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# EE1001 Counting Part II

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# Intended Learning Outcomes

- Upon completion of this session, you will be able to:
  - ✓1 Understanding some basic concepts in set theory
  - ✓2. Solving problems with the inclusion-exclusion principle
  - ✓3. Solving problems with the pigeon-hole principle

# Chapter Contents

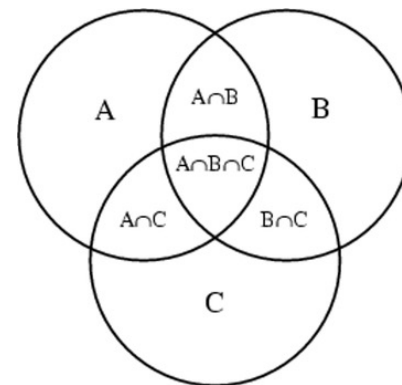
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- **Basic Set Theory**
- **The inclusion-exclusion principle**
- **The pigeon-hole principle**

# Why we study Set Theory

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- **the common language to speak about mathematics, so learning set theory means learning the common language.**
- Studying set theory, even naively, is the technical spine of how to handle infinite sets. Since modern mathematics is concerned with many infinite sets, larger and smaller, it is a good idea to learn about infinite sets if one wishes to understand mathematical objects better.
- We can easily find the relationship between different things or sets with set theory.



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# 1. Basic Set Theory

# Basic Definitions

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- **Set:** a set is an unordered collection of objects. For sets, we'll use variables  $S, T, U, \dots$
- **Element:** The objects used to form a set are called its element. For elements, we'll use variables  $a, b, c \dots$
- There are ways of describing a set
  - **Explicitly:** listing the elements of a set

e.g.  $\{a, b, c\}, \{1, 2, 3\}, \{\text{Woody, Joe, Paul}\}$  are all finite sets  
 $\{1, 2, 3, \dots\}$  is a way we denote an infinite set
  - **Set builder notation:** For any proposition  $P(x)$  over any universe of discourse,  $\{x|P(x)\}$  is the set of all  $x$  such that  $P(x)$ .

e.g.,  $\{x \mid x \text{ is an integer where } x > 0 \text{ and } x < 5\}$   
or  $\{x : x \text{ is an integer where } x > 0 \text{ and } x < 5\}$

# Properties of sets

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- Sets are inherently ***unordered***.

$$\{a, b, c\} = \{a, c, b\} = \{b, a, c\} = \{b, c, a\} = \{c, a, b\} = \{c, b, a\}$$

- All elements are ***distinct (unequal)***: multiple listings make no difference!

$$\{a, b, c\} = \{a, a, b, a, b, c, c, c, c\}.$$

- Two sets are declared to be equal ***if and only if*** they contain exactly the same elements.

The set  $\{1, 2, 3, 4\} =$

$$\{x \mid x \text{ is an integer where } x > 0 \text{ and } x < 5\} =$$

$$\{x \mid x \text{ is a positive integer whose square is } > 0 \text{ and } < 25\}$$

# Infinite Sets

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- Conceptually, sets may be *infinite* (i.e., not *finite*, without end, unending).
- Symbols for some special infinite sets:
  - $\mathbf{N} = \{0, 1, 2, \dots\}$  The natural numbers.
  - $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$  The integers.
  - $\mathbf{R}$  = The “real” numbers, such as 20, 2.14,  $\pi$
  - $\mathbf{Z}^+ = \{1, 2, 3, \dots\}$  set of positive integers

Note: Real number are the numbers that can be represented by an infinite decimal representation, including both **rational**, and **irrational** numbers such as  $\pi$  that can be represented as points along an infinitely long number line.

# Basic Set Relations

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- $x \in S$  (“ $x$  is in  $S$ ”) is the proposition that object  $x$  is an *element* or *member* of set  $S$ .
- $x \notin S$  : “ $x$  is not in  $S$ ”
- $\emptyset$  (“null”, “the empty set”) is the unique set that contains no elements whatsoever.
- $\emptyset = \{\} = \{x | \mathbf{False}\}$
- $\emptyset \neq \{\emptyset\}$ . Since the right term is a set with an element  $\emptyset$ .
- $S \subseteq T$  (“ $S$  is a *subset* of  $T$ ”) means that every element of  $S$  is also an element of  $T$ .
- $\emptyset$  is a subset of any set.  $\emptyset \subseteq S$

# Sets Are Objects

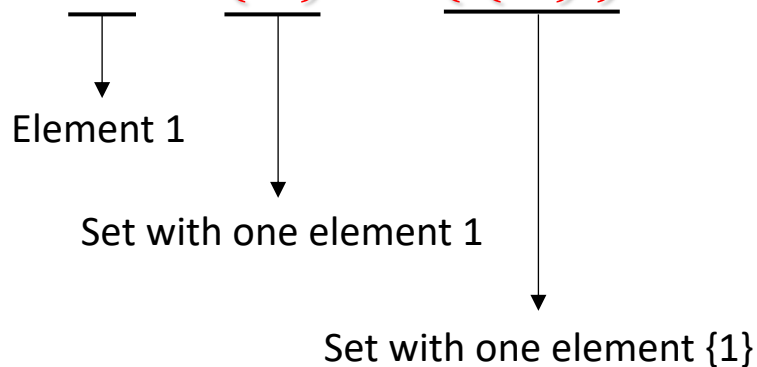
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- The objects that are elements of a set may themselves be sets.

let  $S = \{x \mid x \subseteq \{1,2,3\}\}$

then  $S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

■ Note that  $1 \neq \{1\} \neq \{\{1\}\} !!!!$



# Operation Between Set: Union

- For sets  $A$ ,  $B$ , their *union*  $A \cup B$  is the set containing all elements that are either in  $A$ , **or** (“ $\vee$ ”) in  $B$  (or, of course, in both).

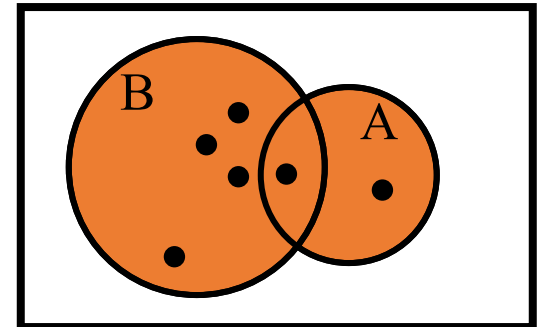
- $$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

- Note that  $A \cup B$  contains all the elements of  $A$  and it contains all the elements of  $B$ :

- $$(A \cup B \supseteq A) \wedge (A \cup B \supseteq B)$$

e.g.

- $\{a,b,c\} \cup \{2,3\} = \{a,b,c,2,3\}$
- $\{2,3,5\} \cup \{3,5,7\} = \{2,3,5,3,5,7\} = \{2,3,5,7\}$



$A \cup B$

# Operation Between Set: Intersection

- For sets  $A$ ,  $B$ , their intersection  $A \cap B$  is the set containing all elements that are simultaneously in  $A$  and (“ $\wedge$ ”) in  $B$ .

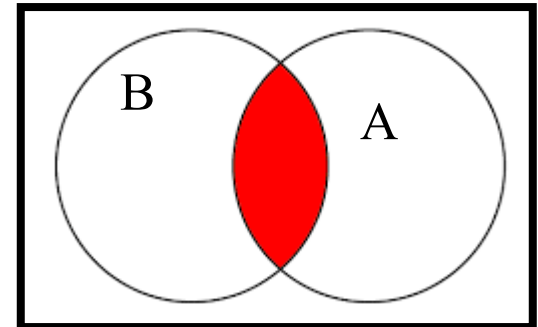
- $A \cap B \equiv \{x \mid x \in A \wedge x \in B\}$

- Note that that  $A \cap B$  is a subset of  $A$  **and** it is a subset of  $B$ :

- $(A \cap B \subseteq A) \wedge (A \cap B \subseteq B)$

e.g.

- $\{a,b,c\} \cap \{2,3\} = \emptyset$
- $\{2,4,6\} \cap \{3,4,5\} = \{4\}$



$A \cap B$

# Operation Between Set: Intersection

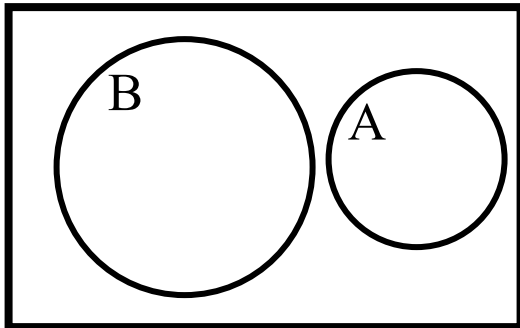
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■ Considering  $A = \{x : x \text{ is a US president}\};$

$B = \{x : x \text{ is in this room}\}$

What is  $A \cap B$  ?

$$A \cap B = \{x : x \text{ is a US president in this room}\} = \emptyset$$



Sets whose  
intersection is  
empty are called  
*disjoint sets*

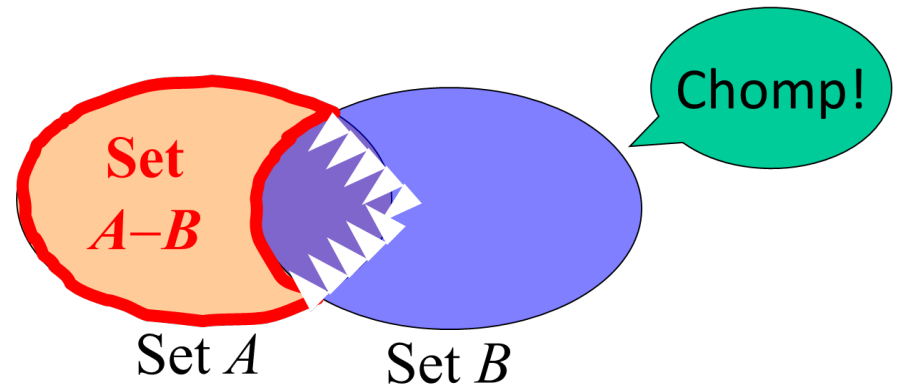
# Operation Between Set: Complement

- Complement of B with respect to A: for sets  $A$ ,  $B$ , the *difference of A and B*, written  $A-B$ , is the set of all elements that are in  $A$  but not  $B$ .

$$\bullet A - B \equiv \{x \mid x \in A \wedge x \notin B\}$$

e.g.

- $\{a,b,c\} - \{a,b,c\} = \emptyset$
- $\{2,4,6\} - \{3,4,5\} = \{2,6\}$
- $\{3,4,5\} - \{2,4,6\} = \{3,5\}$



# Operation Between Set: Complement

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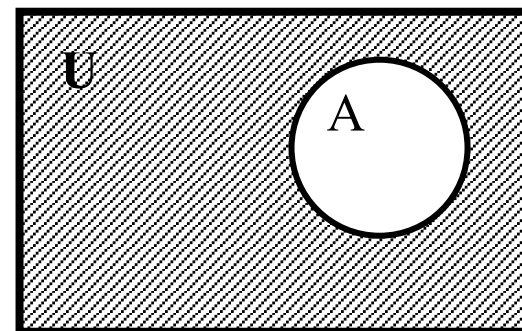
- The *universe of discourse* can itself be considered a set, call it  $U$ .
- The complement of a set  $A$  is:

$$\overline{A} = U - A$$

- Specifically:

$$\overline{\emptyset} = U$$

$$\overline{U} = \emptyset$$

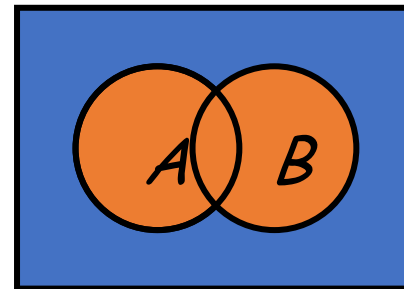


# Set Identities

- Identity:  $A \cup \emptyset = A$   $A \cap U = A$
  - Domination:  $A \cup U = U$   $A \cap \emptyset = \emptyset$
  - Idempotent:  $A \cup A = A = A \cap A$
  - Double complement:  $\overline{\overline{A}} = A$
  - Commutative:  $A \cup B = B \cup A$   $A \cap B = B \cap A$
  - Associative:  $A \cup (B \cup C) = (A \cup B) \cup C$   
 $A \cap (B \cap C) = (A \cap B) \cap C$
- Specifically:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$



Proof by "diagram" (useful!).

# In-Class Exercises:

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- Finish the membership Table:

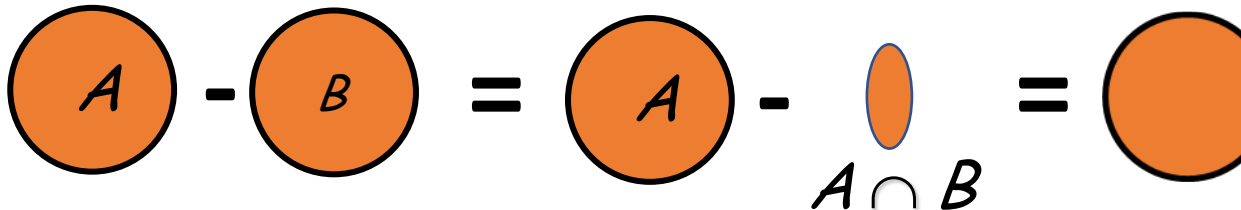
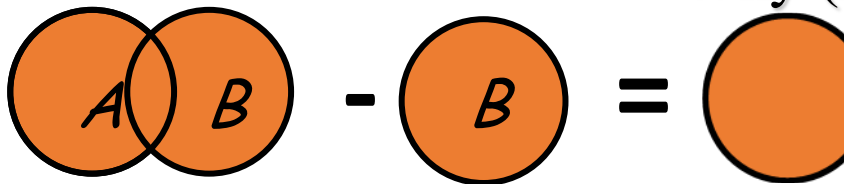
| A | B | $A \cup B$ | $(A \cup B) - B$ | A-B |
|---|---|------------|------------------|-----|
| 0 | 0 |            |                  |     |
| 0 | 1 |            |                  |     |
| 1 | 0 |            |                  |     |
| 1 | 1 |            |                  |     |

# In-Class Exercises:

- Finish the membership Table:

| A | B | $A \cup B$ | $(A \cup B) - B$ | $A - B$ |
|---|---|------------|------------------|---------|
| 0 | 0 | 0          | 0                | 0       |
| 0 | 1 | 1          | 0                | 0       |
| 1 | 0 | 1          | 1                | 1       |
| 1 | 1 | 1          | 0                | 0       |

Why  $(A \cup B) - B = A - B$  ?



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## 2. The inclusion-exclusion principle

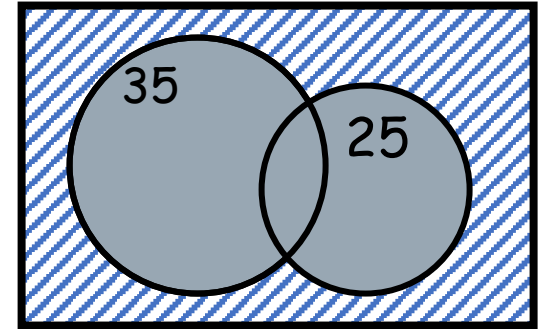
# Inclusion/Exclusion

■ Considering we have 50 students:

35 like apple;  $A$

25 like orange;  $B$

15 like both;  $A \cap B$



How many students don't like the two fruits Neither?

students like orange or apple  $A \cup B$

= students only like apple + students only like orange + students like both

= students only like apple + students like both

(students like apple)  $A$

+ students only like orange + students like both

(students like orange)  $B$

- students like both  $A \cap B$

$$= 35 + 25 - 15 = 45$$

$$A \cup B = A + B - A \cap B$$

Students don't like neither =  $50 - 45 = 5$

# Principle of Inclusion & Exclusion

- This is so-called Principle of Inclusion & Exclusion:

$$A \cup B = A + B - A \cap B$$

- For three disjoint sets

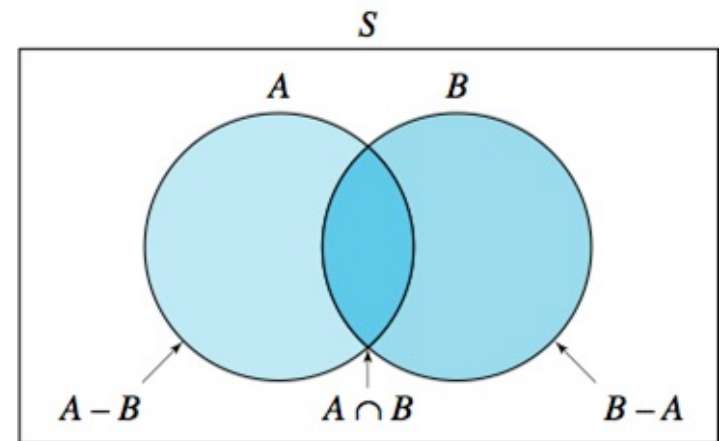
$$A \cup B = (A - B) + (B - A) + (A \cap B)$$

- We have solved for two finite sets  $A$  and  $B$

$$A - B = A - (A \cap B) \quad \text{and} \quad B - A = B - (A \cap B)$$

- Hence,

$$\begin{aligned} A \cup B &= (A - B) + (B - A) + (A \cap B) \\ &= A - (A \cap B) + B - (A \cap B) + (A \cap B) \\ &= A + B - (A \cap B) \end{aligned}$$

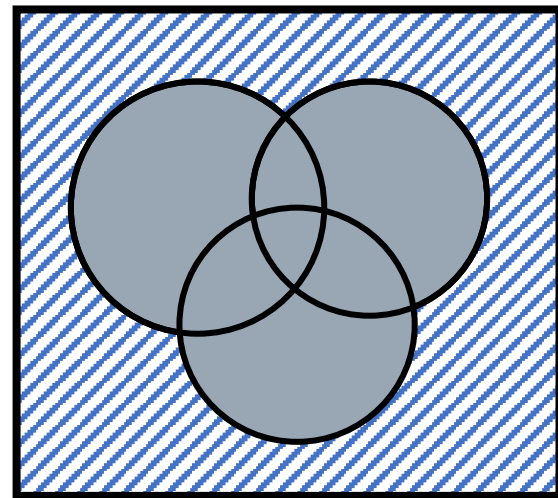


# Principle of Inclusion & Exclusion

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- Similarly, for three sets, we can obtain:

$$A \cup B \cup C = A + B + C - A \cap B - A \cap C - B \cap C + A \cap B \cap C$$



# Applications

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- How many integers from 1 to 1000 are either multiples of 3 or multiples of 5?

**Solutions:**

## **Step 1: Define the sets:**

- Let us assume that  $A$  = set of all integers from 1 to 1000 that are multiples of 3.
- Let us assume that  $B$  = set of all integers from 1 to 1000 that are multiples of 5.
- $A \cup B$  = The set of all integers from 1 to 1000 that are multiples of either 3 or 5.
- $A \cap B$  = The set of all integers that are both multiples of 3 and 5, which also is the set of integers that are multiples of 15.

## **Step 2: Computer the number in each set: ( $|A|$ refers the number of elements in $A$ )**

- From 1 to 1000, every third integer is a multiple of 3, each of this multiple can be represented as  $3p$ , for any integer  $p$  from 1 through 333, Hence  $|A| = 333$ .
- Similarly for multiples of 5, each multiple of 5 is of the form  $5q$  for some integer  $q$  from 1 through 200. Hence, we have  $|B| = 200$ .
- To determine the number of multiples of 15 from 1 through 1000, each multiple of 15 is of the form  $15r$  for some integer  $r$  from 1 through 66.

# Applications

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- How many integers from 1 to 1000 are either multiples of 3 or multiples of 5?

**Solutions:**

**Step 3: To use the inclusion/exclusion principle to obtain  $|A \cup B|$**

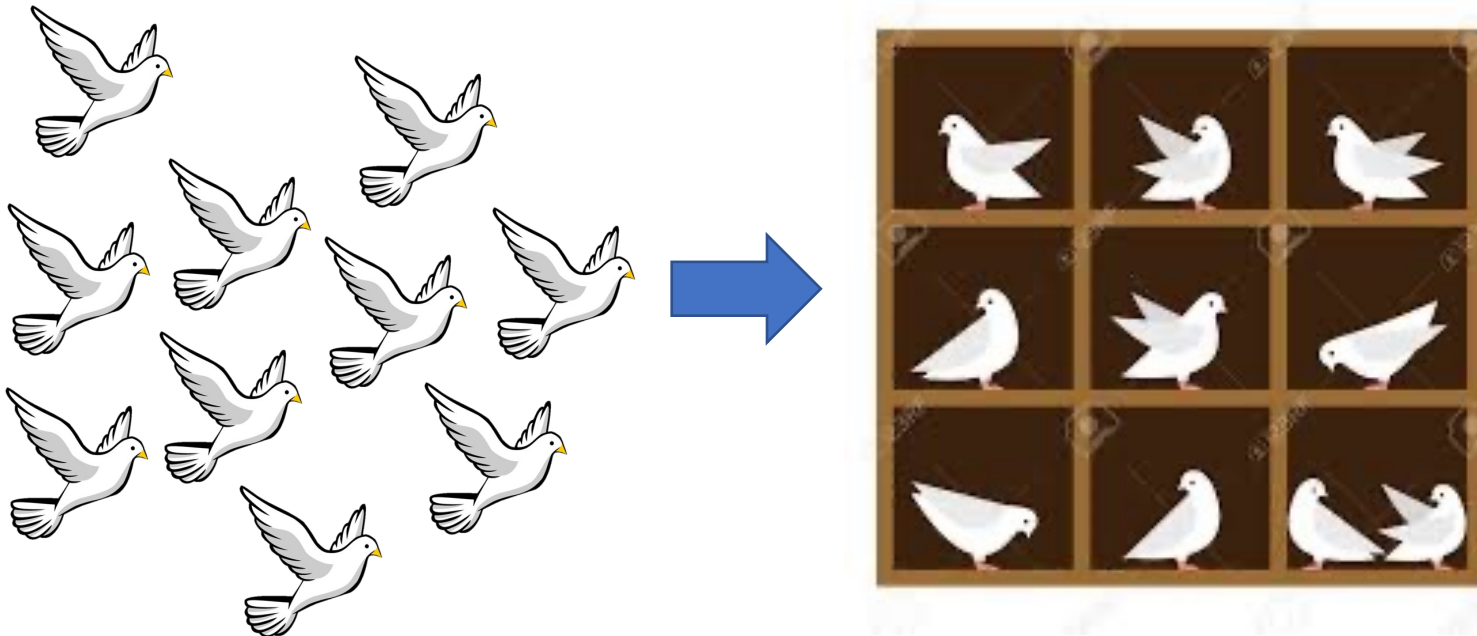
$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\&= 333 + 200 - 66 \\&= 467.\end{aligned}$$

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# 3. The pigeon-hole principle

# The pigeon-hole principle

- Suppose a flock of pigeons fly into a set of pigeonholes to roost;
- If there are more pigeons than pigeonholes, then there must be at least 1 pigeonhole that has more than one pigeon in it
- *If  $k+1$  or more objects are placed into  $k$  boxes, then there is at least one box containing two or more of the objects*



# The applications of pigeon-hole principle

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- Some examples:
- In a group of 13 people, there must be two people with the birthdays in the same month;
- In a group of 367 people, there must be at least two people with the same birthday.
- In a group of 27 English words, at least two words must start with the same letter

# The applications of pigeon-hole principle

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- Furthermore:
- If  $N$  objects are placed into  $k$  boxes, then there is at least one box containing  $\lceil N/k \rceil$  objects.
- $\lceil N/k \rceil$  denotes the ceiling function, which is the smallest integer that is not smaller than  $N/k$ .
- Examples:
- Among 100 people, there are at least  $\lceil 100/12 \rceil = 9$  born on the same month;

# In-Class Exercises:

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- How many students in a class must there be to ensure that 6 students get the same grade (one of A, B, C, D, or F)?

Solution:

The “boxes” are the grades. Thus,  $k = 5$

Thus, we set  $\lceil N/5 \rceil = 6$

Lowest possible value for N is 26

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•END