

In-Class Activity 1 Solutions

Wednesday, September 25

Question 1 (Finding limits): Compute the following

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$$

Solution: The numerator and denominator both tend to zero as x tends to 0. Multiplying both numerator and denominator by $\sqrt{x+1} + 1$, we have

$$\frac{\sqrt{x+1} - 1}{x} = \frac{(\sqrt{x+1} - 1)(\sqrt{x+1} + 1)}{x(\sqrt{x+1} + 1)} = \frac{x+1-1}{x(\sqrt{x+1} + 1)} = \frac{1}{\sqrt{x+1} + 1}$$

Then,

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{2}$$

Question 2 (Application of chain rule or differentiation of composite functions): Find the derivative of the function

$$y = \sqrt{x + \sqrt{x + \sqrt{x}}}$$

Solution: Suppose $u = x + \sqrt{x + \sqrt{x}}$ and $y = \sqrt{u} = u^{1/2}$. Since u is a function of x and y is a function of u , using chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We have

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2}$$

$$\frac{du}{dx} = \frac{d}{dx}(x + (x + x^{1/2})^{1/2}) = \frac{d}{dx}(x) + \frac{d}{dx}(x + x^{1/2})^{1/2},$$

by applying the sum rule. Now, we need to first find $\frac{d}{dx}(x + x^{1/2})^{1/2}$. Let $w = x + x^{1/2}$ and $z = w^{1/2}$. Then, w is a function of x and z is a function of w , so chain rule implies

$$\frac{dz}{dx} = \frac{dz}{dw} \cdot \frac{dw}{dx} = \frac{1}{2}w^{-1/2} \left(1 + \frac{1}{2}x^{-1/2}\right) = \frac{1}{2} \left(x + x^{1/2}\right)^{-1/2} \left(1 + \frac{1}{2}x^{-1/2}\right)$$

Therefore,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2} \left[x + \left(x + x^{1/2} \right)^{1/2} \right]^{-1/2} \left[1 + \frac{1}{2} \left(x + x^{1/2} \right)^{-1/2} \left(1 + \frac{1}{2} x^{-1/2} \right) \right]$$

Question 3 (Application of differentiation involving logarithmic functions): The function f is defined for $x > 1$ by

$$f(x) = \frac{1}{x} \ln \left(\frac{1}{x} \right) + \left(1 - \frac{1}{x} \right) \ln \left(1 - \frac{1}{x} \right)$$

Show that the derivative of f is

$$f'(x) = \frac{\ln(x-1)}{x^2}$$

Solution: Let $f(x) = g(x) + h(x)$, where

$$g(x) = \frac{1}{x} \ln \left(\frac{1}{x} \right) \quad h(x) = \left(1 - \frac{1}{x} \right) \ln \left(1 - \frac{1}{x} \right)$$

$$g'(x) = \frac{1}{x} \frac{d}{dx} \left[\ln \left(\frac{1}{x} \right) \right] + \ln \left(\frac{1}{x} \right) \frac{d}{dx} x^{-1},$$

using product rule.

Let $z = \ln y$ with $y = \left(\frac{1}{x} \right)$. By chain rule

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \left(\frac{1}{y} \right) \left(\frac{-1}{x^2} \right) = \left(\frac{-1}{x} \right)$$

Then,

$$g'(x) = -\frac{1}{x^2} - \frac{1}{x^2} \ln \left(\frac{1}{x} \right)$$

Next, we have to find $h'(x)$.

$$h'(x) = \ln \left(1 - \frac{1}{x} \right) \frac{1}{x^2} + \left(1 - \frac{1}{x} \right) \frac{d}{dx} \ln \left(1 - \frac{1}{x} \right)$$

Let $w = \ln u$ with $u = 1 - \frac{1}{x}$. By chain rule

$$\frac{dw}{dx} = \frac{dw}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{1}{x^2} = \frac{1}{x(x-1)}$$

Then,

$$h'(x) = \ln\left(1 - \frac{1}{x}\right) \frac{1}{x^2} + \left(1 - \frac{1}{x}\right) \frac{1}{x(x-1)} = \frac{1}{x^2} \ln\left(1 - \frac{1}{x}\right) + \frac{1}{x^2}$$

$$f'(x) = g'(x) + h'(x) = -\frac{1}{x^2} - \frac{1}{x^2} \ln\left(\frac{1}{x}\right) + \frac{1}{x^2} \ln\left(1 - \frac{1}{x}\right) + \frac{1}{x^2}$$

$$= \frac{1}{x^2} \left[\ln\left(1 - \frac{1}{x}\right) - \ln\left(\frac{1}{x}\right) \right]$$

$$f'(x) = \frac{1}{x^2} \ln\left(\frac{1 - \frac{1}{x}}{\frac{1}{x}}\right),$$

using $\ln A - \ln B = \ln\left(\frac{A}{B}\right)$. Therefore,

$$f'(x) = \frac{\ln(x-1)}{x^2}$$

[Notes: Please note that some of the answers are shown in great detail. Once you practice more, you will see that not all steps are required to be written explicitly. Because we are learning at this stage, I have provided the very detailed solution here.]