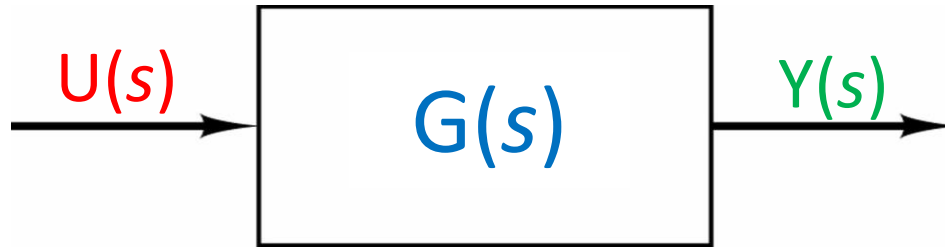


EE3114 Systems and Control

3. Dynamic Response

Semester B, 2017-18

Given Transfer Function $G(s)$ and Input $U(s)$:



$$Y(s) = G(s)U(s)$$

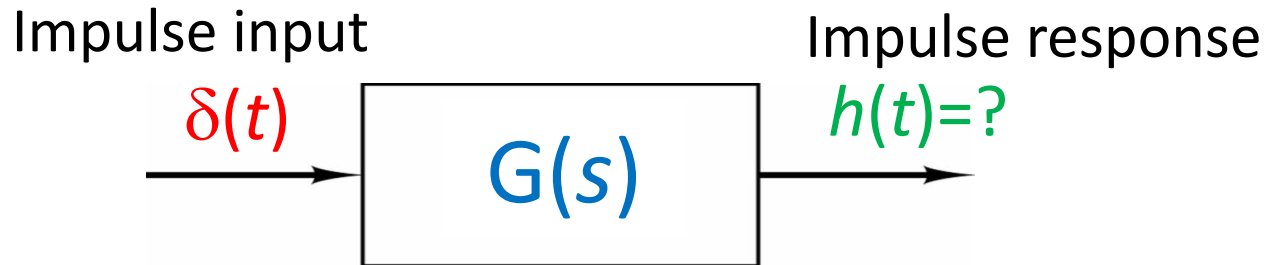
Output obtained by Inverse Laplace Transform:

$$y(t) = g(t) * u(t) = L^{-1}\{G(s)U(s)\}$$

Convolution

$$f(t) * g(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(\tau)g(t - \tau) d\tau$$

Impulse Response



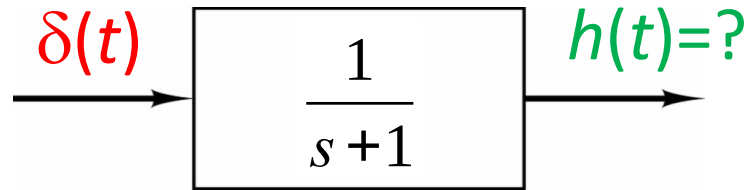
$$U(s) = 1 \quad \text{for } u(t)=\delta(t)$$

$$\longrightarrow H(s) = G(s)U(s) = G(s)$$

$$\longrightarrow h(t) = L^{-1}[G(s)]$$

Impulse response of the system is obtained by
Inverse Laplace Transform of Transfer Function

Example:



$$H(s) = \frac{1}{s+1}$$

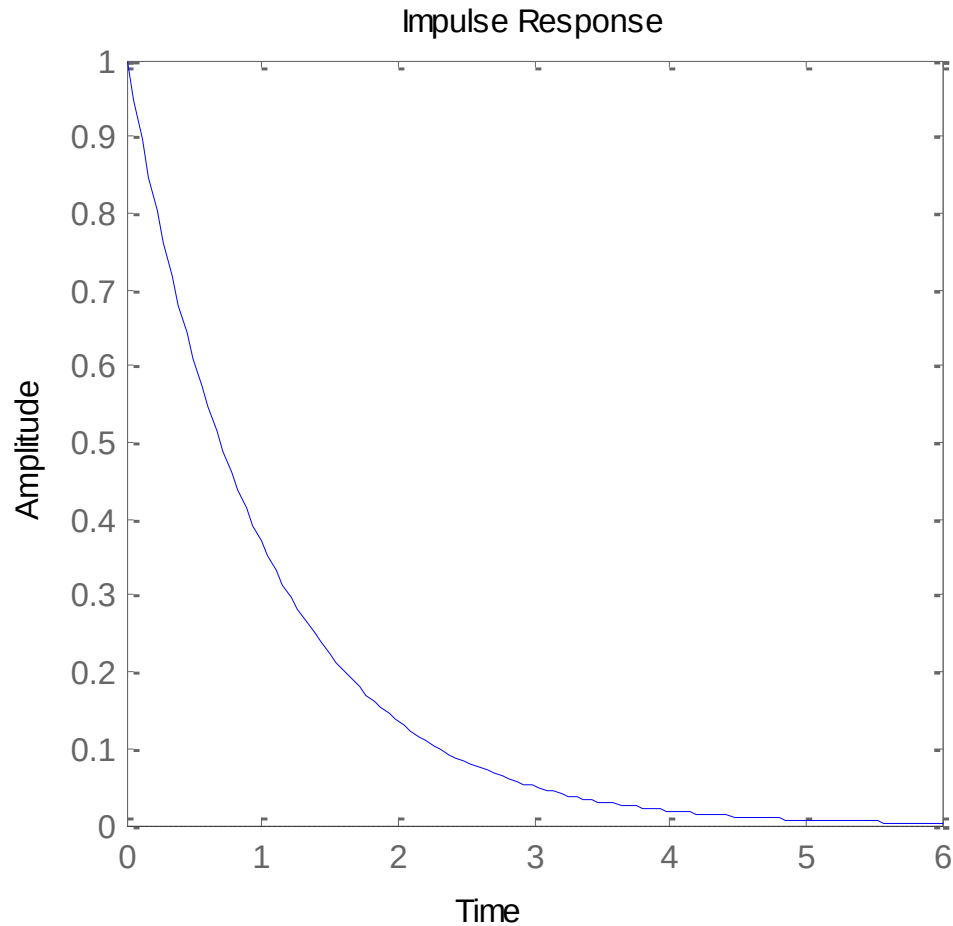
Inverse Laplace
Transform of $H(s)$



$$h(t) = e^{-t}$$

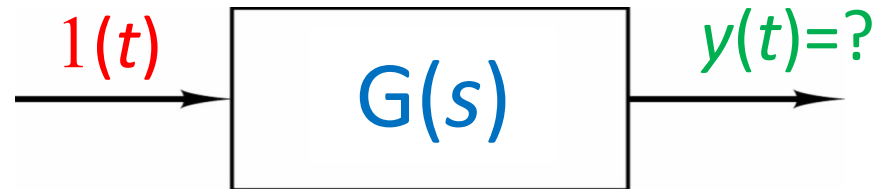


```
%% MatLab code  
Hs = tf(1,[1 1]);  
impz(Hs);
```



Step Response

Step input



Step response

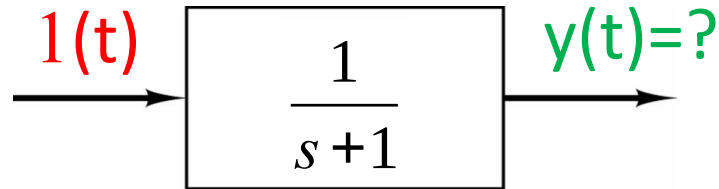
$$U(s) = \frac{1}{s} \quad \text{for } u(t)=1(t)$$

➡ $Y(s) = G(s)U(s) = G(s)\frac{1}{s}$

➡ $y(t) = L^{-1}\left[\frac{G(s)}{s}\right]$

Step response is the integral of the impulse response
(Integration property of Laplace Transform)

Example

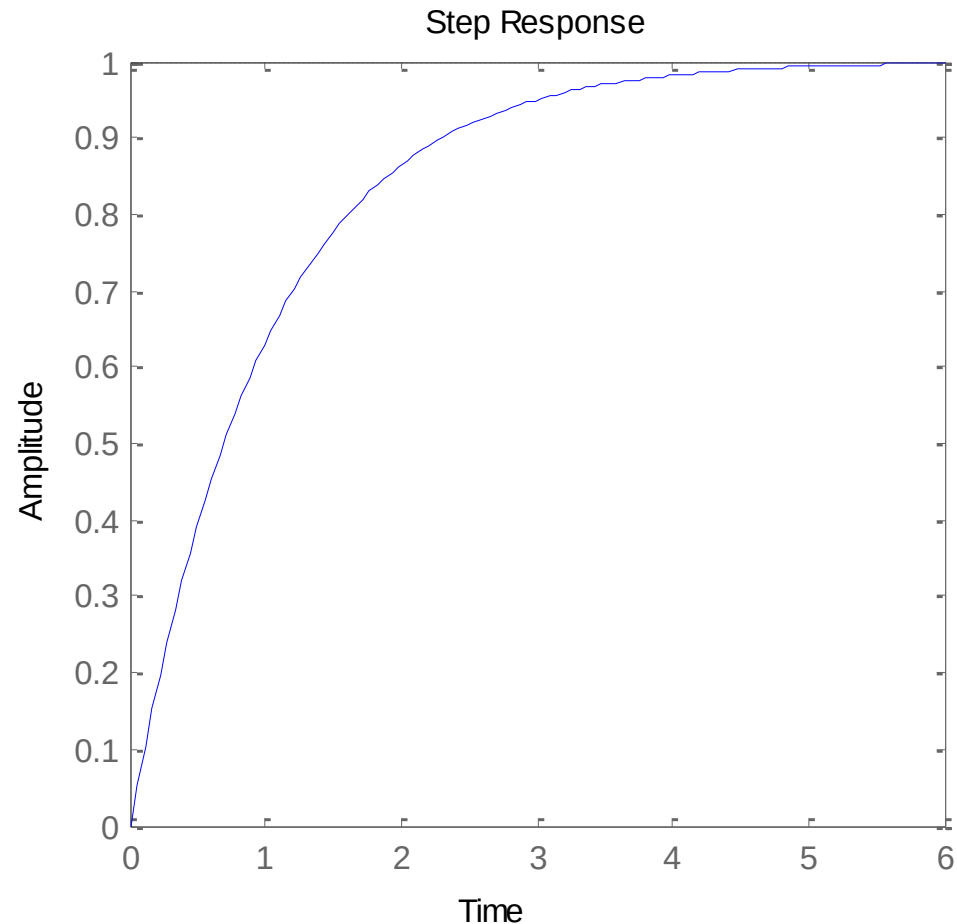


$$Y(s) = \frac{1}{s(s+1)}$$

$$y(t) = 1 - e^{-t}$$



```
%% MatLab code  
Hs = tf(1,[1 1]);  
step(Hs);  
[y,t] = step(Hs,6); %Time [0,6]  
plot (t, y);
```



Poles and Zeros

Transfer Function

$$\begin{aligned} G(s) &= \frac{N(s)}{D(s)} = \frac{b_0 s^m + \dots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} \dots + a_{n-1} s + a_n} \\ &= \frac{k(s - z_1) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)} \end{aligned}$$

p_i are **poles**:

$$D(s) = 0 \rightarrow G(s) \rightarrow \infty$$

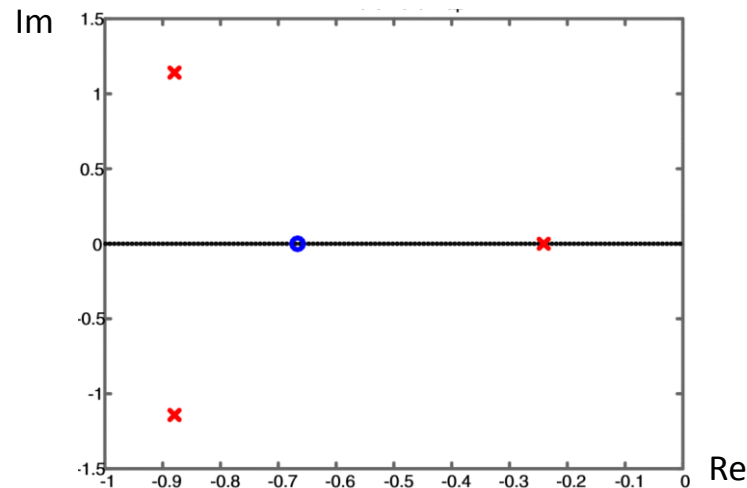
z_i are **zeros**:

$$N(s) = 0 \rightarrow G(s) = 0$$

Pole-Zero Plot

- System dynamics can be represented graphically by plotting the **locations of poles and zeros** on the complex s-plane
- A zero location is marked by a circle (o)
- A pole location is marked by a cross (x)

Example:



Example

A linear system is described by the differential equation

$$\ddot{y} + 4\dot{y} + 8y = \dot{u} + u$$

Plot the system transfer function poles and zeros.

Solution:

System transfer function:

$$\frac{Y(s)}{U(s)} = \frac{s+1}{s^2 + 4s + 8} = \frac{s - (-1)}{(s - (-2 - 2j))(s - (-2 + 2j))}$$

It has zero: $s = -1$

poles:

$$s = -2 - 2j$$

$$s = -2 + 2j$$

$$H = \frac{s+1}{s^2+4s+8}$$

zero: $s = -1$

poles:

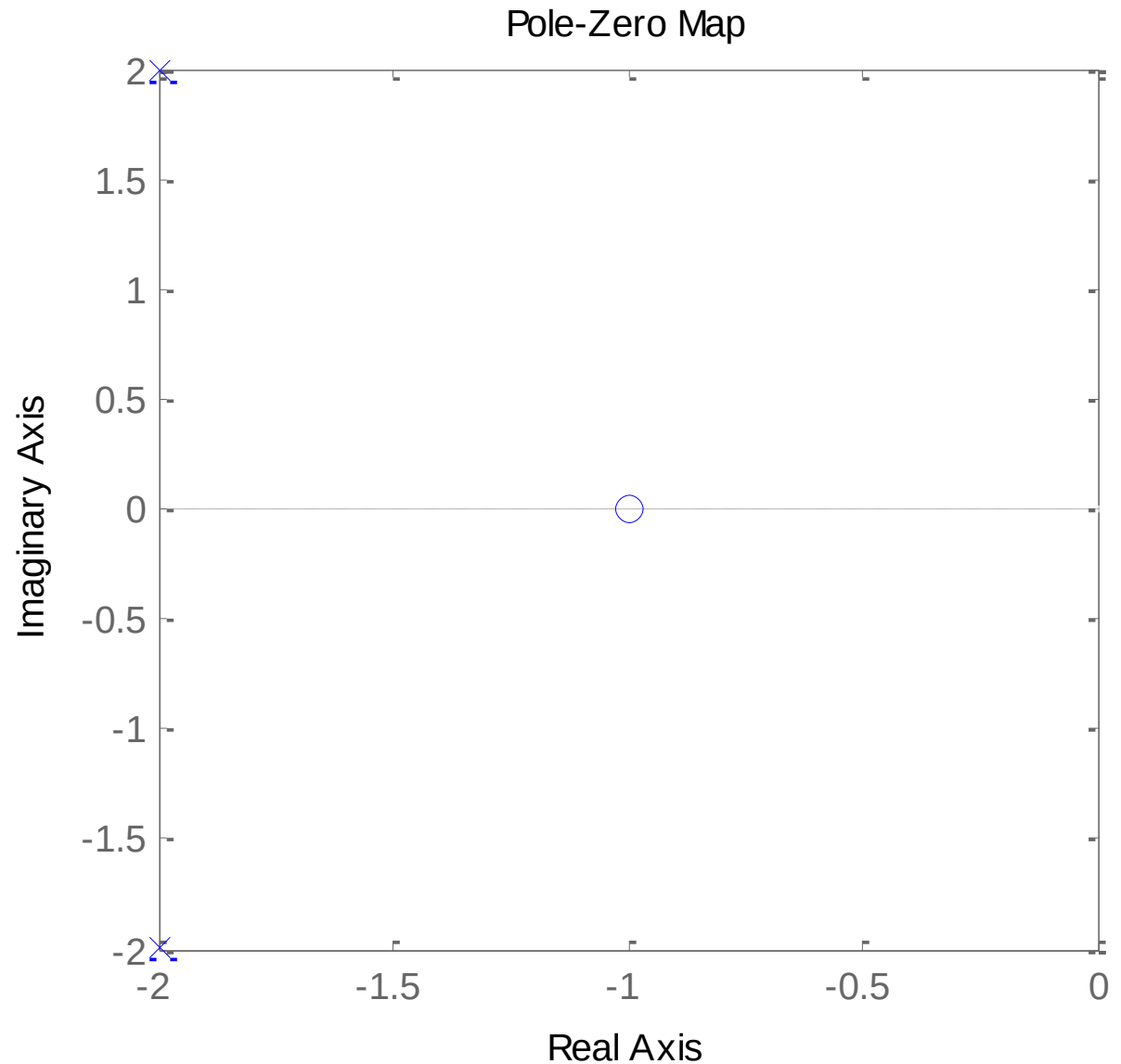
$$s = -2 - 2j$$

$$s = -2 + 2j$$

%% MatLab code

```
H = tf([1 1],[1 4 8]);
```

```
pzmap(H);
```



Inverse Problem

Example

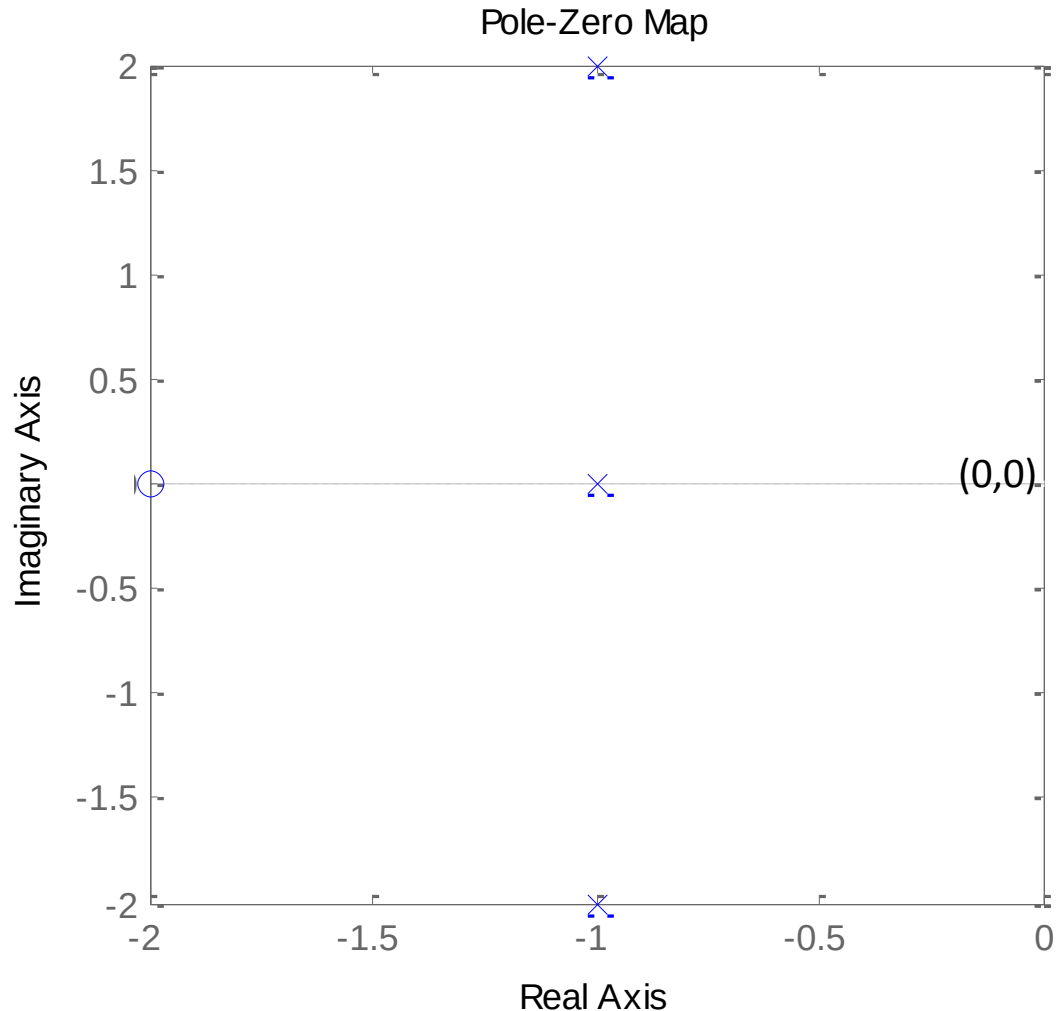
Given the pole-zero plot of a third-order system as shown on the right, with a gain factor k

Find its differential equation.

Solution: First, find

Zero: -2

Poles: -1 , $-1 - 2j$, $-1 + 2j$



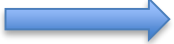
Then, find transfer function

Zero: -2

Poles: $-1, -1 - 2j, -1 + 2j$

MatLab code:

`H = zpk(-2,[-1 -1-2j -1+2j],1);pzmap(H);
tf(H)` 
$$\frac{s + 2}{s^3 + 3s^2 + 7s + 5}$$

 OR
$$\frac{Y(s)}{U(s)} = \frac{k(s + 2)}{(s + 1)(s - (-1 - 2j))(s - (-1 + 2j))}$$
$$= \frac{k(s + 2)}{s^3 + 3s^2 + 7s + 5}$$


$$\frac{d^3 y}{dt^3} + 3 \frac{d^2 y}{dt^2} + 7 \frac{dy}{dt} + 5y = k \frac{du}{dt} + 2ku$$

Final Value Theorem

If all poles of $sY(s)$ are in Left Half s -Plane (LHP), then

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s)$$

Proof:

$$L\left[\frac{dy(t)}{dt}\right] = sY(s) - y(0)$$

$$\int_0^{\infty} \frac{dy(t)}{dt} e^{-st} dt = sY(s) - y(0)$$

$$\lim_{s \rightarrow 0} \int_0^{\infty} \frac{dy(t)}{dt} e^{-st} dt = \lim_{s \rightarrow 0} sY(s) - y(0)$$

$$\int_0^{\infty} \frac{dy(t)}{dt} dt = \lim_{s \rightarrow 0} sY(s) - y(0)$$

$$y(\infty) - y(0) = \lim_{s \rightarrow 0} sY(s) - y(0)$$

$$y(\infty) = \lim_{s \rightarrow 0} sY(s)$$

Initial Value Theorem

For *any* Laplace Transform pair (y, Y) :

$$y(0^+) = \lim_{s \rightarrow \infty} sY(s)$$

Proof:

$$L\left[\frac{dy(t)}{dt}\right] = sY(s) - y(0^+)$$

$$\int_0^{\infty} \frac{dy(t)}{dt} e^{-st} dt = sY(s) - y(0^+)$$

$$\lim_{s \rightarrow \infty} \int_0^{\infty} \frac{dy(t)}{dt} e^{-st} dt = \lim_{s \rightarrow \infty} sY(s) - y(0^+)$$

$$0 = \lim_{s \rightarrow \infty} sY(s) - y(0^+)$$

$$y(0^+) = \lim_{s \rightarrow \infty} sY(s)$$

Example:

$$Y(s) = \frac{3(s+2)}{s(s^2 + 2s + 10)} \Rightarrow sY(s) = \frac{3(s+2)}{s^2 + 2s + 10}$$

Poles of $sY(s)$ are $-2 \pm 3j$, which are in LHP, so

$$y(\infty) = \lim_{s \rightarrow 0} sY(s) = 6/10 = 0.6$$

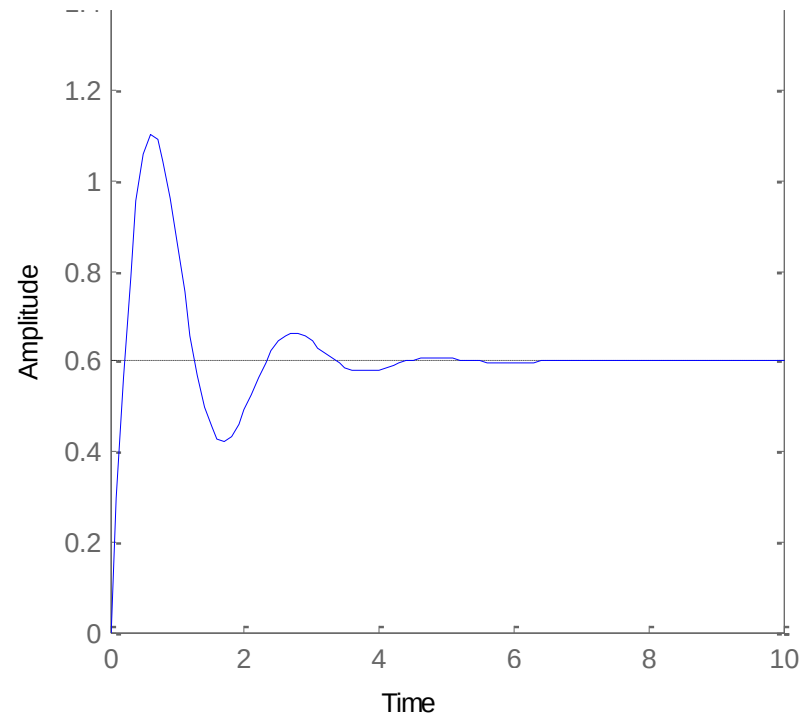
$$y(0^+) = \lim_{s \rightarrow \infty} \frac{3(s+2)}{(s^2 + 2s + 10)} = 0$$

```
%% MatLab code  
num=[3 6];  
den=[1 2 10 0];  
Ys = tf(num,den);  
impulse(Ys)
```

```
%% OR  
num=[3 6];  
den=[1 2 10];  
Ys = tf(num,den);  
step(Ys)
```

$$\frac{3(s+2)}{s(s^2 + 2s + 10)}$$

$$\frac{3(s+2)}{s^2 + 2s + 10}$$



Example: $Y(s) = \frac{s}{s^2 + 1} \Rightarrow sY(s) = \frac{s^2}{s^2 + 1}$

Poles of $sY(s)$ are $\pm j$, which are not in LHP.

So, the final value theorem is not applicable.

But, there is no condition for initial value theorem.

$$y(0^+) = \lim_{s \rightarrow \infty} \frac{s^2}{s^2 + 1} = 1$$

%% MatLab code

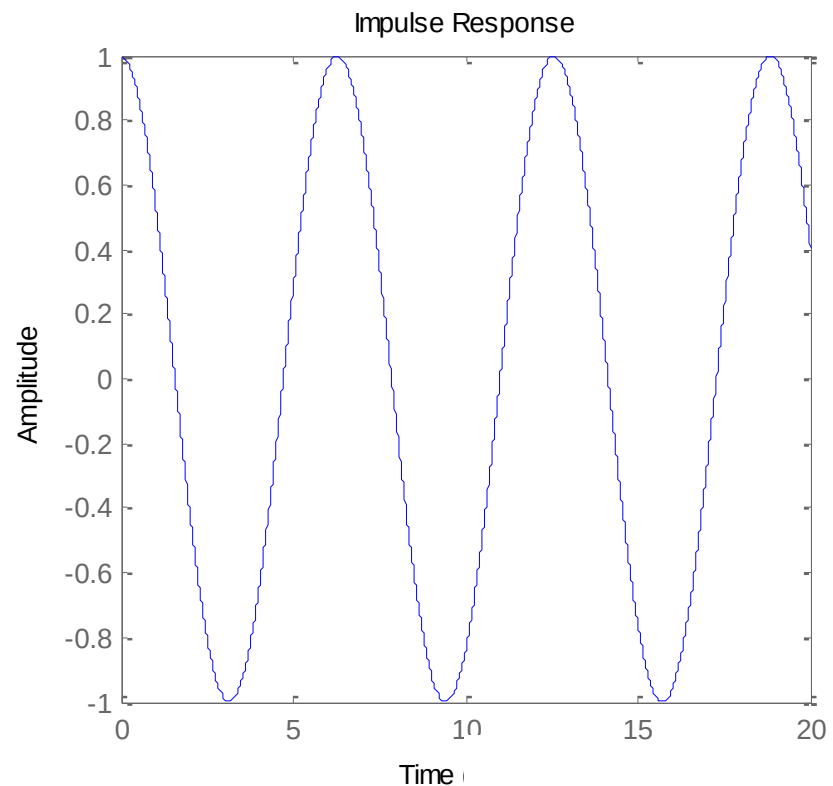
```
num = [1 0];
```

```
den = [1 0 1];
```

```
Ys = tf(num,den)
```

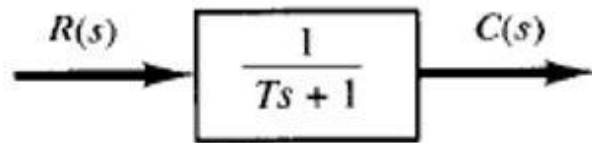
```
impz(Ys, 0:0.01:20);
```

$$\frac{s}{s^2 + 1}$$

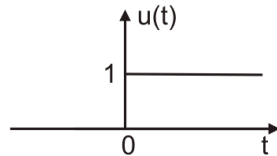


Inverse Laplace Transform

Example: First-Order LTI System



Unit Step Input:



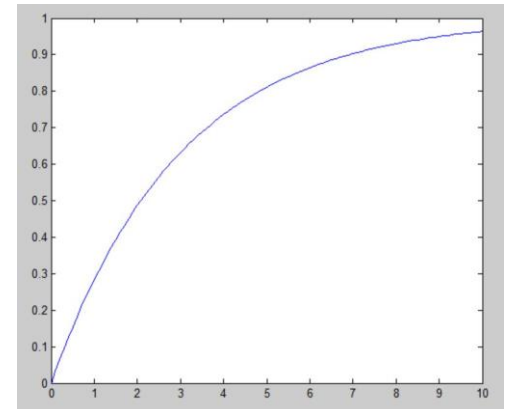
Output:

$$C(s) = \frac{1}{Ts + 1} \frac{1}{s} = \frac{1}{s} - \frac{T}{Ts + 1} = \frac{1}{s} - \frac{1}{s + (1/T)}$$

Inverse Laplace Transform → Unit Step Response

$$c(t) = 1 - e^{-t/T}, \quad \text{for } t \geq 0$$

```
%% MatLab code,  
Hs = tf(1,[3 1]); %T = 3  
step(Hs);  
[y,t] = step(Hs,10); %Time [0,10]  
plot (t, y);
```



Inverse Laplace Transform

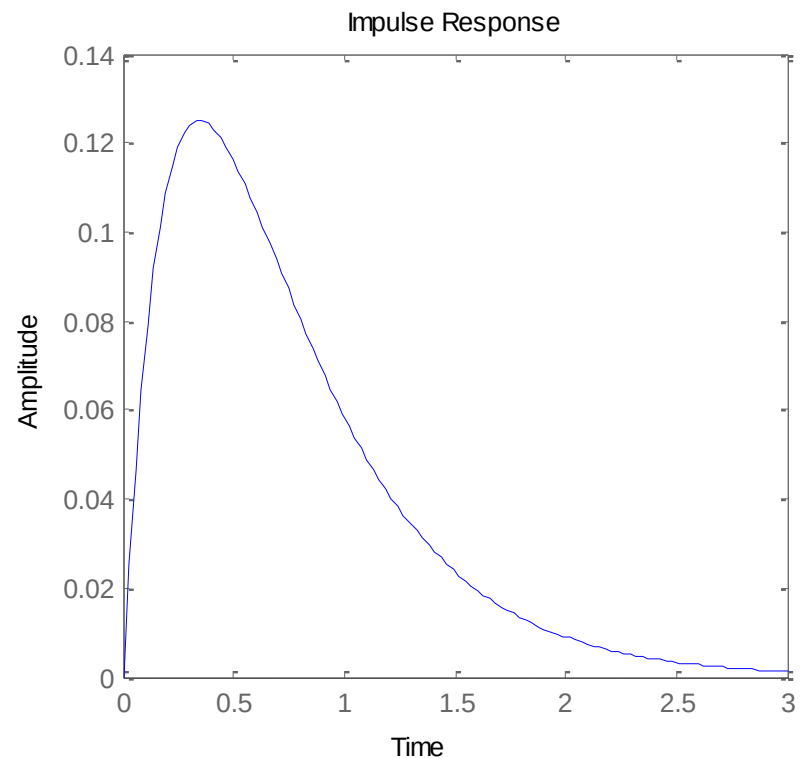
$$y(t) \xleftarrow{L^{-1}} Y(s)$$

- Partial fraction expansion and term-by-term inversion

Example (single poles):

$$\begin{aligned} Y(s) &= \frac{1}{(s+2)(s+4)} \\ &= \frac{1}{2} \left(\frac{1}{s+2} \right) - \frac{1}{2} \left(\frac{1}{s+4} \right) \end{aligned}$$

$$y(t) = \frac{1}{2} e^{-2t} - \frac{1}{2} e^{-4t}$$



```
%%MatLab code  
syms s;  
H=1/((s+2)*(s+4));  
F = ilaplace(H);  
ezplot(F,[0:0.1:3])
```

Example (multiple poles):

$$Y(s) = \frac{s+3}{(s+2)^2(s+1)} = \frac{C_1}{s+2} + \frac{C_2}{(s+2)^2} + \frac{C_3}{s+1}$$

$$C_2 = (s+2)^2 Y(s) \Big|_{s=-2} = \frac{s+3}{s+1} \Big|_{s=-2} = -1$$

$$C_1 = \frac{d}{ds} \left[(s+2)^2 Y(s) \right] \Big|_{s=-2} = \frac{d}{ds} \left(\frac{s+3}{s+1} \right) \Big|_{s=-2} = \frac{(s+1) - (s+3)}{(s+1)^2} \Big|_{s=-2} = -2$$

$$C_3 = (s+1)Y(s) \Big|_{s=-1} = 2$$

%% MatLab code

```
b = [1 3];
```

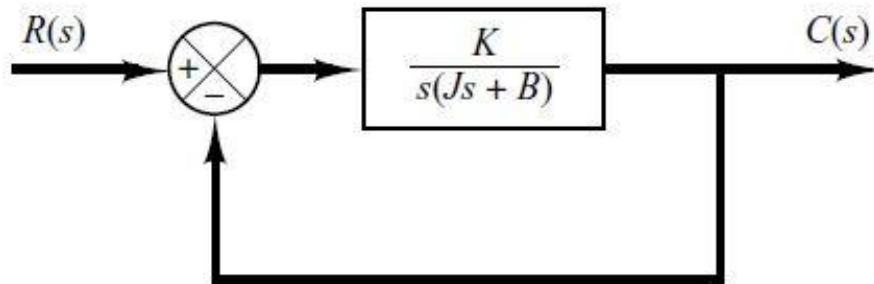
```
a = poly([-2 -2 -1]);
```

```
[r,p,k] = residue(b,a)
```

`[r,p,k] = residue(b,a)` finds the residues, poles, and direct term of a [Partial Fraction Expansion](#)

$$\frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = \frac{r_n}{s-p_n} + \dots + \frac{r_2}{s-p_2} + \frac{r_1}{s-p_1} + k(s).$$

Example: (Complex Conjugate Poles)



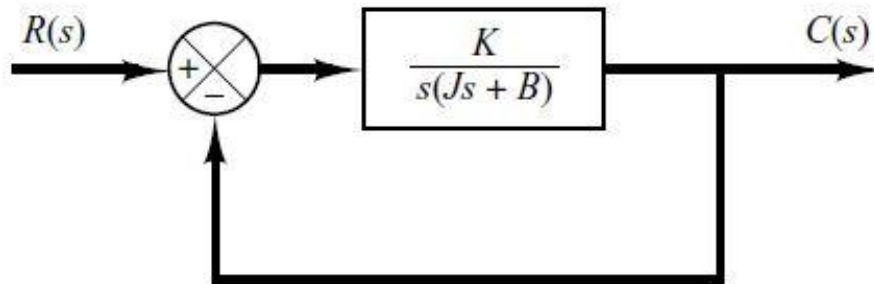
Constants:

$$K > 0, J > 0, B > 0$$

Find Closed-loop Transfer Function = ?

5 minutes

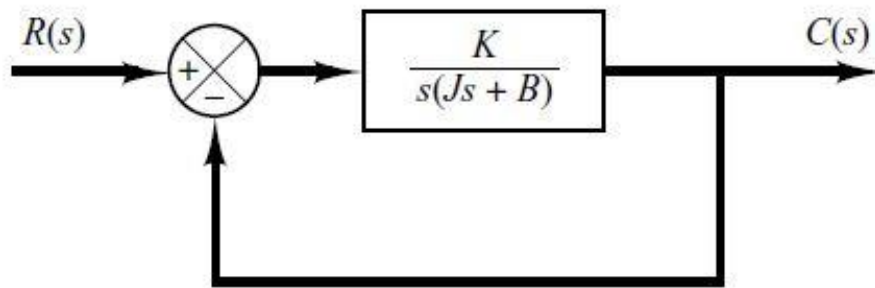
Example: (Complex Conjugate Poles)



Find Closed-loop Transfer Function = ?

Answer:
$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K}$$

Example: (Complex Conjugate Poles)



Closed-loop Transfer Function:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K}$$



$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}} \right] \left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}} \right]}$$

The closed-loop poles are complex conjugates if $B^2 - 4JK < 0$

Let

$$\frac{K}{J} = \omega_n^2, \quad \frac{B}{J} = 2\zeta\omega_n = 2\sigma \quad (\omega_n \text{ -- natural frequency})$$
$$\zeta = \frac{B}{2\sqrt{JK}}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}} \right] \left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}} \right]}$$

The closed-loop poles are complex conjugates if $B^2 - 4JK < 0$

Let

$$\frac{K}{J} = \omega_n^2, \quad \frac{B}{J} = 2\zeta\omega_n = 2\sigma \quad (\omega_n \text{ -- natural frequency})$$

$$\zeta = \frac{B}{2\sqrt{JK}}$$



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \zeta\omega_n - j\omega_d)(s + \zeta\omega_n + j\omega_d)}$$

where

$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$



$$p = -\sigma + j\omega_d \quad \text{and} \quad p^* = -\sigma - j\omega_d$$

Complex Conjugate Poles

(a) Cartesian Representation

$$p = -\sigma + j\omega_d \quad \text{and} \quad p^* = -\sigma - j\omega_d$$

Inverse Laplace Transform:

$$\frac{C}{s-p} + \frac{C^*}{s-p^*} \xrightarrow{L^{-1}} 2|C| e^{-\sigma t} \cos(\omega_d t + \angle C)$$

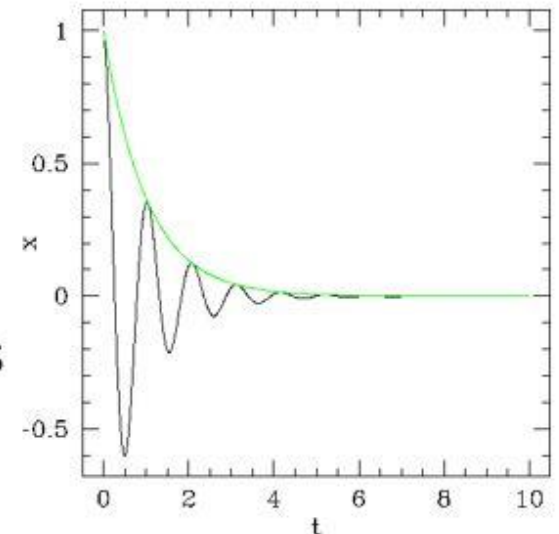
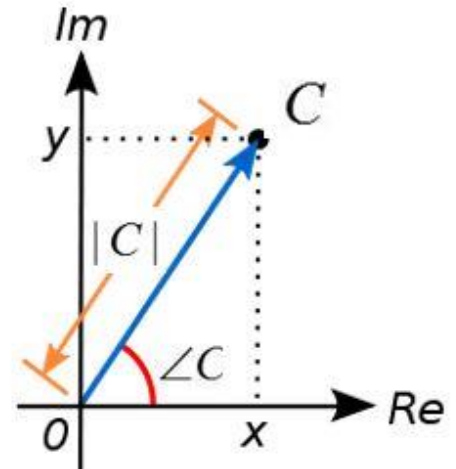
$$C = |C| e^{j\angle C}$$

Time-signal $y(t)$: A sinusoidal signal with

Frequency: ω_d

Decay envelop: $e^{-\sigma t}$

Different C results in different phase shifts and amplitudes of the sinusoidal signal



(b) (ω_n, ζ) Representation

$$p = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

Conversion to/from Cartesian Representation

$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

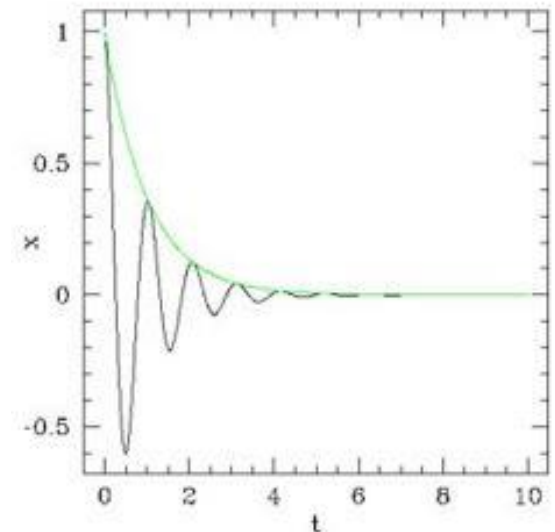
or

$$\left(\omega_n = \sqrt{\sigma^2 + \omega_d^2}, \zeta = \frac{\sigma}{\sqrt{\sigma^2 + \omega_d^2}} \right)$$

Sinusoidal signal $y(t)$ has

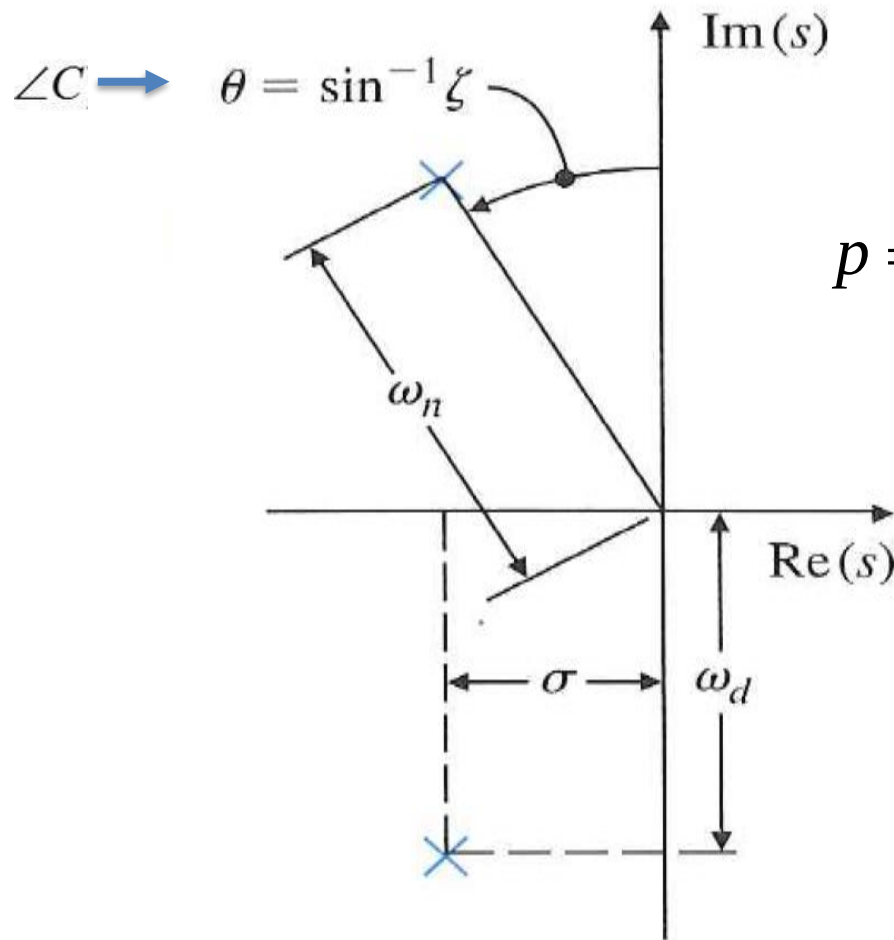
Frequency: $\omega_n\sqrt{1-\zeta^2}$

Decay envelop: $e^{-\zeta\omega_n t}$



Geometric Meaning of Complex Conjugate Poles

$$p = -\sigma + j\omega_d \quad \text{and} \quad p^* = -\sigma - j\omega_d$$



$$p = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

(if $\zeta = 1$ then $\omega_n = \sigma$)

Physical Meaning of Complex Conjugate Poles

$$p = -\sigma + j\omega_d \quad \text{and} \quad p^* = -\sigma - j\omega_d$$

$$p = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

ζ – Damping Ratio

If $\zeta < 1$ then $\omega_n > \sigma$ (under damped)

If $\zeta = 1$ then $\omega_n = \sigma$ (un-damped)

If $\zeta > 1$ then $\omega_n < \sigma$ (over damped)

(As shown below)

More precisely:

$$\begin{aligned} Y(s) &= \frac{1}{(s-p)(s-p^*)} \\ &= \frac{1}{(s+\sigma)^2 + \omega_d^2} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

If damping ratio $\zeta < 1$ (**under damped**)

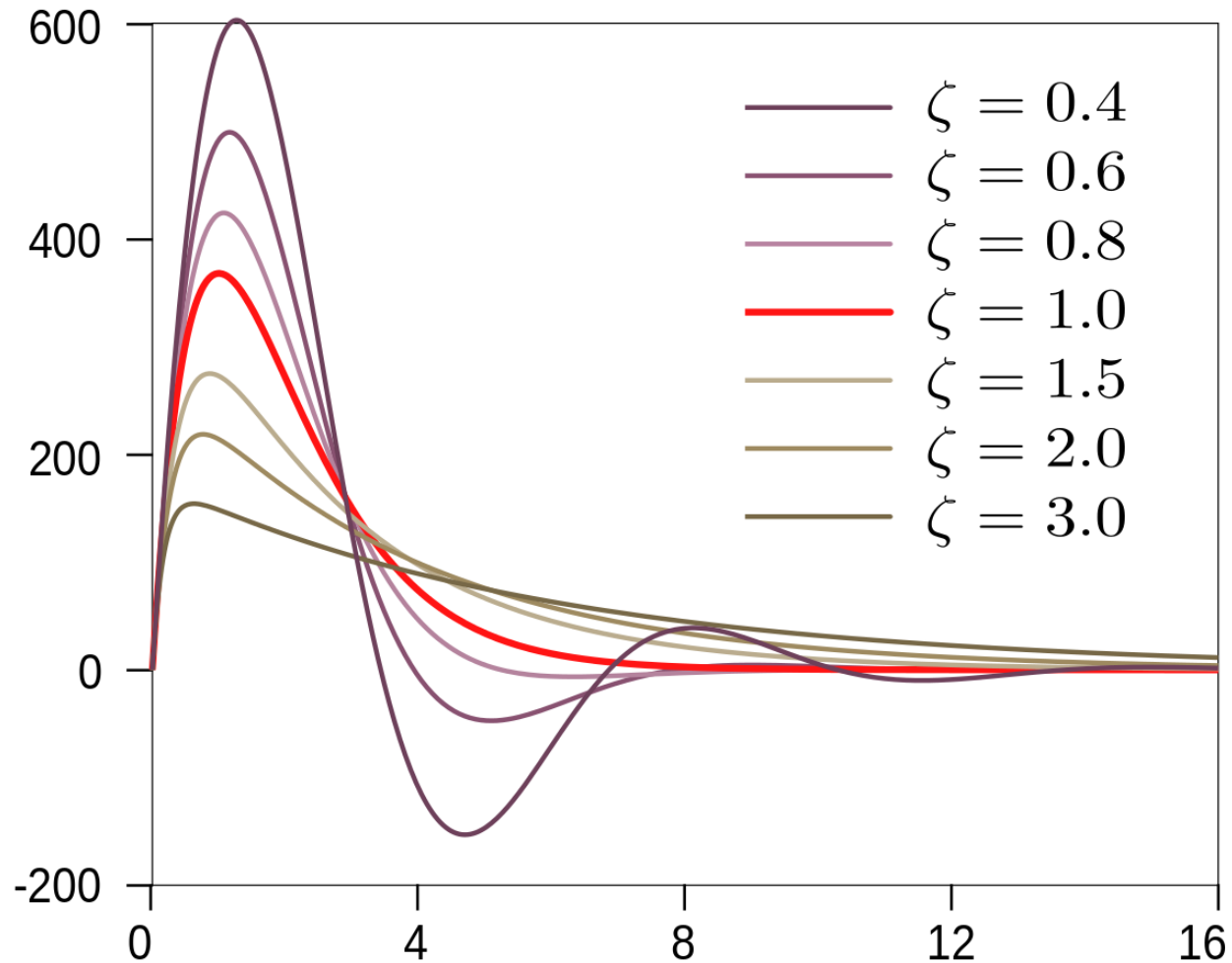
$$y(t) = \frac{1}{\omega_d} e^{-\sigma t} \sin \omega_d t = \frac{1}{\omega_n \sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t)$$

If damping ratio $\zeta = 1 \rightarrow$ **un-damped** natural frequency $\omega_n = \sigma$

$$y(t) = te^{-\sigma t} = te^{-\omega_n t}$$

If $\zeta > 1$ then $\omega_n < \sigma$ (**over damped**) $\rightarrow$ not oscillatory

Performances of Different Damping Ratios



Effects of Poles on System Response

Example: Oscillatory Time Response

Show the relation between the poles of

$$H(s) = \frac{2s+1}{s^2+2s+5}$$

and the **impulse response** of the system, and find the exact impulse response

Solution:

The poles of the transfer function: real part - 1 and imaginary parts $\pm 2j$

and
$$\omega_n^2 = 5 \Rightarrow \omega_n = \sqrt{5} = 2.24 \text{ rad / s}$$

$$2\zeta\omega_n = 2 \Rightarrow \zeta = \frac{1}{\sqrt{5}} = 0.447$$



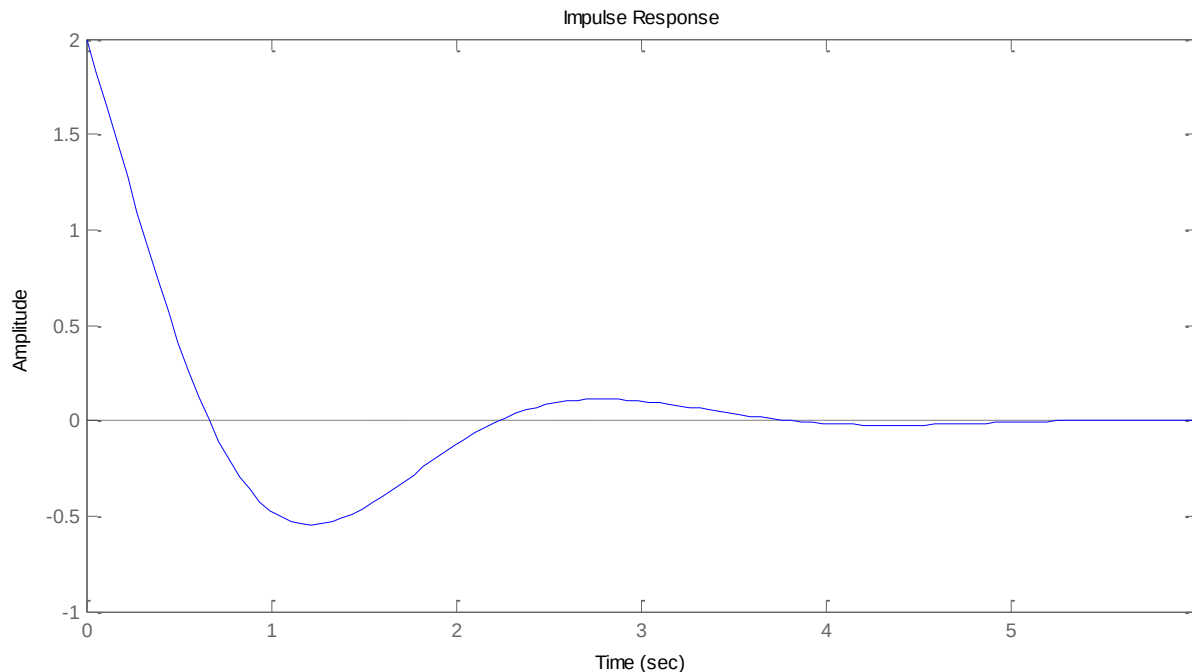
A motion of frequency around 2 *rad/s* with very little oscillatory will be expected

Since

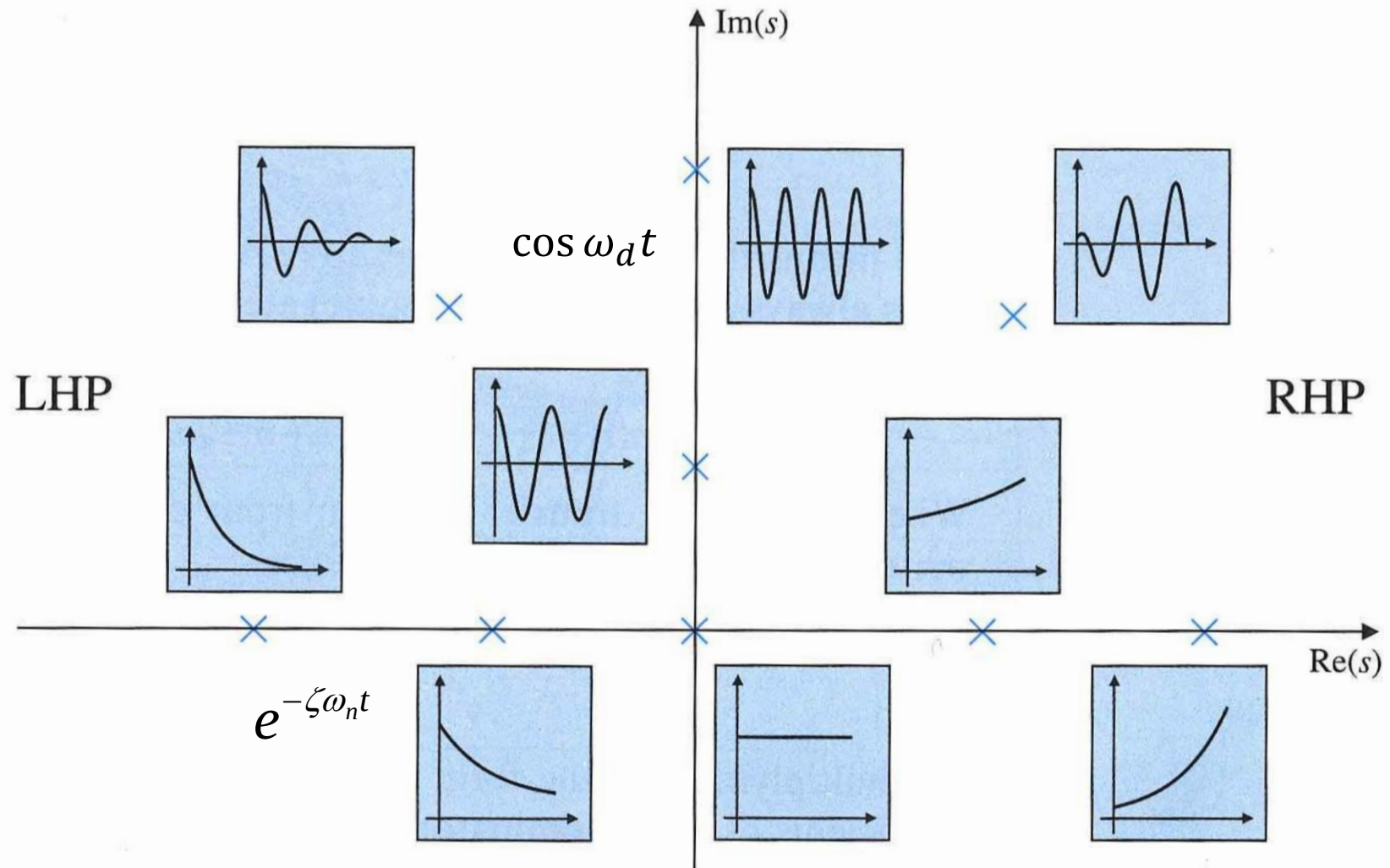
$$H(s) = \frac{2s+1}{(s+1)^2 + 2^2} = 2 \frac{s+1}{(s+1)^2 + 2^2} - \frac{1}{2} \frac{2}{(s+1)^2 + 2^2}$$

the **impulse response** is $h(t) = \left(2e^{-t} \cos 2t - \frac{1}{2}e^{-t} \sin 2t \right) 1(t)$

The plot below shows: (1) how the envelope attenuates the sinusoid, (2) the domination of the $2 \cos 2t$ term, (3) the small phase shift caused by the $-0.5 \sin 2t$ term.



Pole Location and System Response



Effects of Zeros on System Response

Zeros exert their influence by modifying the coefficients of the exponential terms whose shapes are determined by the poles

Example:

$$H_1(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}$$
$$H_2(s) = \frac{2(s+1.1)}{1.1(s+1)(s+2)} = \frac{0.18}{s+1} + \frac{1.64}{s+2}$$

Both have the same DC gain (= transfer function value at $s = 0$)

A zero near a pole reduces the effect of the pole on response:

$(s-p)$ reduces its effect on $H(s)$: $C_1 = (s - p_1)H(s)|_{s=p_1}$

Example: A second-order system with zero $s = -\alpha\zeta\omega_n = -\alpha\sigma$

$$H(s) = \frac{(s / \alpha\zeta\omega_n) + 1}{(s / \omega_n)^2 + 2\zeta(s / \omega_n) + 1}$$

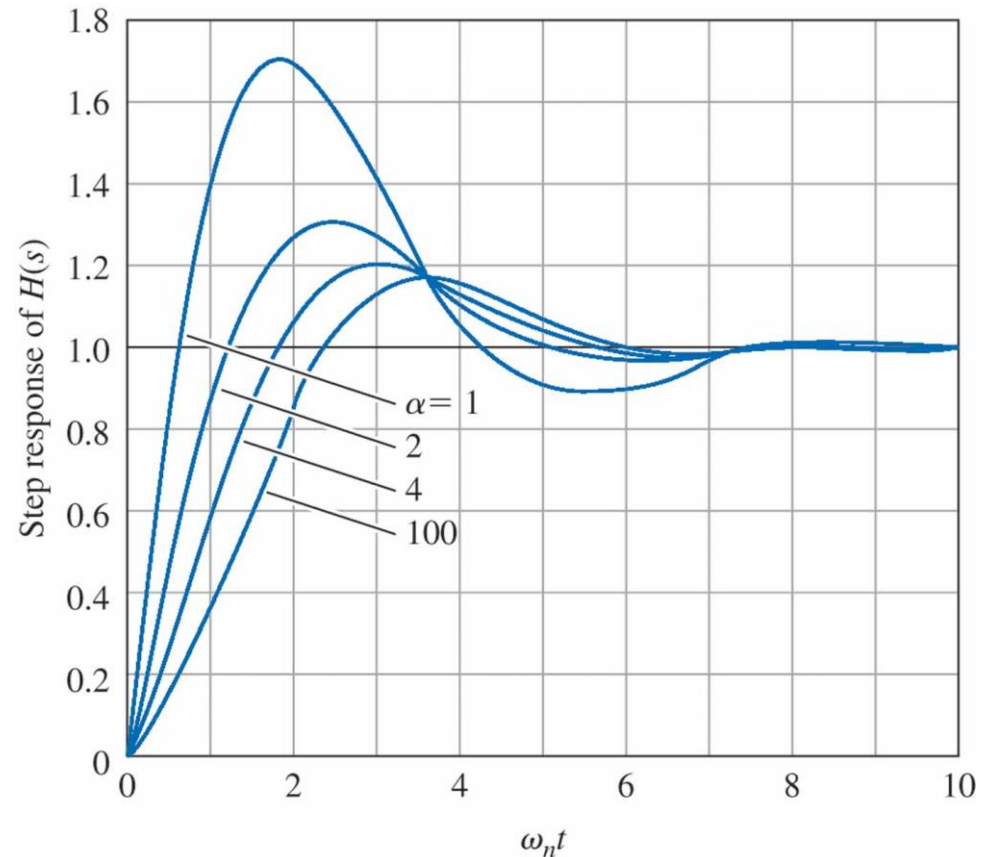
If $\alpha > 0 \rightarrow$ a zero in the LHP:

If $\alpha \cong 1$

- $\rightarrow$ value of zero is close to the real part of the poles
- $\rightarrow$ substantial influence on the response

If α is large

- $\rightarrow$ zero is far from poles
- $\rightarrow$ zero has only a little effect on the response



Plots of the step response of a second-order system with a zero $-\alpha\sigma$ ($\zeta = 0.5$)

In particular, if $\omega_n=1$:

$$H(s) = \frac{(s/\alpha\zeta) + 1}{s^2 + 2\zeta s + 1}$$

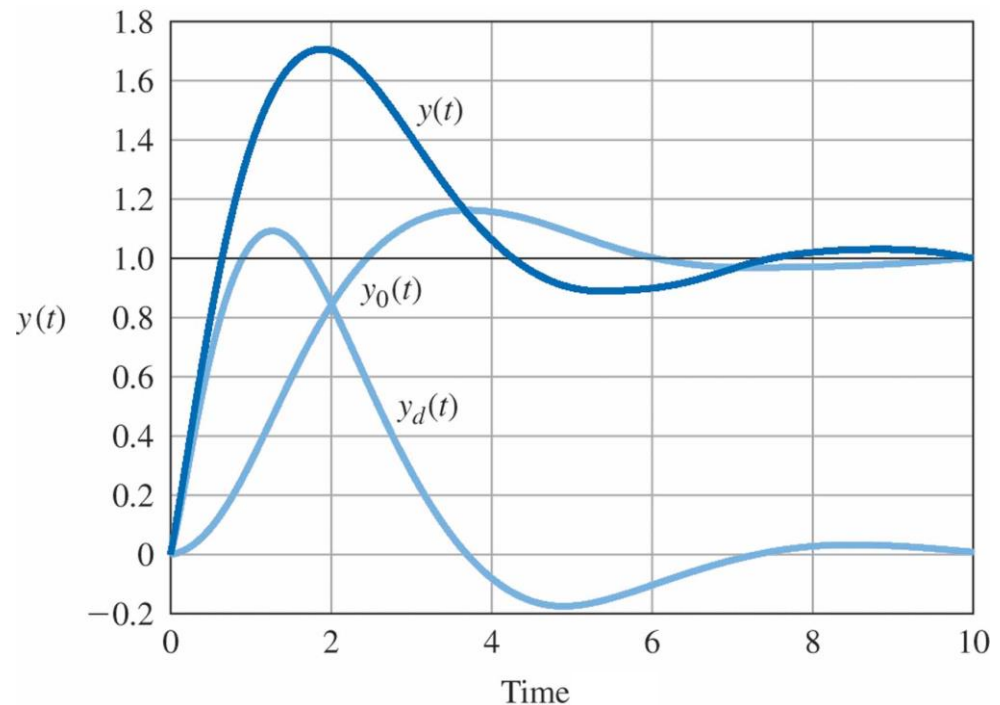
$$= \underbrace{\frac{1}{s^2 + 2\zeta s + 1}}_{H_0} + \underbrace{\frac{1}{\alpha\zeta} \frac{s}{s^2 + 2\zeta s + 1}}_{H_d}$$

If $\alpha > 0$
overshoot

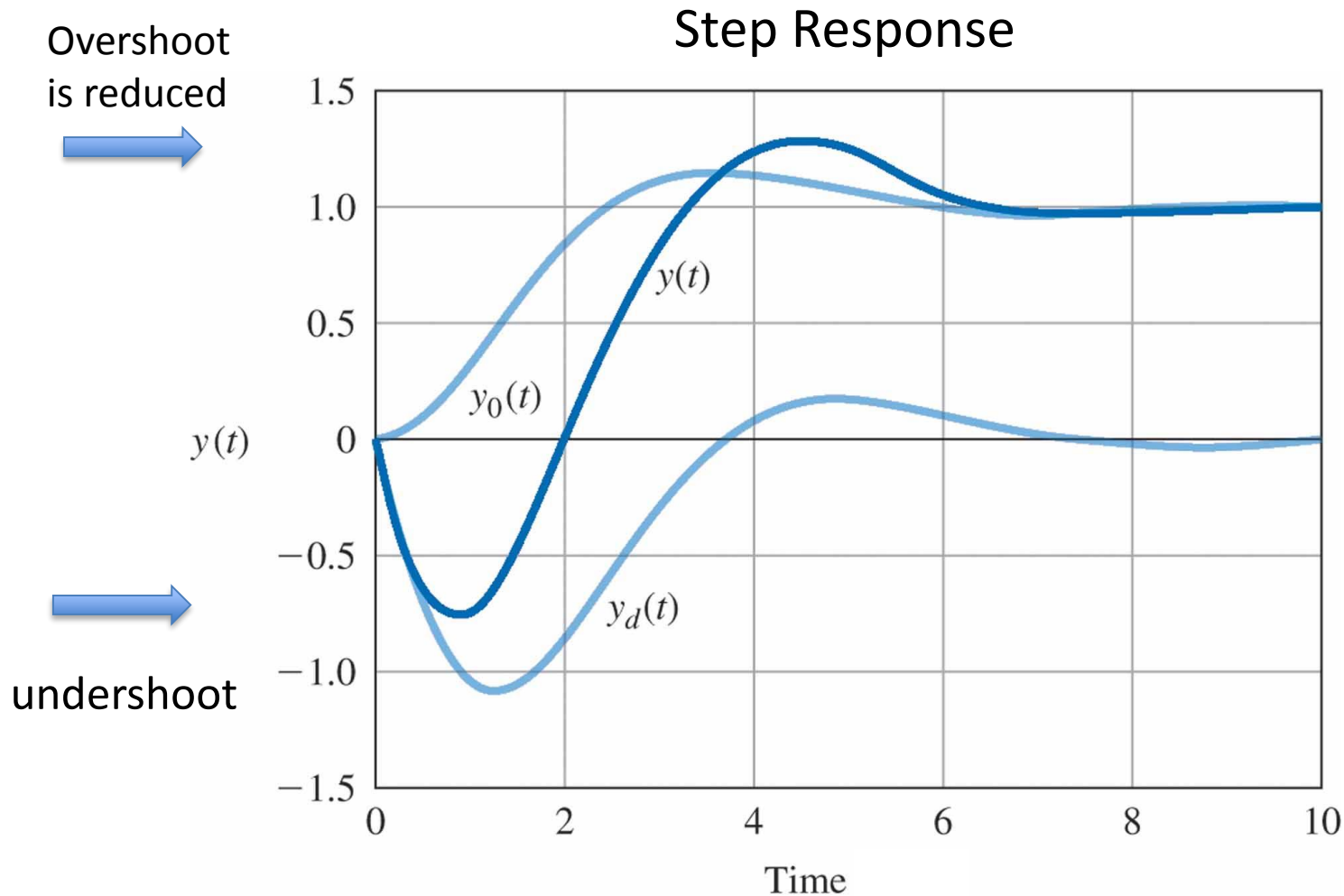


Step Response:

$$\begin{aligned} y(t) &= y_0(t) + y_d(t) \\ &= y_0(t) + \frac{1}{\alpha\zeta} \dot{y}_0(t) \end{aligned}$$



If $\alpha < 0 \rightarrow$ a zero in the RHP: non-minimum-phase system



Note: An LTI system is said to be **minimum-phase** if the system and its inverse are causal and stable.

Effect of Zeros on Transient Response

Example:

A second-order system with a finite zero and a unit DC gain:

$$H(s) = \frac{24}{z} \frac{(s + z)}{(s + 4)(s + 6)}$$

Determine the effect of the zero location ($s = -z$) on the **unit step response** when $z = \{1, 2, 3, 4, 5\}$

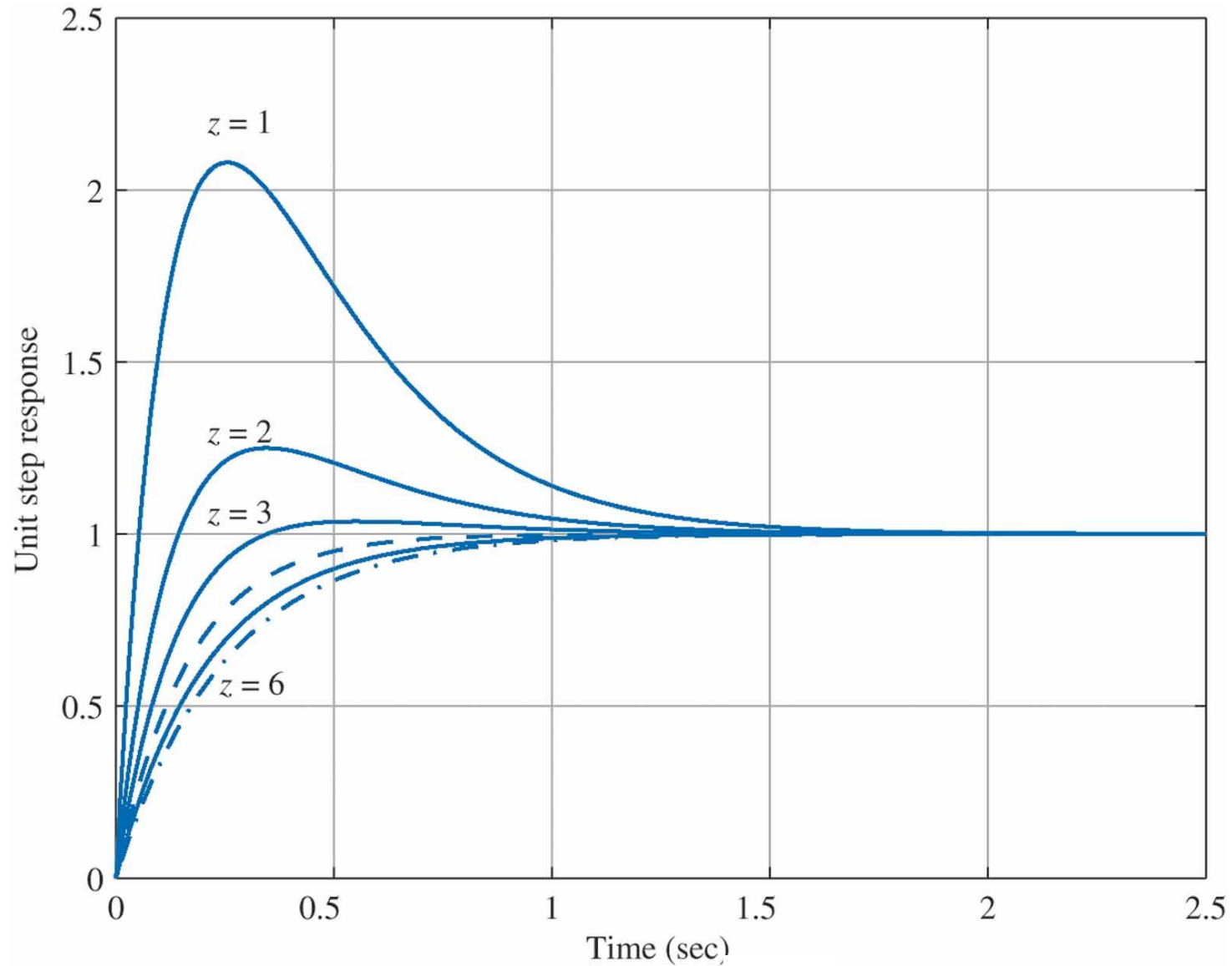
Solution:

$$Y(s) = \frac{24}{z} \frac{s}{s(s + 4)(s + 6)} + \frac{24}{s(s + 4)(s + 6)}$$

Step Response:

$$y(t) = 1 + \left(\frac{12}{z} - 3 \right) e^{-4t} + \left(2 - \frac{12}{z} \right) e^{-6t}$$

Effect of Zero z on Transient Response



Effect of Complex Zeros on Transient Response

Example:

Consider a third-order feedback system with the transfer function having a pair of lightly-damped poles and a pair of complex zeros:

$$H(s) = \frac{(s + \alpha)^2 + \beta^2}{(s + 1)[(s + 0.1)^2 + 1]}$$

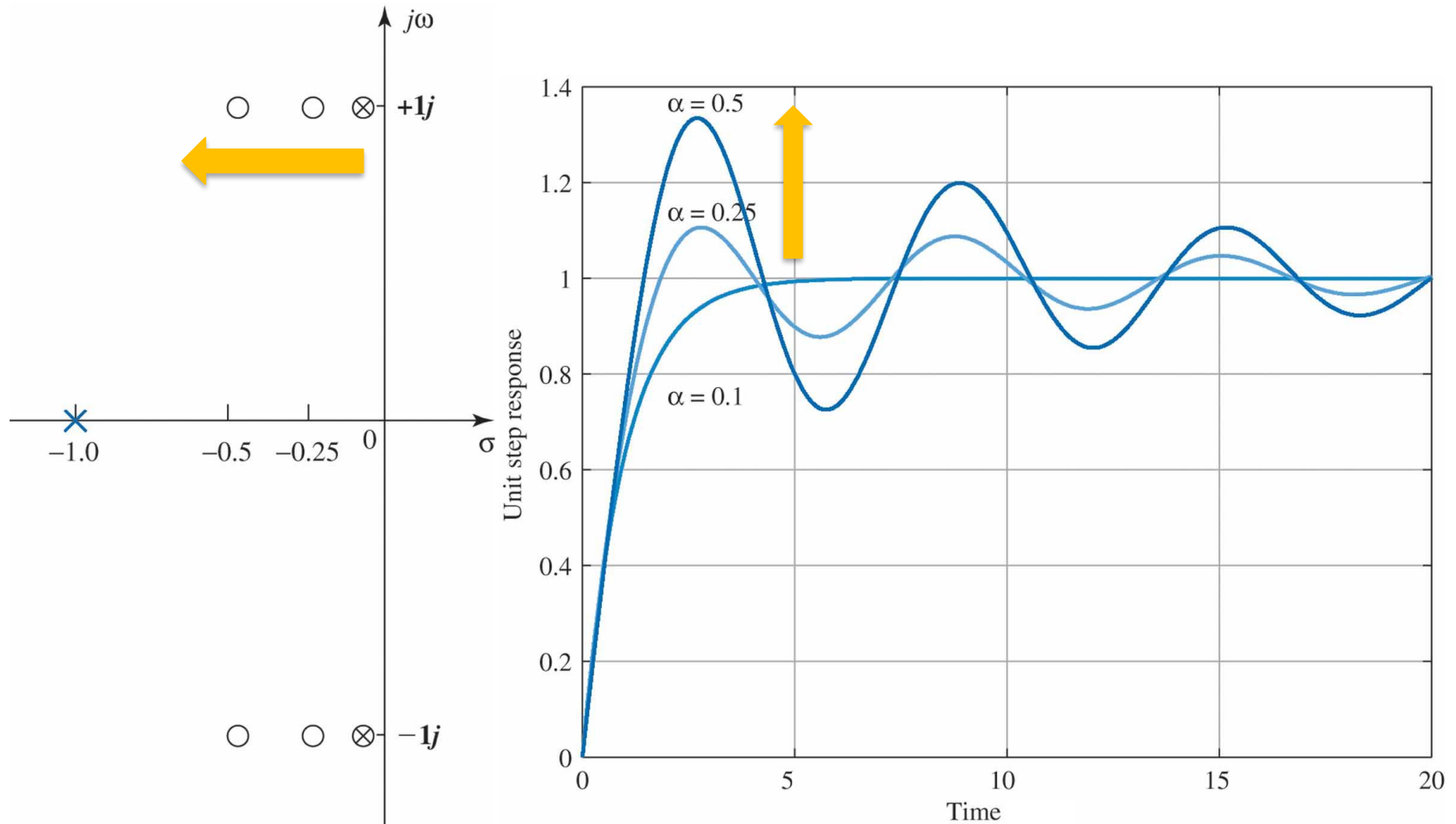
Determine the effect of the complex zero locations on the **unit step response** of the system, when

$$(\alpha, \beta) = (0.1, 1)$$

$$(\alpha, \beta) = (0.25, 1)$$

$$(\alpha, \beta) = (0.5, 1)$$

Effect of Complex Zeros on Transient Response



Effect of Real Poles on Transient Response

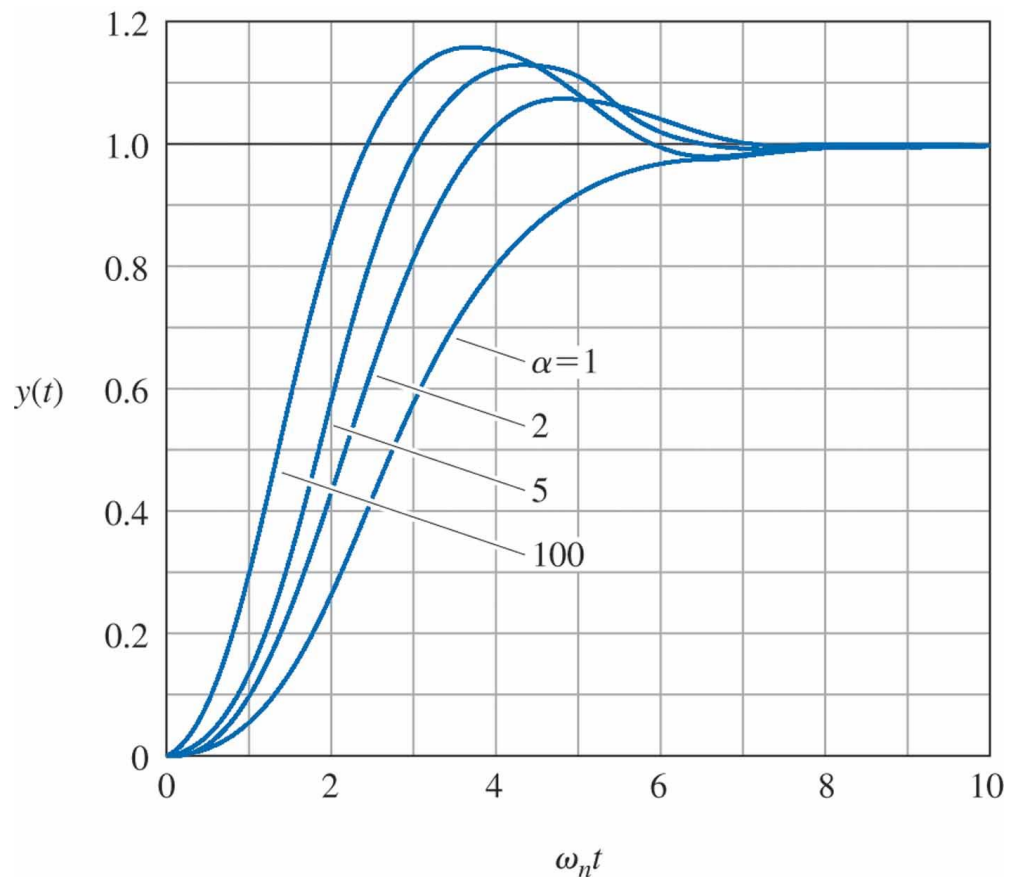
Example:
$$H(s) = \frac{1}{[(s / \alpha \zeta \omega_n) + 1][(s / \omega_n)^2 + 2\zeta(s / \omega_n) + 1]}$$

Real pole: $-\alpha \zeta \omega_n$

Step response of the
third-order systems
with $\zeta = 0.5$

Major effect:

Smaller $\alpha > 0$
(pole still in LHP)
increases rise time



Summary

Effects of Pole-Zero Patterns on Dynamic Response

- Locations of poles in the s -plane determine the characteristics of the response
- A zero in LHP will increase the overshoot if the zero is within a factor of 4 of the real part of the complex poles
- A zero in RHP will reduce the overshoot (but may cause the step response to move to a wrong direction: undershoot)
- A real pole in LHP will increase the rise time significantly if the pole is within a factor of 4 of the real part of the complex poles

END