sin \omega t + [a\omega C + (b - \omega^2)D] \cos \omega t = bV_0 \sin \omega t. \quad (110)$$

- Equation (110) must be true for all t .
- When $\omega t = 2n\pi$, $\sin \omega t = 0$, $\cos \omega t = 1$ and we see that Equation (110) implies

$$a\omega C + (b - \omega^2)D = 0. \quad (111)$$

5.3.3.2 Example: RLC Circuit with Sine-Wave Source

- When $\omega t = 2n\pi + \pi/2$, $\sin \omega t = 1$, and $\cos \omega t = 0$, in which case Equation (110) implies that

$$(b - \omega^2)C - a\omega D = bV_0. \quad (112)$$

- Equations (111) and (112) are two simultaneous linear equations in the two unknowns C and D .
- Their solutions are

$$\begin{aligned} C &= \frac{b - \omega^2}{(b - \omega^2)^2 + (a\omega)^2} bV_0, \\ D &= \frac{-a\omega}{(b - \omega^2)^2 + (a\omega)^2} bV_0. \end{aligned} \quad (113)$$

- We can simplify the form of the answer. Following Section 2.2.3, instead of writing

$$V(t) = C \sin \omega t + D \cos \omega t.$$

5.3.3.2 Example: RLC Circuit with Sine-Wave Source

■ We can define

$$\begin{aligned}\cos \theta &= \frac{C}{\sqrt{C^2 + D^2}}, \\ \sin \theta &= \frac{-D}{\sqrt{C^2 + D^2}},\end{aligned}$$

and write

$$\begin{aligned}V(t) &= \sqrt{C^2 + D^2}(\cos \theta \sin \omega t - \sin \theta \cos \omega t), \\ V(t) &= \sqrt{C^2 + D^2} \sin(\omega t - \theta).\end{aligned}\tag{114}$$

■ The steady-state solution to Equation (106), then, is

$$V(t) = GV_0 \sin(\omega t - \theta),\tag{115}$$

where

$$G = \frac{b}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}},\tag{116}$$

$$\theta = \tan^{-1}(a\omega, b - \omega^2).\tag{117}$$

5.3.3.2 Example: RLC Circuit with Sine-Wave Source

- As discussed in Section 2.2.3 for a water pipe temperature problem, Equations (116) and (117) provide the **gain** and **phase lag** of the frequency response of the RLC circuit in Fig. 5.7.
- Note that we have derived these results without knowing the magnitude of $a = R/L$ and $b = 1/LC$.
- We have neither factored the ODEs characteristic equation nor determined the ODEs fundamental solutions.

5.3.3.3 Resonance with a Sine-Wave Source

- Let us examine the behavior of the gain G in Equation (116) as a function of the input frequency ω .
 - Recall that Section 2.2.3 determined that the gain of the frequency response for a first-order LTI system decreases monotonically with input frequency.
 - We examine here the possibility that, for second-order LTI systems, a certain input frequency may maximize the gain.
- The gain is maximized when the denominator

$$F(\omega) = (b - \omega^2)^2 + (a\omega)^2 \quad (118)$$

is minimized. We let $x = \omega^2$ and expand Equation (118):

$$F(x) = b^2 - 2bx + x^2 + a^2x = b^2 + (a^2 - 2b)x + x^2.$$

5.3.3.3 Resonance with a Sine-Wave Source

- To minimize $F(x)$ we set its derivative equal to zero:

$$\frac{dF}{dx} = (a^2 - 2b) + 2x = 0.$$

resulting in

$$x = b - a^2/2 > 0. \quad (119)$$

- Recalling that $x = \omega^2$ we see that we must have

$$x = b - a^2/2 > 0$$

to have a real input frequency maximizing the gain.

5.3.3.3 Resonance with a Sine-Wave Source

- Looking back at Section 5.1.3 we see from Equations (40) and (41) that the gain has a maximum only when the system is underdamped.
- For this case, when $\omega_m = \sqrt{b - a^2/2}$ is real, the maximum gain is given by

$$G_m = \frac{b}{\sqrt{(b - \omega_m^2)^2 + (a\omega_m)^2}} = \frac{b}{\sqrt{(a^2/2)^2 + a^2(b - a^2/2)^2}},$$
$$G_m = \frac{b}{a\sqrt{b - a^2/4}}. \quad (120)$$

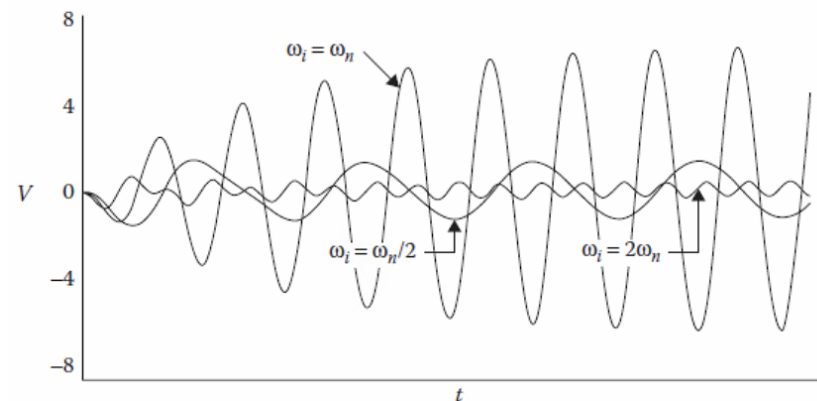
- When $a^2/4 \ll b$ Equation (120) becomes

$$G_m = \frac{\sqrt{b}}{a} \gg \frac{1}{2} \quad (121)$$

and we see that the gain can become very large.

5.3.3.3 Resonance with a Sine-Wave Source

- That is, when a system is only lightly damped, an input frequency near the natural frequency of the system results in a very large magnitude of the output, a phenomenon known as **resonance**.



- Fig. 5.8 Output of an example underdamped second-order system of natural frequency ω_n when excited by sine-wave inputs of unit magnitude but different frequencies ω_n .

5.3.3.3 Resonance with a Sine-Wave Source

- Fig. 5.8 presents the output of the underdamped second-order system whose natural response is shown in Fig. 5.2 when that system is excited by three sine-wave inputs, each with unit magnitude but with different frequencies.
- When the input frequency ω_i equals the systems natural frequency ω_n the magnitude of the output in steady state is much greater than it is when the input frequency is a factor of two greater or less than the natural frequency.
- In summary, when driven by a sine-wave source, an underdamped second-order system can experience a sharp rise in its gain when the input frequency approaches its natural frequency, in contrast to the monotonically decreasing gain in the frequency response of a first-order system.

5.3.3.4 Resonance with a Periodic Input Other than a Sine Wave

- The quantitative results of Sections 5.3.3.2 and 5.3.3.3 apply to a sine-wave input, but the qualitative conclusions apply much more generally.
- We will see in Chapter 7 that:
 - the steady-state response of a stable, damped n th order LTI system to any periodic input is periodic;
 - and that an underdamped system can experience resonance when the input frequency approaches the systems natural frequency, independent of the inputs waveform.

5.3.3.5 Satisfying Initial Conditions Using the Undetermined Coefficients Method

- Suppose that the values of the solution $y(t)$ and its derivative $\dot{y}(t)$ are specified at time t_0 as y_0 and $\dot{y}_0$.
 - Let us call the solution we derive using the undetermined coefficients method $y_C(t)$.
 - In contrast to the particular solution generated by the variation of parameters method, which has zero initial value and zero initial derivative, typically $y_C(t_0) \neq 0$, and $\dot{y}_C(t_0) \neq 0$.
- To satisfy given initial conditions we must find the ODEs fundamental solutions $y_1(t)$ and $y_2(t)$ and then choose coefficients c_1 and c_2 to satisfy

$$\begin{aligned}y_0 &= y_C(t_0) + c_1 y_1(t_0) + c_2 y_2(t_0), \\ \dot{y}_0 &= \dot{y}_C(t_0) + c_1 \dot{y}_1(t_0) + c_2 \dot{y}_2(t_0).\end{aligned}$$

5.3.3.6 Limitations of the Undetermined Coefficients Method

- Note that the undetermined coefficients method applies only to LTI ODEs.
 - It also applies only to a certain class of input functions, namely those that are themselves fundamental solutions to the class of LTI ODEs.
 - One cannot successfully apply the undetermined coefficients method when the input function is, for example, $\ln(1 + t)$ or $1/(1 + t)$ or $\tan \omega t$.
- Finally, the undetermined coefficients method is, in practice, a trial and error method.
 - Even if one recalls Table 5.1 faithfully (e.g., try $Ct + D$, not Ct , when the input is a multiple of t) the trial solutions in Table 5.1 fail when the input is itself a fundamental solution of the particular ODE.
 - For second-order equations the solution will then involve either t or t^2 times the trial solution suggested in Table 5.1.

5.4 Existence and Uniqueness of Solutions to Second-Order Linear Ordinary Differential Equations

■ **Theorem:**

Consider the second-order linear ODE

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t). \quad (122)$$

Suppose that the values of $y(t)$ and dy/dt are specified to be y_0 and $\dot{y}_0$ at time t_0 . If the functions $p(t)$, $q(t)$, and $g(t)$ are continuous on an open interval I that contains the point t_0 then a solution to Equation (122) exists throughout the interval I and the solution is unique.

■ **Corollary:**

If the functions $p(t)$, $q(t)$, and $g(t)$ are continuous everywhere then Equation (122) has a unique solution at every instant of time. In particular, LTI equations have unique solutions at every instant of time.

End