

2. Dynamic Circuits: First-Order Transient

I. Components

- Capacitors (C) [Section 6.2, 6.3]
- Inductors (L) [Section 6.4, 6.5]
- C & L vs. R $\rightarrow$ Storage vs. Dissipation [Section 6.1]

Alexander & Sadiku,
“Fundamentals of Electric Circuits”
5th Edition Chapters 6, 9

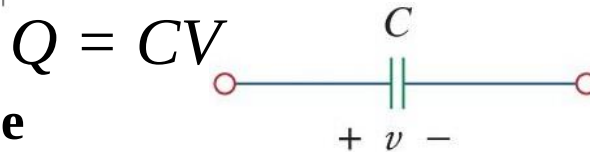
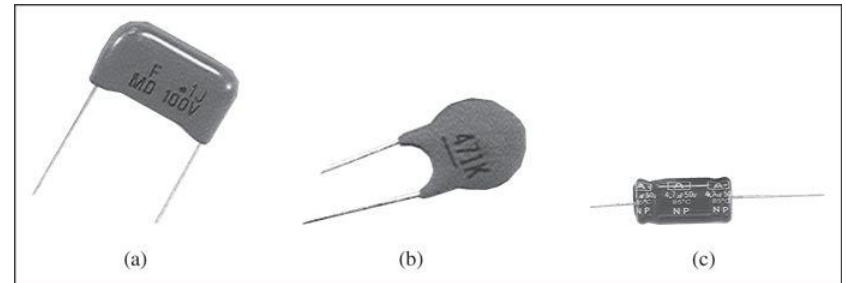
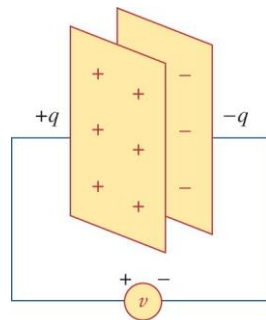
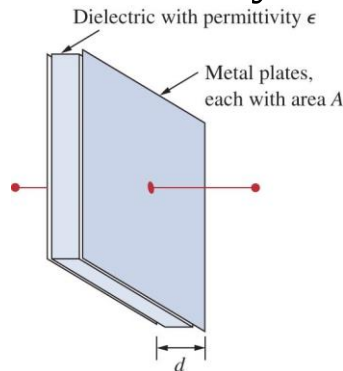
II. Simple RC and RL Circuits

III. Transient Solutions

Capacitor and Capacitance

Capacitor

- Simplest way to form a capacitor is to sandwich an insulator (technical term is dielectric) between a pair of parallel conducting plates
- Hence the symbol for a capacitor is two parallel lines separated by a gap

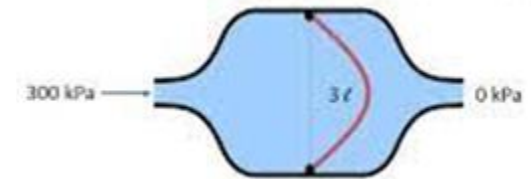
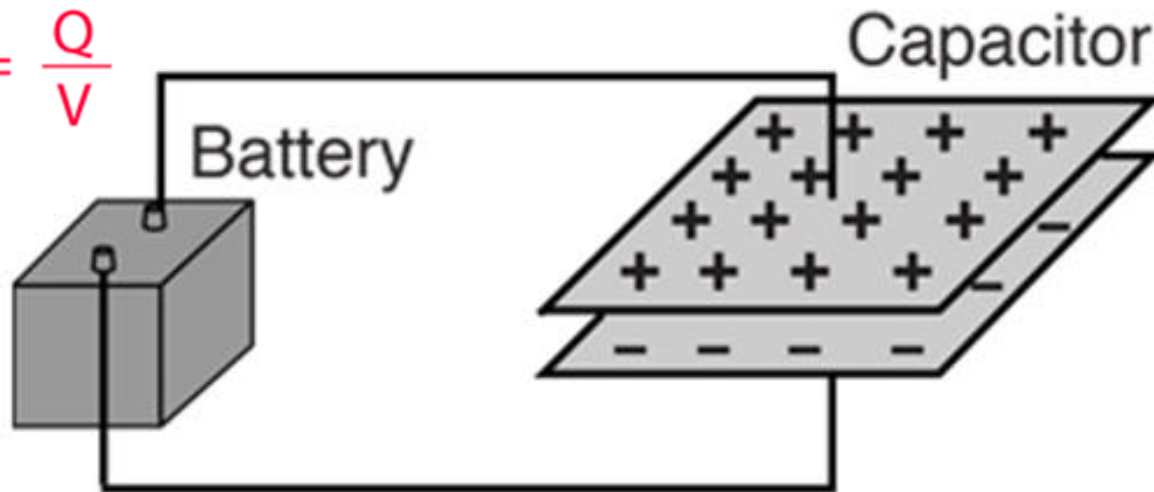


Capacitance

- Commonly symbolized by the letter C with unit of Farads (F)
- Relates the amount of charge stored for a given voltage applied
- No energy dissipated unlike resistors
- Energy is stored (in keeping the plates apart) $E = \frac{1}{2}Cv^2$ J

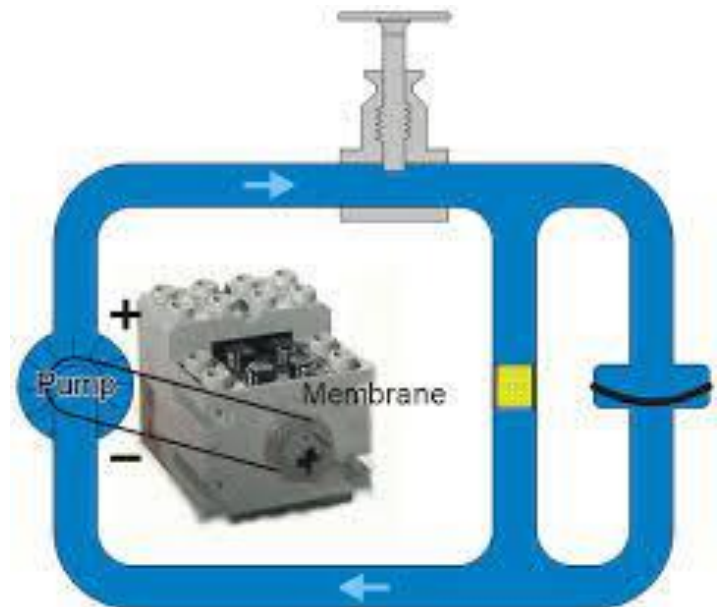
Analogy of a Capacitor

$$C = \frac{Q}{V}$$



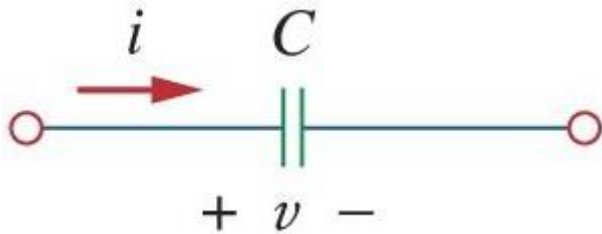
$$\text{Capacitance (C)} = \frac{\text{charge stored (Q)}}{\text{voltage applied (V)}}$$

Since more charges can be stored at a higher voltage, so capacitance is expressed at charges per unit voltage



I-V relation in a capacitor

If the voltage across the capacitor is time-varying, then the charge stored on the capacitor must also be time-varying:



$$Q(t) = CV(t) \quad \text{Differentiating with respect to time (t):}$$

$$i(t) = C \frac{dV(t)}{dt}$$

Current depends on the
rate of change of voltage

Case study to consider:

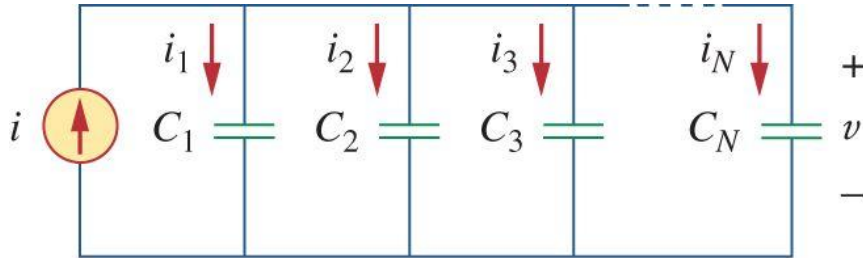
Given that $C = 1 \text{ nF}$

If $V = 1\text{V} \rightarrow Q = \underline{1 \text{ nC}}$

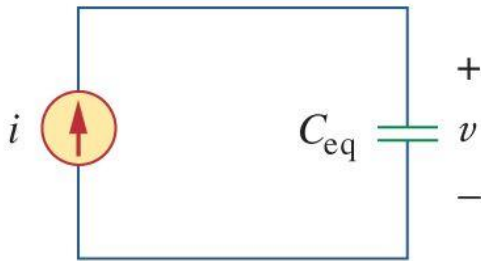
If we reverse V , such that now $V = -1\text{V} \rightarrow Q = \underline{\hspace{2cm}}$

If the above change was made gradually over 1 ms , what would be the resulting current?

Capacitors in parallel



(a)

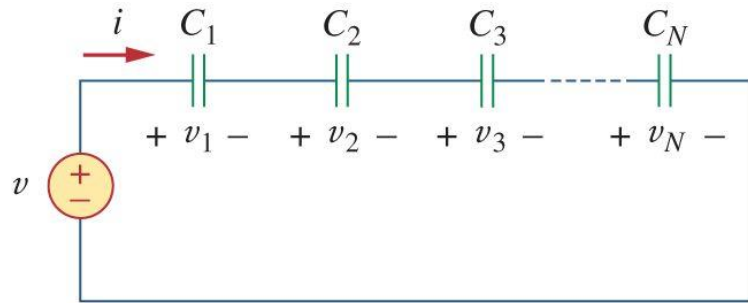


(b)

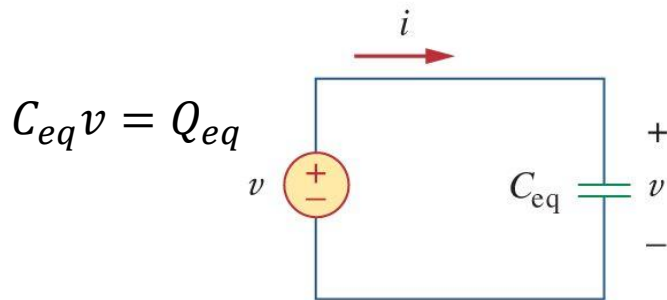
$$C_{eq} = C_1 + C_2 + C_3 + \dots + C_N$$

What is the value of Q_{eq} ?

Capacitors in series



(a)



(b)

$$v = v_1 + v_2 + v_3 + v_4$$

$$\Rightarrow \frac{Q_{eq}}{C_{eq}} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \frac{Q_3}{C_3} + \frac{Q_4}{C_4}$$

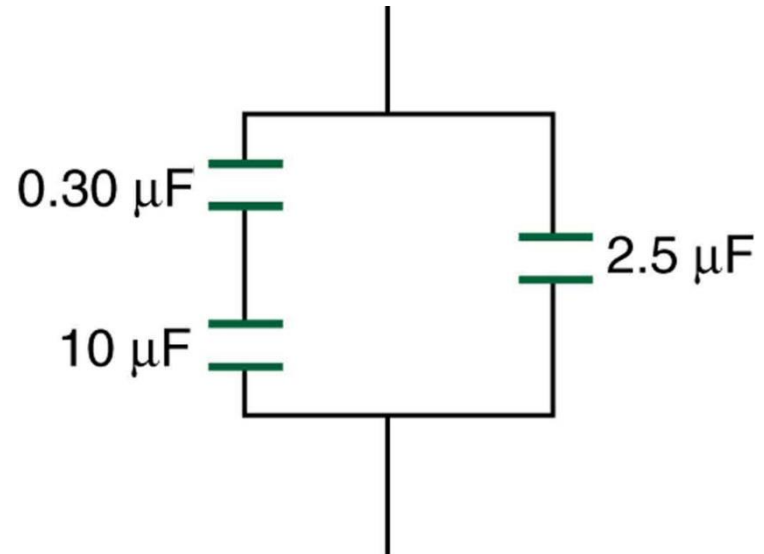
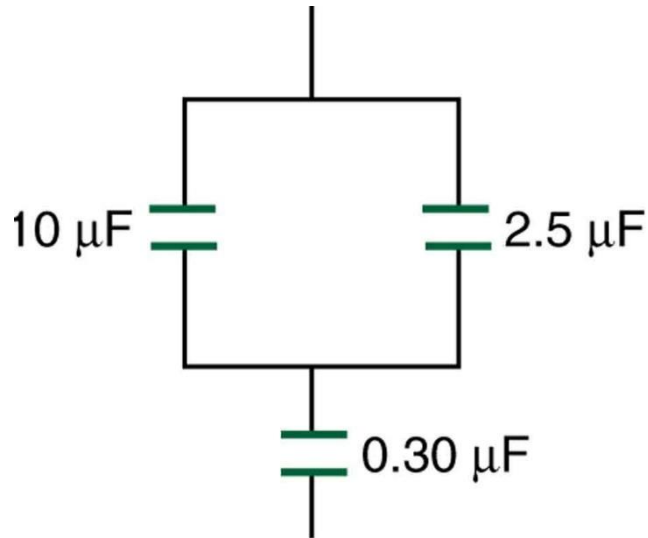
$$\Rightarrow \frac{1}{C_{eq}} \frac{dQ_{eq}}{dt} = \frac{1}{C_1} \frac{dQ_1}{dt} + \frac{1}{C_2} \frac{dQ_2}{dt} + \frac{1}{C_3} \frac{dQ_3}{dt} + \frac{1}{C_4} \frac{dQ_4}{dt}$$

$$\text{Since } i = \frac{dQ_{eq}}{dt} = \frac{dQ_1}{dt} = \frac{dQ_2}{dt} = \frac{dQ_3}{dt} = \frac{dQ_4}{dt} \Rightarrow$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_N}$$

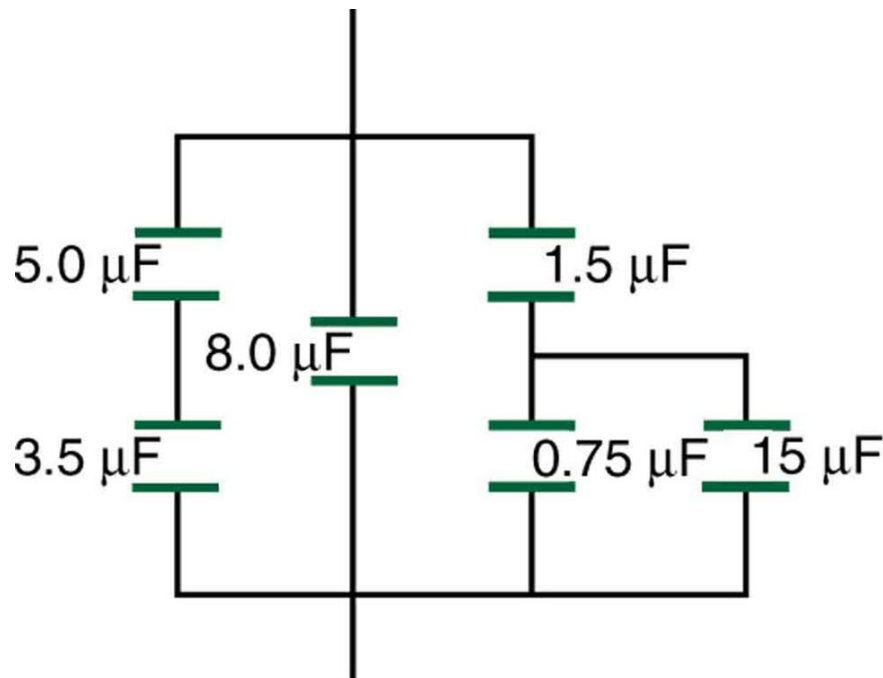
Example on Capacitor in parallel and series

What is the equivalent of the following capacitance circuit?



Example on Capacitor in parallel and series

What is the equivalent of the following capacitance circuit?



Response in Capacitor

$$+ \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ - \end{array} V \quad Q(t) = CV(t) \xrightarrow[\text{with respect to time}]{\text{Differentiate}} i(t) = C \frac{dV(t)}{dt}$$

If voltage is constant with time: $dV/dt = 0 \rightarrow I = 0$

No voltage change $\rightarrow$ No current

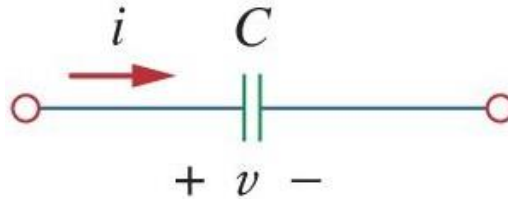
If voltage changes with time: $dV/dt \neq 0 \rightarrow I \neq 0$

Voltage change $\rightarrow$ Charge changes $\rightarrow$ Current

Since $i(t)$ must be finite and continuous, the **voltage changes must be continuous with time**. Otherwise, i will be infinite if V has discontinuities.

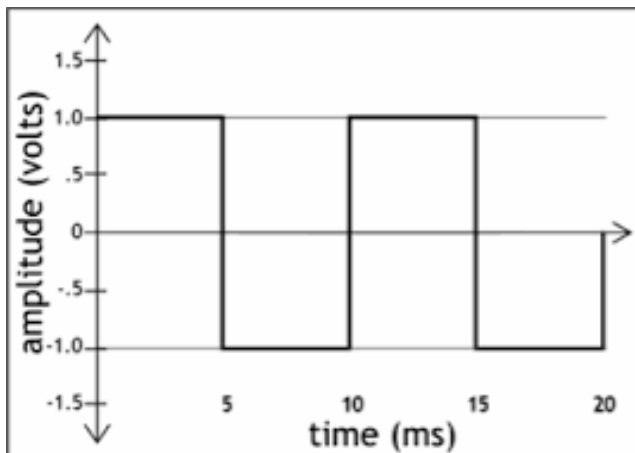
Examples

$$i(t) = C \frac{dv(t)}{dt}$$



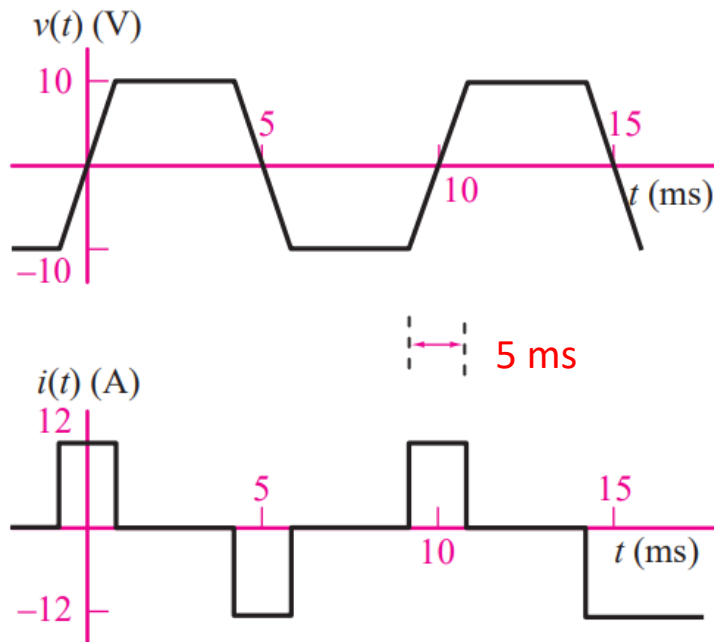
Case 1: If $v(t) = 10 \sin(2t)$ V, $C = 10$ μ F, what is $i(t)$?

Case 2: If $v(t)$ is shown in the figure below, what is $i(t)$?



Example

If the plots shown in Figure P4.18 are the voltage across and the current through an ideal capacitor, determine the capacitance.



Summary: Capacitor response to DC

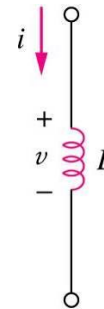
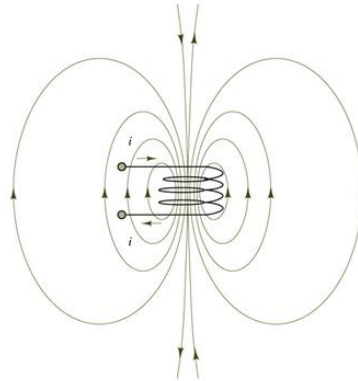
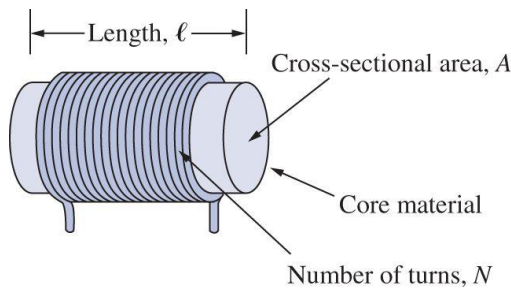
If the applied voltage is DC

- Insulating dielectric blocks the current from flowing through
- Plates will charge up
- At DC, capacitor blocks current from flowing through

Inductors and Inductance

Inductor

- Simplest way to form an inductor is to winding a coil around a core that concentrates magnetic field lines (flux)
- Hence symbol of an inductor is coil between two terminals



$$\phi = Li$$

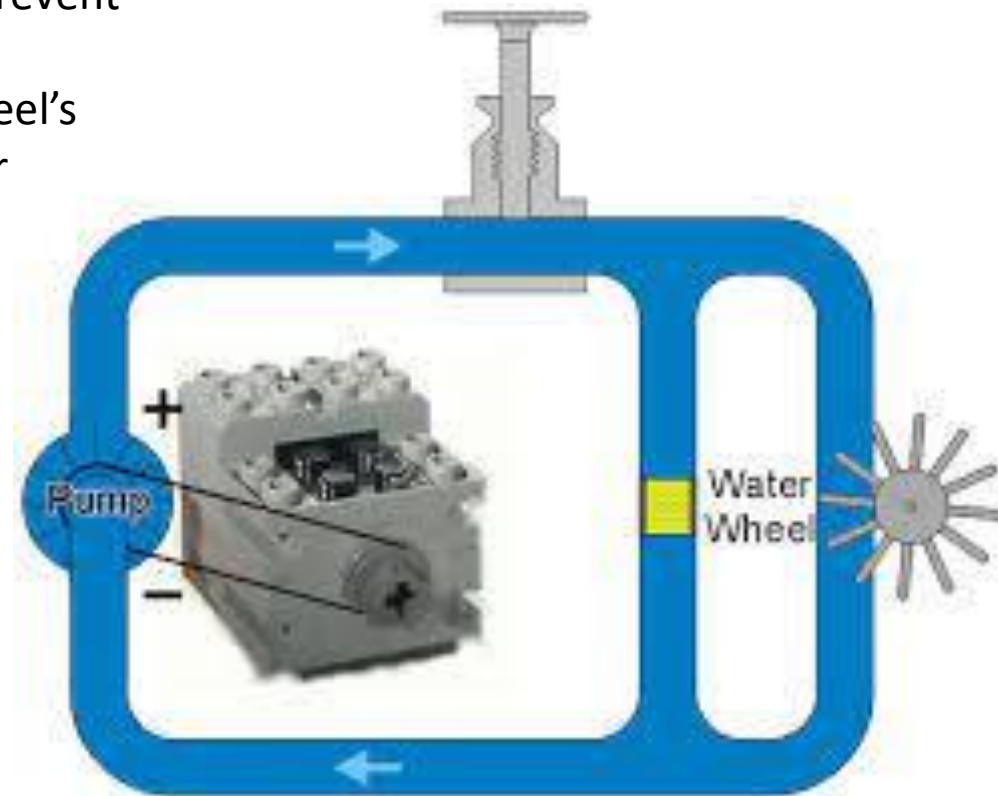
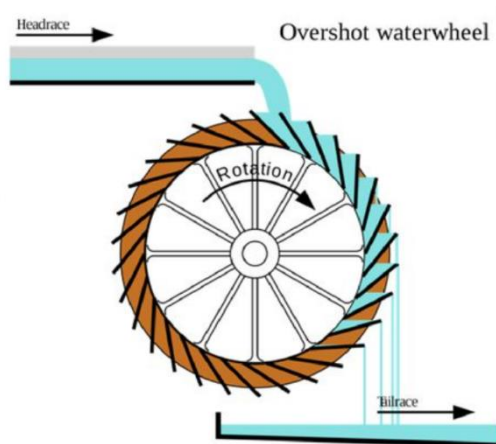


Inductance

- Inductance is commonly symbolized by letter L with unit of Henrys (H)
- Passing a current through an inductor produces a magnetic flux (ϕ) that is related to the inductance (L)
- No energy dissipated unlike resistors (note that wires are assumed to have no resistance by definition)

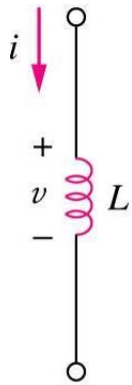
Analogy of an Inductor

- An inductor likes a water wheel. As the water starts flowing, the wheel will tend to prevent the flow but will gradually speed up.
- When the water stops flowing, the wheel's momentum will try and keep the water flowing for a bit.



I-V relation in an inductor

If the current through an inductor is time-varying, then the generated voltage must also be time-varying:



$$\text{Flux: } \phi = Li$$

$$\text{Faraday's Law: } v(t) = \frac{d\phi(t)}{dt}$$

$$\text{Differentiating with respect to time (t): } v(t) = L \frac{di(t)}{dt}$$

Voltage depends on the **rate of change** of current

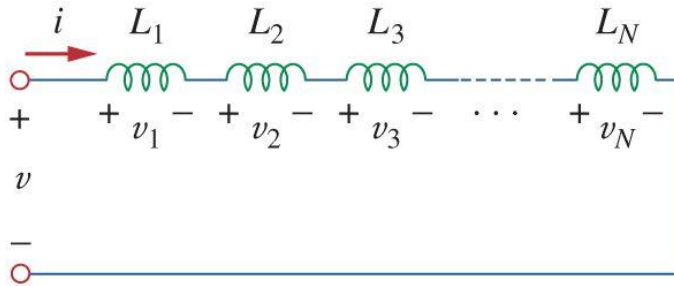
If current is constant with time $\rightarrow di/dt = 0 \rightarrow V = 0$ (No voltage)

No current change $\rightarrow$ No voltage difference

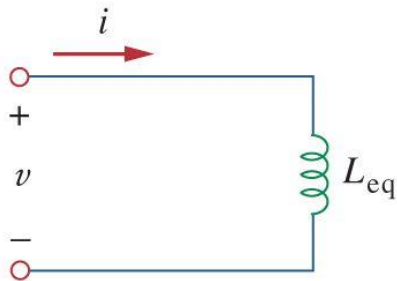
If current changes with time $\rightarrow di/dt \neq 0 \rightarrow V \neq 0$ (There is voltage)

Current change $\rightarrow$ Voltage difference

Inductors in series



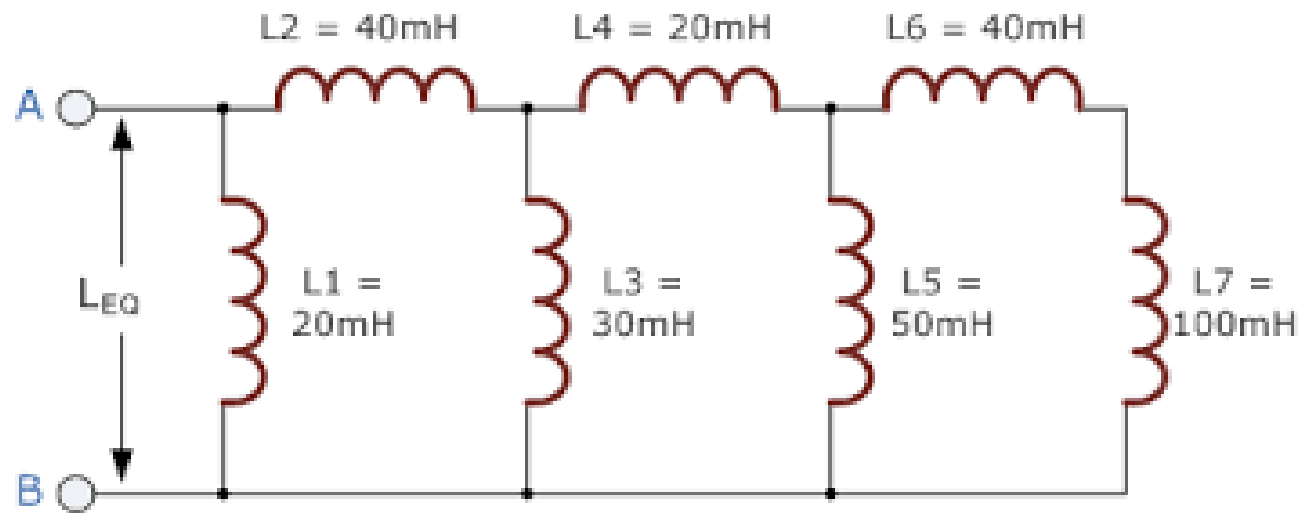
(a)



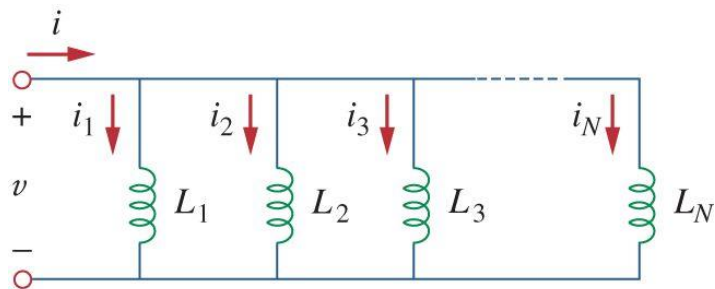
(b)

$$L_{eq} = L_1 + L_2 + L_3 + \dots + L_N$$

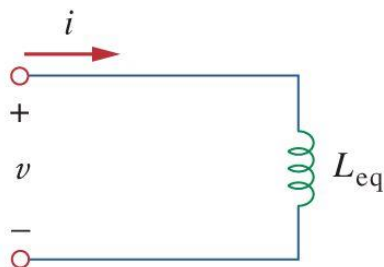
Example on Inductor in parallel and series



Inductors in parallel



(a)



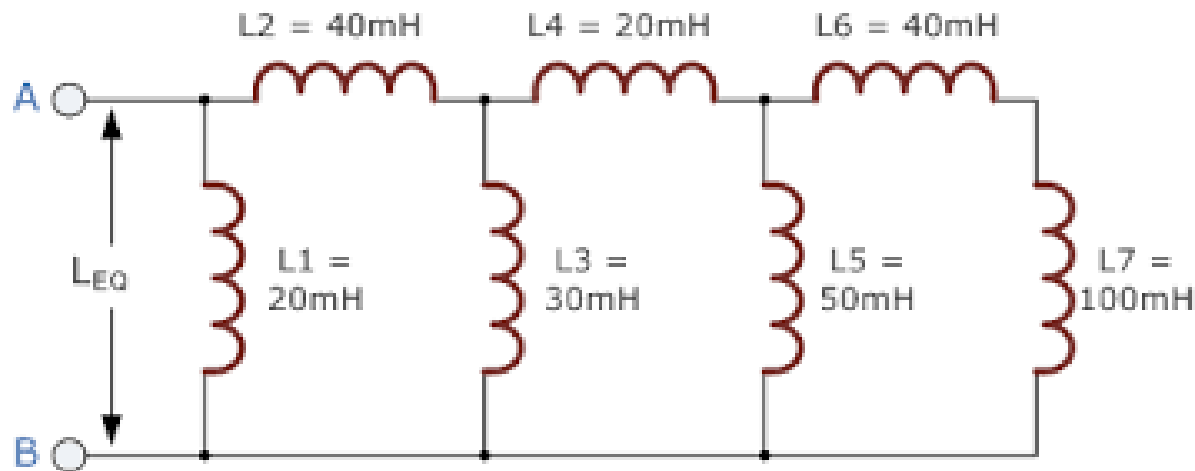
(b)

$$\frac{1}{L_{eq}} \int v(t) dt = \frac{1}{L_1} \int v(t) dt + \frac{1}{L_2} \int v(t) dt + \frac{1}{L_3} \int v(t) dt + \cdots + \frac{1}{L_N} \int v(t) dt$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}$$

Example on Inductor in parallel

- What is the equivalent of the following inductance circuit?



Important characteristics of the basic elements

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i	$v = iR$	$v = \frac{1}{C} \int_{t_0}^t i(\tau) d\tau + v(t_0)$	$v = L \frac{di}{dt}$
i-v	$i = v/R$	$i = C \frac{dv}{dt}$	$i = \frac{1}{L} \int_{t_0}^t v(\tau) d\tau + i(t_0)$
Two in series	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel	$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$
At DC	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly	Not applicable	v	i

When a circuit has C and/or L, the circuit becomes dynamic.

- Voltage and/or current is a function of time.
- Voltage and/or current is described by differential equation.
- The circuit has
 - › **transient response** (circuit response immediately after certain initial condition) AND
 - › **steady state response** (as $t \rightarrow \infty$)

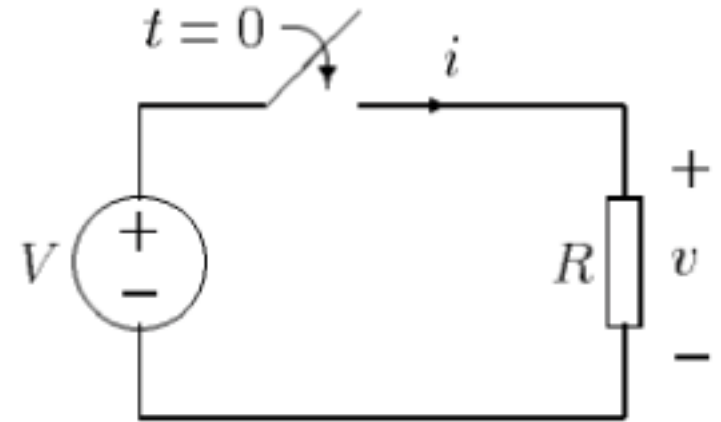


Note that pure resistive circuits have no transient!

When we turn on the switch at $t = 0$, potential difference across the resistor R becomes V immediately.

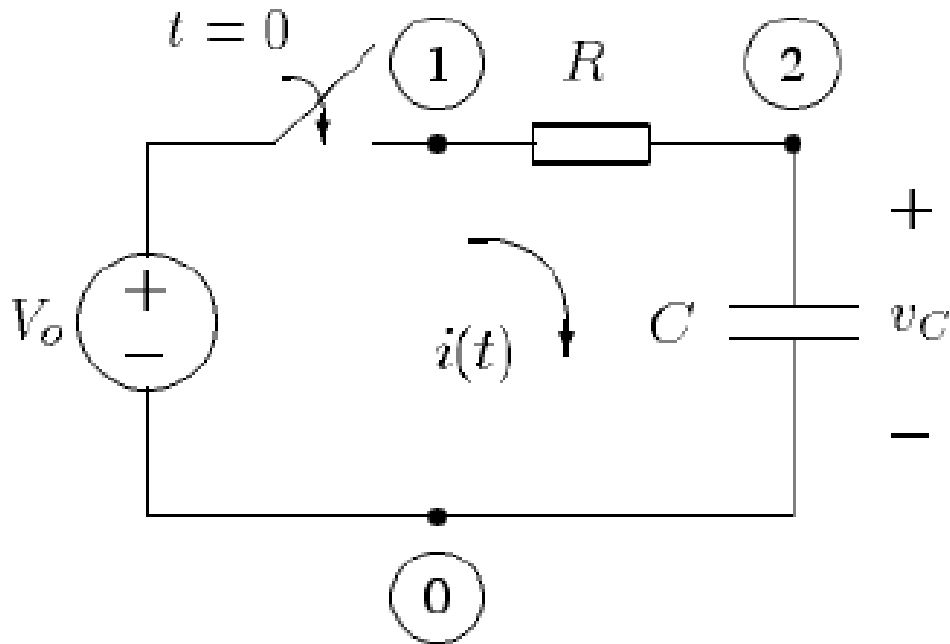
For all $t \geq 0$,

- $v = V = iR$
- $i = V/R$



Simple first-order RC circuit

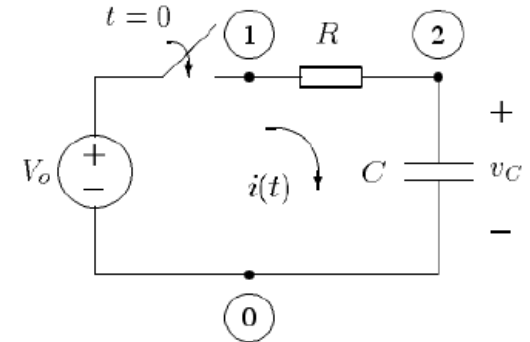
If V_o is a constant voltage, how the current $i(t)$ and the voltage across the capacitor C , $v_c(t)$ will change when the switch is closed at $t = 0$



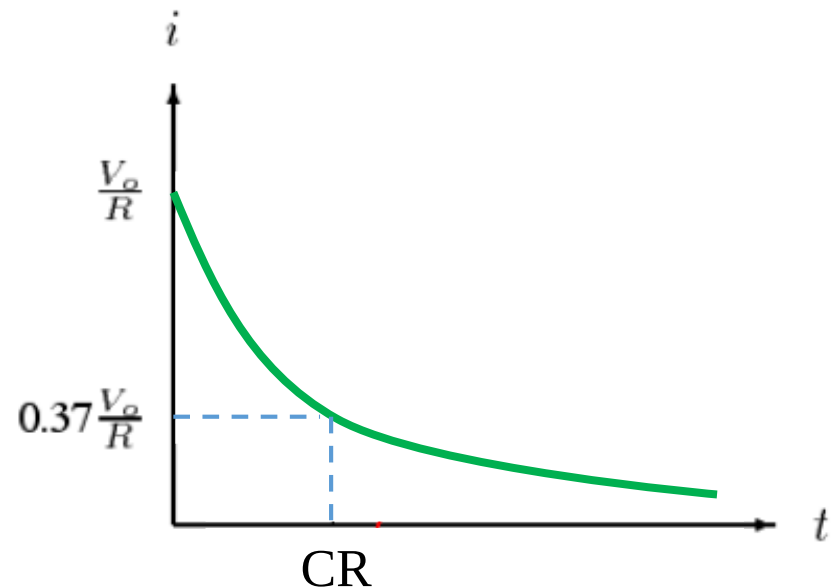
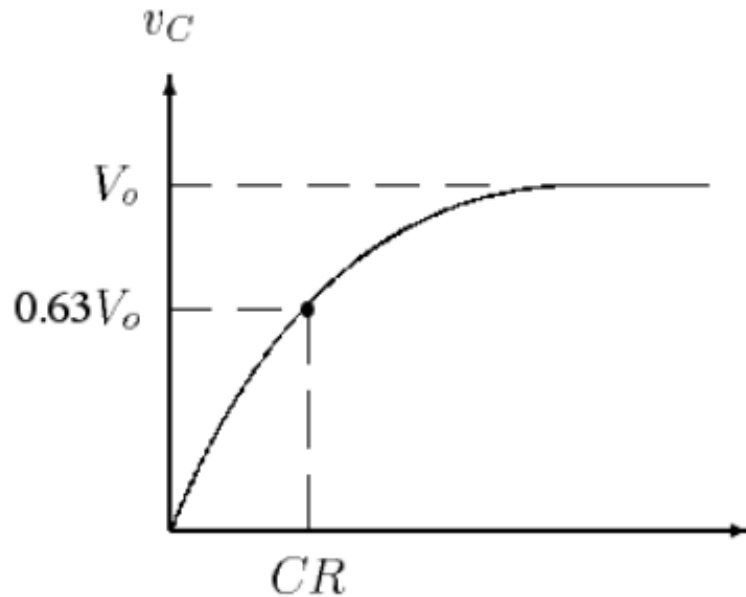
Simple first-order RC circuit

- At $t = 0$, $v_c(0) = 0$
- After $t = 0$, the circuit is closed:
 - › $i(t) = \frac{v_R}{R} = C \frac{dv_c}{dt}$, and
 - › $v_R(t) = V_0 - v_c(t)$
- Hence, $\frac{V_0 - v_c(t)}{CR} = \frac{dv_c}{dt} \Rightarrow \frac{dv_c}{dt} + \frac{v_c}{CR} = \frac{V_0}{CR}$
- The general solution is $v_c(t) = Ae^{-\frac{t}{CR}} + V_0$ for $t \geq 0$ (A is a constant)
- As the **initial condition is $v_c(0^+) = 0$** , $A + V_0 = 0 \Rightarrow A = -V_0$
- Hence,

$$v_c(t) = V_0 \left(1 - e^{-\frac{t}{CR}} \right)$$



Transient response of a simple RC circuit



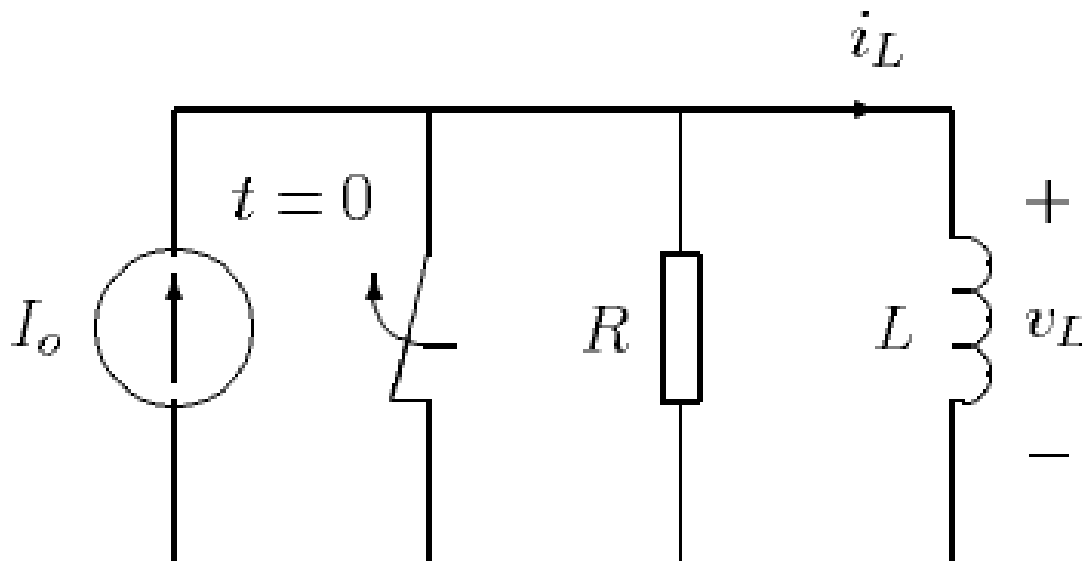
$$v_c(t) = V_0 \left(1 - e^{-\frac{t}{CR}}\right) \quad \Rightarrow \quad i(t) = C \frac{dv_c}{dt} = \frac{V_0}{R} e^{-\frac{t}{CR}}$$

When $t = CR$,

$$v_c(CR) = V_0(1 - e^{-1}) = 0.63V_0 \quad \text{and} \quad i(CR) = \frac{V_0}{R} e^{-1} = 0.37V_0$$

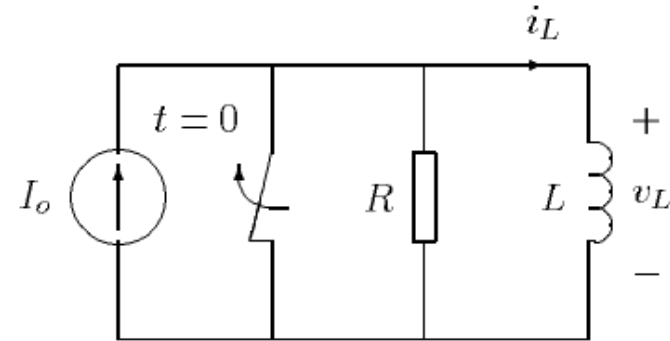
Simple first-order RL circuit

If I_o is a constant current, how the current $i_L(t)$ and the voltage across the inductor $v_L(t)$ will change when the switch is opened at $t = 0$



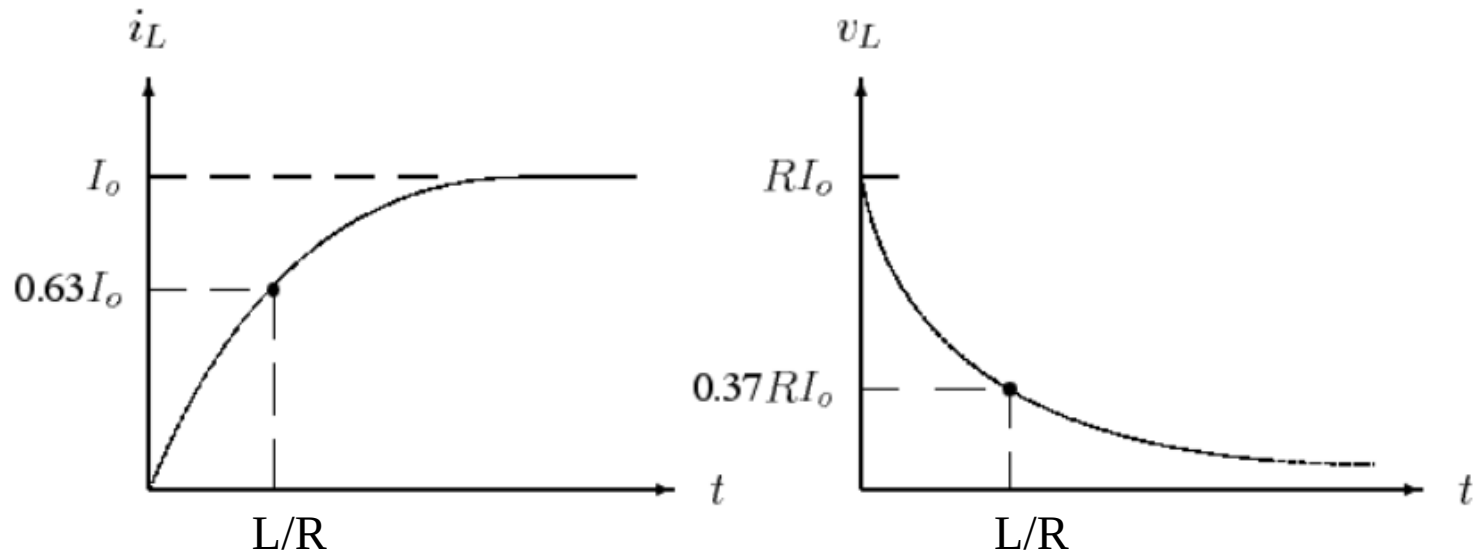
Simple first-order RL circuit

- When $t < 0$, short-circuit at switch, $i_L = 0$
- For $t \geq 0$,
 - › $v_L(t) = L \left(\frac{di_L}{dt} \right)$, and
 - › $v_R(t) = (I_0 - i_L(t))R$
- Hence, $(I_0 - i_L(t))R = L \left(\frac{di_L}{dt} \right) \Rightarrow \frac{di_L}{dt} + \frac{R}{L} i_L(t) = \frac{RI_0}{L}$
- The general solution is $i_L(t) = Ae^{-\frac{Rt}{L}} + I_0$ where A is a constant
- $i_L(0^+) = 0 = A + I_0 \Rightarrow A = -I_0$
- Hence, for $t \geq 0$, we obtain



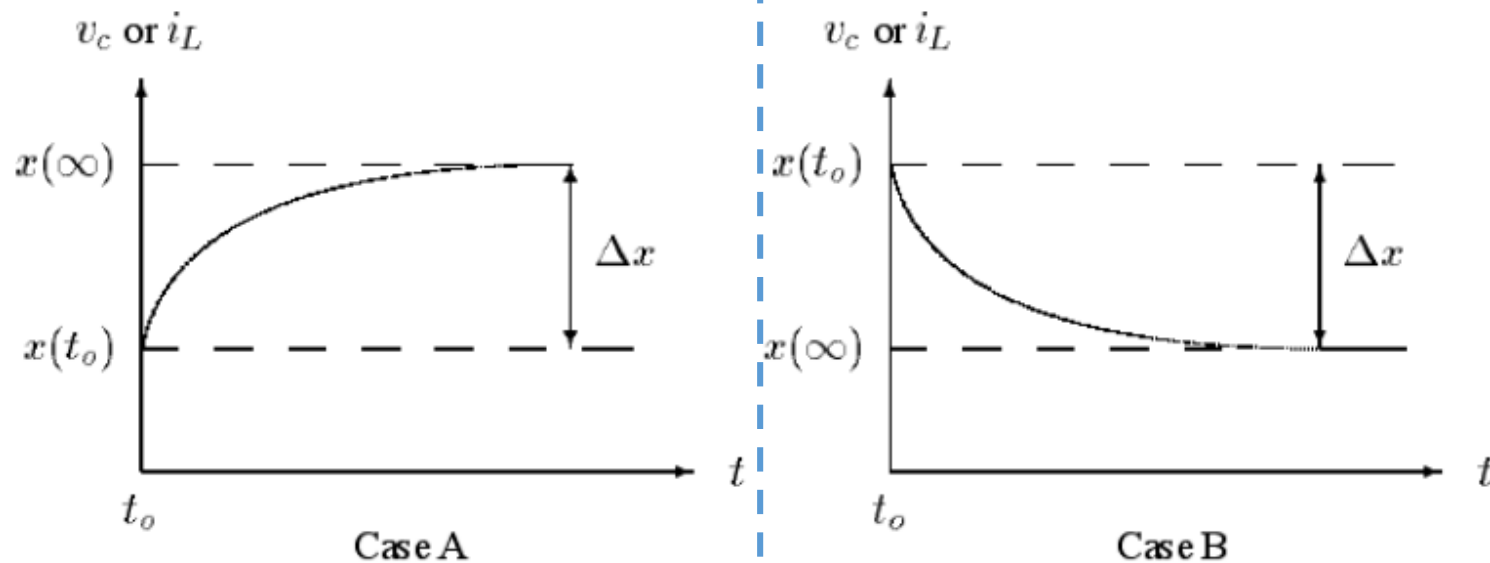
$$i_L(t) = I_0 \left(1 - e^{-\frac{Rt}{L}} \right)$$

Transient response of a simple RL circuit



$$i_L(t) = I_0 \left(1 - e^{-\frac{Rt}{L}}\right) \quad \Rightarrow \quad v_L(t) = L \frac{di_L(t)}{dt} = RI_0 e^{-\frac{Rt}{L}}$$

General first-order solution



$$x(t) = x(t_0) + \Delta x(1 - e^{-(t-t_0)/\tau}) \quad x(t) = x(\infty) + \Delta x e^{-(t-t_0)/\tau}$$

1. Find the time constant τ
2. Find initial value $x(t_0)$ and final value $x(\infty)$
3. Determine whether Case A or Case B expression shall be used

Time constant τ

- Simple 1st order RC circuit

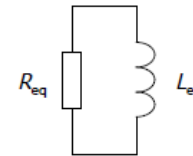
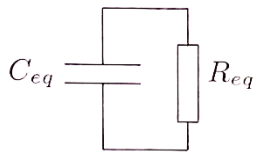
$$\tau = CR$$

- Simple 1st order RL circuit

$$\tau = \frac{L}{R}$$

To find τ from an equivalent simple RC or RL circuit

1. Short-circuit all voltage sources and open-circuit all current sources
2. Place switches in their final positions
3. Reduce resistances to one equivalent resistance R_{eq} , if possible



Reduce capacitances to one equivalent capacitance C_{eq} (if possible) $\rightarrow$

Time constant of any RC circuit

$$\tau = C_{eq}R_{eq}$$

Reduce inductances to one equivalent inductance L_{eq} (if possible) $\rightarrow$

Time constant of any RL circuit

$$\tau = \frac{L_{eq}}{R_{eq}}$$

Example 1: Initial and final values given

- At $t = 0$, the switch is thrown to the right
- Initial and final values of capacitor voltage are:

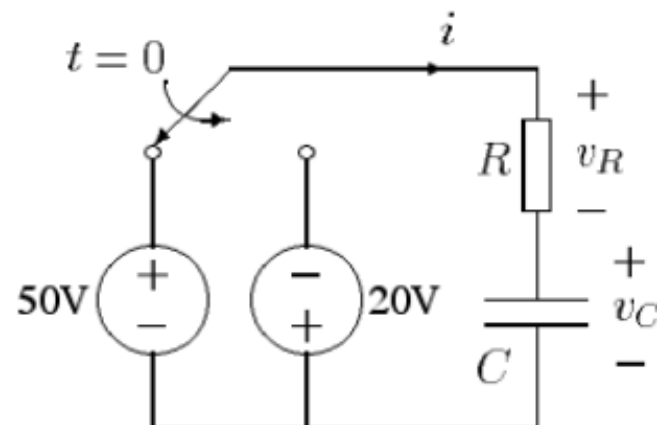
$$v_C(0^+) = 50V \text{ and } v_C(\infty) = -20V$$

- Decrease in $v_C \rightarrow$ Case B ($\tau = CR$)

$$v_C(t) = -20 + 70e^{-\frac{t}{CR}}$$

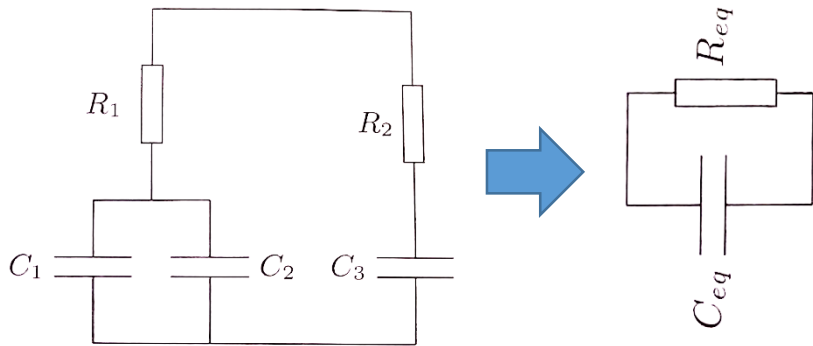
- $v_R(t) = -20 - v_C(t) = -70e^{-\frac{t}{CR}}$

- $i(t) = \frac{v_R(t)}{R} = -\frac{70}{R}e^{-\frac{t}{CR}}$



Example 2 (non-trivial boundary conditions)

- At $t = 0$, the switch is thrown to the right

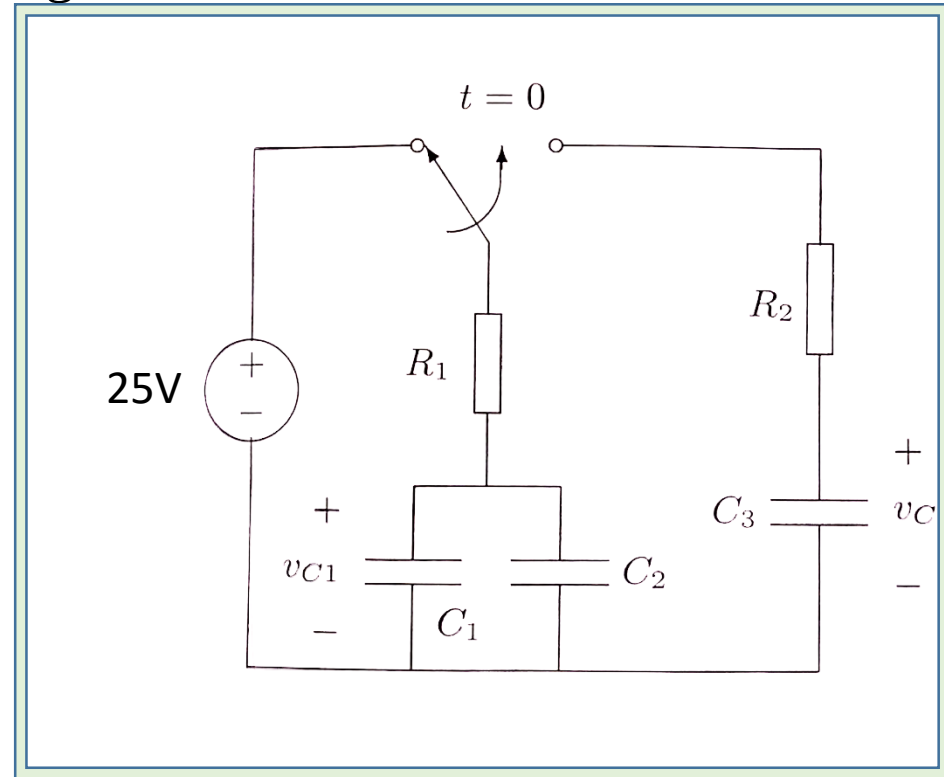


→ Find equivalent resistance:

$$R_{eq} = R_1 + R_2$$

→ Find equivalent capacitance

$$\frac{1}{C_{eq}} = \frac{1}{C_1 + C_2} + \frac{1}{C_3} \Rightarrow$$
$$C_{eq} = \frac{C_3(C_1 + C_2)}{(C_1 + C_2 + C_3)}$$



Example 2 (non-trivial boundary conditions)

- Time constant $\tau = \frac{C_3(C_1+C_2)(R_1+R_2)}{C_1+C_2+C_3}$

- Initial values are

$$v_{c_1}(0^+) = 25, \quad v_{c_2}(0^+) = 25, \quad v_{c_3}(0^+) = 0$$

- To find the final values after the switch throw to right

$$(C_1 + C_2) \frac{dv_{c_1}}{dt} + C_3 \frac{dv_{c_3}}{dt} = 0$$

$\Rightarrow (C_1 + C_2)v_{c_1}(t) + C_3v_{c_3}(t) = \text{constant}$ for all $t \geq 0$

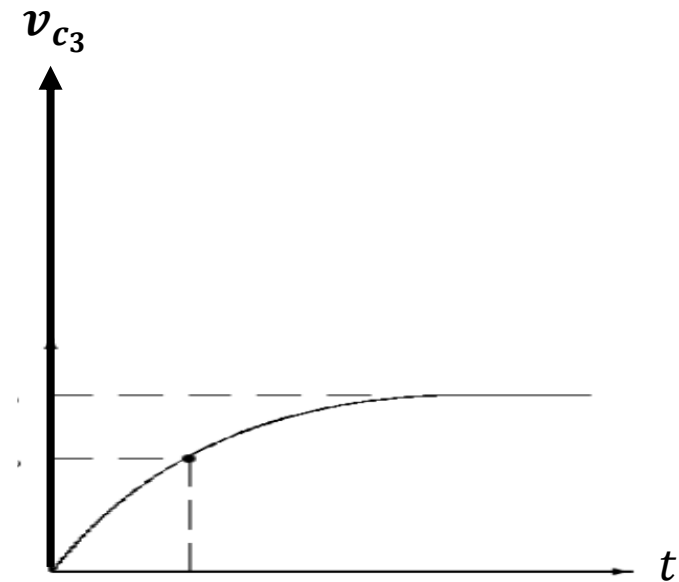
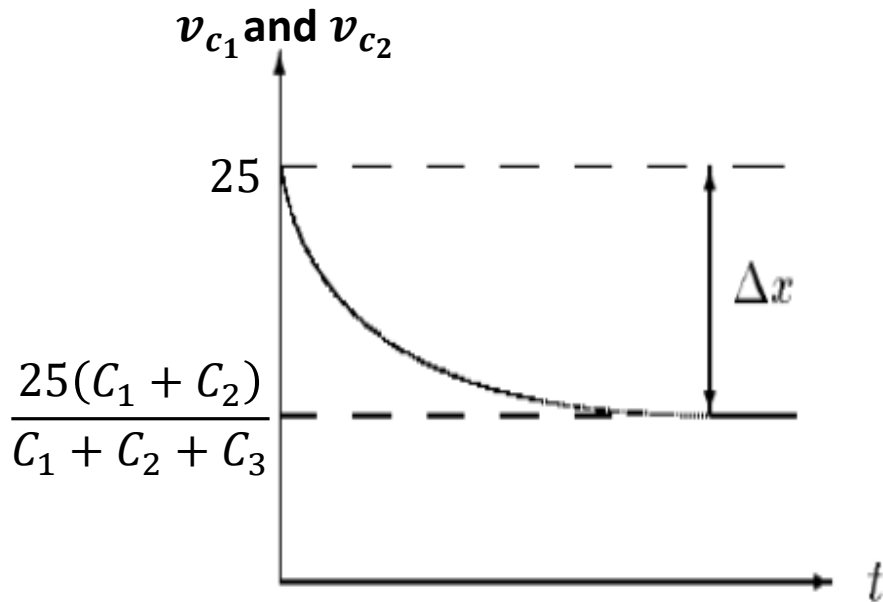
- $v_{c_1}(0^+) = 25$ and $v_{c_3}(0^+) = 0 \Rightarrow 25(C_1 + C_2) = \text{constant}$

$\Rightarrow (C_1 + C_2)v_{c_1}(t) + C_3v_{c_3}(t) = 25(C_1 + C_2)$

- The final values must satisfy $v_{c_1}(\infty) = v_{c_3}(\infty) \Rightarrow$

$$v_{c_1}(\infty) = v_{c_2}(\infty) = v_{c_3}(\infty) = \frac{25(C_1 + C_2)}{C_1 + C_2 + C_3}$$

Example 2 (non-trivial boundary conditions)



■ v_{c1} and v_{c2} : Case B

$$v_{c1}(t) = v_{c2}(t)$$

$$= \frac{25(C_1 + C_2)}{C_1 + C_2 + C_3} + \frac{25C_3}{C_1 + C_2 + C_3} e^{-\frac{t}{C_{eq}R_{eq}}}$$

■ v_{c3} : Case A

$$v_{c3}(t) = \frac{25(C_1 + C_2)}{C_1 + C_2 + C_3} (1 - e^{-\frac{t}{C_{eq}R_{eq}}})$$